Graph Theory Problem Set - Solutions

1. \[ \alpha \leftrightarrow \beta \rightarrow \gamma \]

2. Attach with a twist.

Second Solution

\[ \alpha \rightarrow \beta \rightarrow \gamma \]

\[ \alpha \rightarrow \beta \rightarrow \gamma \]

\[ \alpha \rightarrow \beta \rightarrow \gamma \]

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\[ \alpha \rightarrow \beta \rightarrow \gamma \]

\[ \alpha \rightarrow \beta \rightarrow \gamma \]
2. **Background**

\[
[X] = [X] - [X]
\]

\[
[06] = 3 [6]
\]

\[
[Y] = -[Y]
\]

Also \[
\begin{array}{c}
\begin{array}{c}
\text{also}
\end{array}
\end{array}
\]

follows from the definition of \([6]\) given in the class notes#2.

\[
\begin{array}{c}
\begin{array}{c}
\text{Thus}
\end{array}
\end{array}
\]

Here we have used the facts:

\[
\begin{array}{c}
\begin{array}{c}
\text{both consequences of the definition of the Penrose evaluation.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{of the Penrose evaluation.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{of the Penrose evaluation.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{of the Penrose evaluation.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{of the Penrose evaluation.}
\end{array}
\end{array}
\]
2. (continued)

\[
\begin{pmatrix}
Y_{11} & Y_{12} & Y_{13} \\
Y_{21} & Y_{22} & Y_{23} \\
Y_{31} & Y_{32} & Y_{33}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
1 & -1 & 1
\end{pmatrix}
+ \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\]

(applying Penrose formula to \(a + b\):

\[
\begin{pmatrix}
Y_{11} & Y_{12} & Y_{13} \\
Y_{21} & Y_{22} & Y_{23} \\
Y_{31} & Y_{32} & Y_{33}
\end{pmatrix}
= \begin{pmatrix}
Y_{11} \\
Y_{21} \\
Y_{31}
\end{pmatrix}
+ \begin{pmatrix}
Y_{12} \\
Y_{22} \\
Y_{32}
\end{pmatrix}
+ \begin{pmatrix}
Y_{13} \\
Y_{23} \\
Y_{33}
\end{pmatrix}.
\]

This is the best we can do. There is no way to eliminate the crossing in the third term.

3. (a) In a planar endgame of Brussels Sprouts we have one marker in each region.

\[\text{e.g.} \quad \begin{array}{c}
\text{A} \quad \text{B} \\
\text{C} \quad \text{R}
\end{array} \]

single marker in outer region

single marker in \(A\)

single marker in \(B\)

single marker in \(C\)

\( \text{single marker in region} R \)

(If there were \(>1\) marker in any region, then one could make a move in that region, connecting two markers. Thus we would not have an endgame position.)
Note that in this endgame position there are $\frac{4}{i + j + k + l}$ markers.

- At the beginning of a game with $n$ starting nodes, there are $4n$ markers.
  
  e.g. \[ \{ + + \} \quad n=2 \quad \text{markers} = 8. \]

- The number $M$ of markers is unchanged when you move.

\[ + + + \underbrace{\rightarrow} \quad + + + \]

\[ M = 8 \]

- Therefore, if a game starts with $M$ markers, any endgame position will have $M$ regions.

\[ + \underbrace{\rightarrow} \quad \bigcirc \underbrace{\rightarrow} \quad \bigcirc \]

\[ M=4 \]

\[ M=4 \rightarrow \frac{5}{6} \]

- Each endgame has a corresponding graph as shown.
3. (continued)

Let \( G \) be the graph of an endgame \( E \). Then

\[
\begin{align*}
\text{\( f(G) = M = 4n \)} \\
\text{(\( n = \# \text{ of starting nodes} \))} \\
\text{\( e(G) = 2M \text{ where} \)} \\
\text{\( m = \# \text{ of moves in the game} \)} \\
\text{\( v(G) = n + m \)}
\end{align*}
\]

In a planar game, \( v - e + f = 2 \).
Thus \( (n + m) - 2m + 4n = 2 \)
\[ \Rightarrow \ 5n - m = 2 \]
\[ \Rightarrow \ 5n - m = 2 \]

Thus we have proved that for a planar game of Brussels Sprouts,
every game (starting with \( n \) nodes)
lasts for exactly \( 5n - 2 \) moves.

(b) This solves (b) as well.

(c) Suppose you play on an orientable surface \( S_g \) of genus \( g \). Suppose your endgame position has only disk-like regions. Then \( v - e + f = 2 - 2g \)

and (next page)
\[(n+m) - 2m + 4n = 2 - 2g\]
\[5n - m = 2 - 2g\]
\[m = (5n - 2) + 2g\]

Thus playing on a surface can yield up to 2g more moves. If you simply play and get to an endgame (possibly with some non-disk regions), then you will find that \[5n - 2 \leq m \leq (5n - 2) + 2g\].

If you are on the projective plane and the endgame has only disk-like regions, then \[v - e + f = 1\] (see notes on Euler characteristic).

\[5n - m = 1\]
\[m = 5n - 1\] is possible on \(\mathbb{P}^2\).

\[A 4 \text{ move game with } n = 1, \text{ on } \mathbb{P}^2.\]
4. These two colorings have opposite parity.

5. (a) This diagram represents $K_6 \subset \text{Torus}$.

(b) $v - e + \chi = 2 - 2g$

$g = 1$, $v = 6$, $e = 6 \cdot \frac{5}{2} = 15$

$6 - 15 + \chi = 0$

$\Rightarrow \chi = 9$
6. Solution omitted here. Will discuss in the problem session or come by in the fall.
7. Consider possible colorings of J.

Case 1: \( a, b, c \) all distinct.

\[
\begin{align*}
\text{Suppose} \quad r & \quad \text{are forced} \quad b \quad \text{are forced} \\
\text{given} \quad & \\
\end{align*}
\]

These are both odd permutations. Any odd product of odd permutations is odd. \( \therefore \)

\[
\begin{align*}
\text{will produce an odd permutation of distinct input colors} & \quad \therefore \text{cannot} \\
\text{yield a coloring of} \quad & \quad \text{with distinct input colors.}
\end{align*}
\]
Case 2. Allow some inputs to be the same.

\[
\begin{align*}
  r & \quad \rightarrow \quad b \\
  \quad \downarrow \quad r & \quad \rightarrow \quad b & \quad \rightarrow \quad p \\
  \quad b & \quad \rightarrow \quad p \\
\end{align*}
\]

for example

You will find in all cases, the color missed on input appears on output. Thus an odd composition of J's cannot have same colors as inputs and outputs. This completes the proof.