1. Recall the rules for sprouts:
   (a) start with no dots in the plane, e.g. $n=3$

   (b) A legal move connects two dots and puts a dot in the middle of the connecting arc.

   (c) No dot can have more than 3 arcs touching it.

   (d) Winner makes the last move.

For example:

- start with 3

- $ightarrow$

- $ightarrow$

- $ightarrow$

- $ightarrow$
(i) Play a number of 2 dot games.

\[\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}\]

\[1, 2, 3, 4, 5\] are the orders of moves (first, second, third...).

Here 1st player won.

Who should win the 2 dot game?

Notice that in the game above there were 5 moves. 
\[5 = 2 \times 3 - 1.\]

(ii) Give an example of a 3 dot sprouts game that has \(3 \times 3 - 1 = 8\) moves.

(iii) I claim that the longest possible \(n\) dot sprouts game has \((3n - 1)\) moves.

Can you give an example of such a game for any given value of \(n\)? Explore examples.
2. A perfect number \( n \) is a number that is equal to the sum of all its divisors (\( \neq \) itself).
For example, \( 6 = 1 + 2 + 3 \) is perfect because the numbers that divide \( 6 = 1 \times 2 \times 3 \) are 1, 2, and 3.
(i) Show that 28 is a perfect number.
(ii) Show that 496 is a perfect number.
(iii) Euclid proved that if \( 2^p - 1 \) is a prime number, then \( N = 2^{p-1}(2^p - 1) \) is perfect. For example,
\[
6 = 2 \times 3 = 2^{1}(2^2 - 1)
\]
and
\[
28 = 2^2 \times 7 = 2^{3}(2^2 - 1)
\]
Show that 496 also has this form for \( p = 5 \).
(iv) Find a perfect number larger than 496.
3. A prisoner has been given the following task. Place one grain of sand on the first square of a chessboard, 2 grains on the second square, 4 grains on the third square, ..., and so on until you put $2^{64}$ grains on the last square.

Thus the prisoner must place $N = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{64}$ grains of sand on the chessboard.

(i) I claim that $\frac{1 + 2 + 2^2 + 2^3 + \ldots + 2^{64}}{2} = 2^{65} - 1$. Can you explain why this is so?

(ii) Assuming that the prisoner places one grain per second on the chessboard, how many years will it take for him to place $(2^{65} - 1)$ sand grains on the board?
Max found the following "proof" in a very old book.

**Theorem.** $-1 = (1 + 2 + 2^2 + 2^3 + 2^4 + ...)$

**Proof.** Let $S = 1 + 2 + 2^2 + 2^3 + ...$

$S = 1 + 2 + 4 + 8 + 16 + ...$

$\Rightarrow S = 1 + 2(1 + 2 + 4 + 8 + ...)$

$\Rightarrow S = 1 + 2S$

$\Rightarrow 2S + 1 = S$

$\Rightarrow S + 1 = 0$

$\Rightarrow S = -1.$

Thus $-1 = 1 + 2 + 4 + 8 + 16 + ...$ Q.E.D

Max said "Well I would say that $\infty = 1 + 2 + 4 + 8 + 16 + ...$"

So this manuscript shows that **Infinity Equals Minus One**.

I do not know what this means, but it must be very deep.

Write a dialogue between yourself and Max discussing this "proof."
4. This is a "knot diagram":

\[ \text{The trefoil knot.} \]

Knot diagrams are composed of arcs put together so that there are crossings where the arcs meet.

I "color" a knot diagram with numbers by the following rule:

\[ \begin{align*}
    c & \Rightarrow c = 2b - a. \\
    b & \quad \text{Each arc receives a color.}
\end{align*} \]

Try to color the trefoil:

\[ \begin{align*}
    b + c & = 2 \quad \text{but } 2 \times 2 - 1 = 3 \\
    \text{So I need } b = 0. \\
    \text{OK! } b \equiv 0 \pmod{3}.
\end{align*} \]
(i) Color this knot using \( \frac{zb-a}{a+b} \) and in mod 3.

(ii) Show that can be colored modulo 5.

and that it cannot be colored modulo 3 (with more than one color).

(iii) Show that these rings cannot be colored with more than one color modulo 3.
(iv) Note that for mod 3 colorings we have the possibilities at crossings:

\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{array}
\]

Check that mod 3 coloring is the same as satisfying the rule that every crossing, e.g., \( \frac{1}{1} \) has either 3 colors (e.g., \( 1 \)) or 1 color (e.g., \( \frac{1}{2} \)).

\( \frac{2 \times 2 - 2 \times 2 = 4 - 2 = 2}{\text{mod } 3} \)

(v) You can see that if you move the knot around by moves like

\[
\begin{array}{ccc}
\text{I: } & \text{II: } & \text{III: } \\
\text{C } & \text{C } & \text{C }
\end{array}
\]

then if one diagram is colored then so will the other ones.
Use this method to induce colorations modulo 3 on each of the above knots.
5. Recall Wilson's Theorem:
   \[ P \text{ is prime if and only if } P \text{ divides } (p-1)!+1. \]

Verify by direct calculation that \( B \) divides \( 12!+1 \).

Note: This means you want to show that \( 12! \equiv -1 \mod 13 \).

Thus you want to calculate
\[
12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1
\mod 13.
\]

Start forming the products and reduce modulo 13 at each stage.

[Here is an example for \( 6!+1 \mod 7 \):

\[
6 \times 5 \equiv 30 \equiv 2 \mod 7
\]
\[
6 \times 5 \times 4 \equiv 2 \times 4 \equiv 8 \equiv 1 \mod 7
\]
\[
6 \times 5 \times 4 \times 3 \equiv 1 \times 3 \equiv 3 \mod 7
\]

\( \Rightarrow 6! \equiv 3 \times 2 = 6 \equiv -1 \mod 7 \)\]