CHAPTER 13

From: "Graph Theory"
by Frank Harary

[For Harary,
a graph is a simple graph.
No loops and no multiple
edges.]

MATRICES

In orderly disorder they
Wait coldly columned, dead, prosaic.
Poet, breathe on them and pray
They burn with life in your mosaic.

J. LUZZATO

A graph is completely determined by either its adjacencies or its incidences. This information can be conveniently stated in matrix form. Indeed, with a given graph, adequately labeled, there are associated several matrices, including the adjacency matrix, incidence matrix, cycle matrix, and cocycle matrix. It is often possible to make use of these matrices in order to identify certain properties of a graph. The classic theorem on graphs and matrices is the Matrix-Tree Theorem, which gives the number of spanning trees in any labeled graph. The matroids associated with the cycle and cocycle matrices of a graph are discussed.

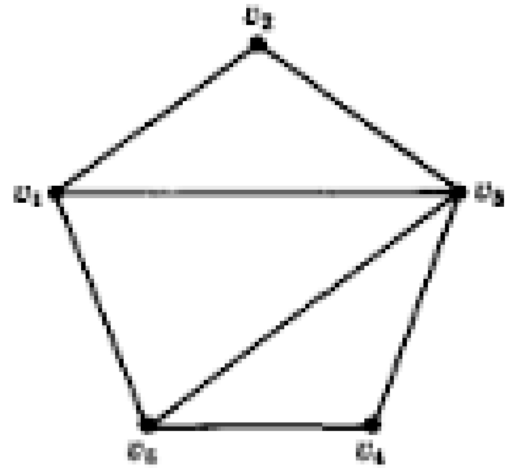
THE ADJACENCY MATRIX

The *adjacency matrix* $A = [a_{ij}]$ *of a labeled graph* G with p points is the $p \times p$ matrix in which $a_{ij} = 1$ if v_i is adjacent with v_j and $a_{ij} = 0$ otherwise. Thus there is a one-to-one correspondence between labeled graphs with p points and $p \times p$ symmetric binary matrices with zero diagonal.

Figure 13.1 shows a labeled graph G and its adjacency matrix A. One immediate observation is that the row sums of A are the degrees of the points of G. In general, because of the correspondence between graphs and matrices, any graph-theoretic concept is reflected in the adjacency matrix. For example, recall from Chapter 2 that a graph G is connected if and only if there is no partition $V = V_1 \cup V_2$ of the points of G such that no line joins a point of V_1 with a point of V_2. In matrix terms we may say that G is connected if and only if there is no labeling of the points of G such that its adjacency matrix has the reduced form

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

G:



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Fig. 13.1. A labeled graph and its adjacency matrix.

where A_{11} and A_{22} are square. If A_1 and A_2 are adjacency matrices which correspond to two different labelings of the same graph G, then for some permutation matrix P, $A_1 = P^{-1}A_2P$. Sometimes a labeling is irrelevant, as in the following results which interpret the entries of the powers of the adjacency matrix.

Theorem 13.1 Let G be a labeled graph with adjacency matrix A. Then the i, j entry of A^n is the number of walks of length n from v_i to v_j.

Corollary 13.1(a) For $i \neq j$, the i, j entry of A^2 is the number of paths of length 2 from v_i to v_j. The i, i entry of A^2 is the degree of v_i and that of A^3 is twice the number of triangles containing v_i.

Corollary 13.1(b) If G is connected, the distance between v_i and v_j for $i \neq j$ is the least integer n for which the i, j entry of A^n is nonzero.

The *adjacency matrix of a labeled digraph D* is defined similarly: $A = A(D) = [a_{ij}]$ has $a_{ij} = 1$ if arc $v_i v_j$ is in D and is 0 otherwise. Thus $A(D)$ is not necessarily symmetric. Some results for digraphs using $A(D)$ will be given in Chapter 16. By definition of $A(D)$, the adjacency matrix of a given graph can also be regarded as that of a symmetric digraph. We now apply this observation to investigate the determinant of the adjacency matrix of a graph, following [H27].

A *linear subgraph of a digraph D* is a spanning subgraph in which each point has indegree one and outdegree one. Thus it consists of a disjoint spanning collection of directed cycles.

Theorem 13.2 If D is a digraph whose linear subgraphs are D_i, $i = 1, \cdots, n$, and D_i has e_i even cycles, then

$$\det A(D) = \sum_{i=1}^{n} (-1)^{e_i}.$$

Every graph G is associated with that digraph D with arcs v_iv_j and v_jv_i whenever v_i and v_j are adjacent in G. Under this correspondence, each linear subgraph of D yields a spanning subgraph of G consisting of a point disjoint collection of lines and cycles, which is called a *linear subgraph of a graph*.

Those components of a linear subgraph of G which are lines correspond to the 2-cycles in the linear subgraph of D in a one-to-one fashion, but those components which are cycles of G correspond to two directed cycles in D. Since $A(G) = A(D)$ when G and D are related as above, the determinant of $A(G)$ can be calculated.

Corollary 13.2(a) If G is a graph whose linear subgraphs are G_i, $i = 1, \cdots, n$, where G_i has e_i even components and c_i cycles, then

$$\det A(G) = \sum_{i=1}^{n} (-1)^{e_i} 2^{c_i}.$$

THE INCIDENCE MATRIX

A second matrix, associated with a graph G in which the points and lines are labeled, is the *incidence matrix* $B = [b_{ij}]$. This $p \times q$ matrix has $b_{ij} = 1$ if v_i and x_j are incident and $b_{ij} = 0$ otherwise. As with the adjacency matrix, the incidence matrix determines G up to isomorphism. In fact any $p - 1$ rows of B determine G since each row is the sum of all the others modulo 2.

The next theorem relates the adjacency matrix of the line graph of G to the incidence matrix of G. We denote by B^T the transpose of matrix B.

Theorem 13.3 For any (p, q) graph G with incidence matrix B,

$$A(L(G)) = B^T B - 2I_q.$$

Let M denote the matrix obtained from $-A$ by replacing the ith diagonal entry by $\deg v_i$. The following theorem is contained in the pioneering work of Kirchhoff [K7].

Theorem 13.4 (Matrix-Tree Theorem) Let G be a connected labeled graph with adjacency matrix A. Then all cofactors of the matrix M are equal and their common value is the number of spanning trees of G.

Proof. We begin the proof by changing either of the two 1's in each column of the incidence matrix B of G to -1, thereby forming a new matrix E. (We will see in Chapter 16 that this amounts to arbitrarily orienting the lines of G and taking E as the incidence matrix of this oriented graph.)

The i, j entry of EE^T is $e_{i1}e_{j1} + e_{i2}e_{j2} + \cdots + e_{iq}e_{jq}$, which has the value $\deg v_i$ if $i = j$, -1 if v_i and v_j are adjacent, and 0 otherwise. Hence $EE^T = M$.

Consider any submatrix of E consisting of $p - 1$ of its columns. This $p \times (p - 1)$ matrix corresponds to a spanning subgraph H of G having $p - 1$ lines. Remove an arbitrary row, say the kth, from this matrix to obtain a square matrix F of order $p - 1$. We will show that $|\det F|$ is 1 or 0 according as H is or is not a tree. First, if H is not a tree, then because H has p points and $p - 1$ lines, it is disconnected, implying that there is a component not containing v_k. Since the rows corresponding to the points of this component are dependent, det $F = 0$. On the other hand, suppose H is a tree. In this case, we can relabel its lines and points other than v_k as follows: Let $u_1 \neq v_k$ be an endpoint of H (whose existence is guaranteed by Corollary 4.1(a)), and let y_1 be the line incident with it; let $u_2 \neq v_k$ be any endpoint of $H - u_1$ and y_2 its incident line, and so on. This relabeling of the points and lines of H determines a new matrix F' which can be obtained by permuting the rows and columns of F independently. Thus $|\det F'| = |\det F|$. However, F' is lower triangular with every diagonal entry $+1$ or -1; hence, $|\det F| = 1$.

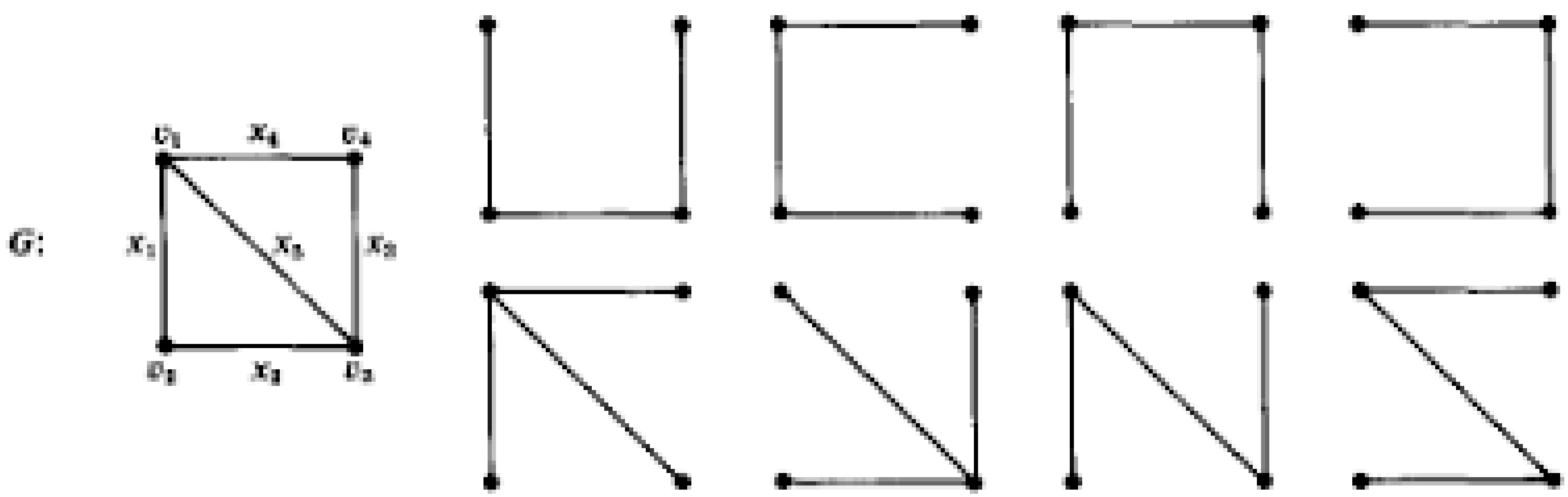


Fig. 13.2. $K_4 - x$ and its spanning trees.

The following algebraic result, usually called the Binet-Cauchy Theorem, will now be very useful.

Lemma 13.4(a) If P and Q are $m \times n$ and $n \times m$ matrices, respectively, with $m \leq n$, then det PQ is the sum of the products of corresponding major determinants of P and Q.

(A major determinant of P or Q has order m, and the phrase "corresponding major determinants" means that the columns of P in the one determinant are numbered like the rows of Q in the other.)

We apply this lemma to calculate the first principal cofactor of M. Let E_1 be the $(p - 1) \times q$ submatrix obtained from E by striking out its first row. By letting $P = E_1$ and $Q = E_1^T$, we find, from the lemma, that the first principal cofactor of M is the sum of the products of the corresponding major determinants of E_1 and E_1^T. Obviously, the corresponding major determinants have the same value. We have seen that their product is 1 if the columns from E_1 correspond to a spanning tree of G and is 0 otherwise. Thus the sum of these products is exactly the number of spanning trees.

The equality of all the cofactors, both principal and otherwise, holds for every matrix whose row sums and column sums are all zero, completing the proof.

To illustrate the Matrix-Tree Theorem, we consider a labeled graph G taken at random, say $K_4 - x$. This graph, shown in Fig. 13.2, has eight spanning trees, since the 2,3 cofactor, for example,

$$\text{of} \quad M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \qquad \text{is} \quad -\begin{vmatrix} 3 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & 0 & 2 \end{vmatrix} = 8.$$

The number of labeled trees with p points is easily found by applying the Matrix-Tree Theorem to K_p. Each principal cofactor is the determinant of order $p - 1$:

$$\begin{vmatrix} p-1 & -1 & \cdot & \cdot & \cdot & -1 \\ -1 & p-1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot & \cdot & p-1 \end{vmatrix}$$

Subtracting the first row from each of the others and adding the last $p - 2$ columns to the first yields an upper triangular matrix whose determinant is p^{p-2}.

Corollary 13.4(a) The number of labeled trees with p points is p^{p-2}.