Introduction to higher homotopy groups and obstruction theory

Michael Hutchings

February 17, 2011

Abstract

These are some notes to accompany the beginning of a second-semester algebraic topology course. The goal is to introduce homotopy groups and their uses, and at the same time to prepare a bit for the study of characteristic classes which will come next. These notes are not intended to be a comprehensive reference (most of this material is covered in much greater depth and generality in a number of standard texts), but rather to give an elementary introduction to selected basic ideas.

Contents

0 Prerequisites

The prerequisites for this course are the fundamental group, covering spaces, singular homology, cohomology, cup product, CW complexes, manifolds, and Poincaré duality.

Regarding covering spaces, recall the Lifting Criterion: if $\widetilde{X} \xrightarrow{p} X$ is a covering space, if $f : (Y, y_0) \to (X, x_0)$, and if $p(\widetilde{x}_0) = x_0$, then f lifts to a map $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ with $p \circ \widetilde{f} = f$ if and only if

$$f_* \pi_1(Y, y_0) \subset p_* \pi_1(\widetilde{X}, \widetilde{x}_0).$$

Regarding homology, there are many ways to define it. All definitions of homology that satisfy the Eilenberg-Steenrod axioms give the same answer for any reasonable space, e.g. any space homotopy equivalent to a CW complex. We will often use *cubical* singular homology, because this is convenient for some technical arguments. To define the homology of a space X, we consider "singular cubes"

$$\sigma : I^k \to X$$

where $I = [0, 1]$. The cube I^k has $2k$ codimension 1 faces

$$F_i^0 = \{(t_1, \ldots, t_k) \mid t_i = 0\},$$
$$F_i^1 = \{(t_1, \ldots, t_k) \mid t_i = 1\},$$

for $1 \leq i \leq k$, each of which is homeomorphic to I^{k-1} in an obvious manner. We now define a boundary operator on formal linear combinations of singular cubes by

$$\partial\sigma := \sum_{i=1}^{k}(-1)^i \left(\sigma|_{F_i^0} - \sigma|_{F_i^1}\right). \tag{0.1}$$

This satisfies $\partial^2 = 0$. We could try to define $C_k(X)$ to be the free $\mathbb{Z}$-module generated by singular k-cubes and $H_k(X) = \operatorname{Ker}(\partial)/\operatorname{Im}(\partial)$. However this is not quite right because then the homology of a point would be $\mathbb{Z}$ in every nonnegative degree. To fix the definition, we mod out by the subcomplex of *degenerate* cubes that are independent of one of the coordinates on I^k, and then it satisfies the Eilenberg-Steenrod axioms.

1 Higher homotopy groups

Let X be a topological space with a distinguished point x_0. The fundamental group $\pi_1(X, x_0)$ has a generalization to *homotopy groups* $\pi_k(X, x_0)$, defined for each positive integer k.

The definition of π_k is very simple. An element of $\pi_k(X, x_0)$ is a homotopy class of maps

$$f : (I^k, \partial I^k) \longrightarrow (X, x_0).$$

Here $I = [0, 1]$. Equivalently, an element of $\pi_k(X, x_0)$ is a homotopy class of maps $(S^k, p) \to (X, x_0)$ for some distinguished point $p \in S^k$.

The group operation is to stack two cubes together and then shrink:

$$fg(t_1, \ldots, t_k) := \begin{cases} f(2t_1, t_2, \ldots, t_k), & t_1 \leq 1/2, \\ g(2t_1 - 1, t_2, \ldots, t_k), & t_1 \geq 1/2. \end{cases}$$

[Draw picture.] It is an exercise to check that $\pi_k(X, x_0)$ is a group, with the identity element given by the constant map sending I^k to x_0.

A map $\phi : (X, x_0) \to (Y, y_0)$ defines an obvious map

$$\begin{aligned} \phi_* : \pi_k(X, x_0) &\longrightarrow \pi_k(Y, y_0), \\ f &\longmapsto f \circ \phi. \end{aligned}$$

This makes π_k a functor from pointed topological spaces to groups. Moreover, ϕ_* is clearly invariant under based homotopy of ϕ, so if ϕ is a based homotopy equivalence then ϕ_* is an isomorphism. (We will see in §3 how to remove the "based" condition.)

We also define $\pi_0(X)$ to be the set of path components of X, although this has no natural group structure.

Here are two nice properties of the higher homotopy groups.

Proposition 1.1. *(a)* $\pi_k(X \times Y, (x_0, y_0)) = \pi_k(X, x_0) \times \pi_k(Y, y_0)$.

(b) If $k > 1$, then $\pi_k(X, x_0)$ is abelian.

Proof. (a) Exercise. (b) [Draw picture] $\square$

Example 1.2. If $k > 1$, then $\pi_k(S^1, p) = 0$.

Proof. Recall that there is a covering space $\mathbb{R} \to S^1$. Any map $f : S^k \to S^1$ lifts to a map $\widetilde{f} : S^k \to \mathbb{R}$, by the Lifting Criterion, since S^k is simply connected. Since $\mathbb{R}$ is contractible, $\widetilde{f}$ is homotopic to a constant map. Projecting this homotopy to S^1 defines a homotopy of f to a constant map. $\square$

Example 1.3. More generally, the same argument shows that if the universal cover of X is contractible, then $\pi_k(X, x_0) = 0$ for all $k > 1$. For example, this holds if X is a Riemann surface of positive genus. This argument is a special case of the long exact sequence in homotopy groups of a fibration, which we will learn about later.

Example 1.4. We will prove shortly that[1]

$$\pi_k(S^n, x_0) \simeq \begin{cases} 0, & 1 \leq k < n, \\ \mathbb{Z}, & k = n. \end{cases} \tag{1.1}$$

The higher homotopy groups $\pi_k(S^n)$ for $k > n$ are very complicated, and despite extensive study are not completely understood, even for $n = 2$. This is the bad news about higher homotopy groups: despite their simple definition, they are generally hard to compute[2]. However, for any given

[1]This can also be seen using some differential topology. Choose a point $x_1 \in S^n$ with $x_0 \neq x_1$. Let $f : (S^k, p) \to (S^n, x_0)$. By a homotopy we can arrange that f is smooth. Moreover if $k < n$ then we can arrange that $x_1 \notin f(S^k)$. Then f maps to $S^n \setminus \{x_1\} \simeq \mathbb{R}^n$, which is contractible, so f is homotopic to a constant map. If $k = n$ then we can arrange that f is transverse to x_1 so that x_1 has finitely many inverse images, to each of which is associated a sign. It can then be shown that the signed count $\#f^{-1}(x_1) \in \mathbb{Z}$ determines the homotopy class of f in $\pi_k(S^k, x_0)$. See e.g. Guillemin and Pollack.

[2]For example, recall that the Seifert-Van Kampen theorem gives an algorithm for computing, or at least finding a presentation of, $\pi_1(X, x_0)$ whenever X can be cut into simple pieces. The idea is that a loop in X can be split into paths which live in the various pieces. This fails for higher homotopy groups because if one attemps to cut a sphere into pieces, then these pieces might be much more complicated objects.

space we can usually compute at least the first few homotopy groups. And homotopy groups have important applications, for example to obstruction theory as we will see below.

2 The Hurewicz isomorphism theorem

To compute some higher homotopy groups, we begin by studying the relation between higher homotopy groups and homology. The key ingredient is the *Hurewicz homomorphism*

$$\Phi : \pi_k(X, x_0) \longrightarrow H_k(X),$$

defined as follows. Recall that the standard orientation of S^k determines a canonical isomorphism

$$H_k(S^k) \simeq Z.$$

The generator is the *fundamental class* $[S^k] \in H_k(S^k)$. If $f : (S^k, p) \to (X, x_0)$ represents $[f] \in \pi_k(X, x_0)$, we define

$$\Phi[f] := f_*[S_k] \in H_k(X).$$

Alternatively, if we use cubical singular homology, then a map $f : (I^k, \partial I^k) \to (X, x_0)$, regarded as a singular cube, defines a cycle in the homology class $\Phi[f]$. By the homotopy invariance of homology, $\Phi[f]$ is well-defined, i.e. depends only on the homotopy class of f. It is an exercise to check that Φ is a homomorphism.

Also, it follows immediately from the definition that the Hurewicz map Φ is natural, in the following sense: If $\psi : (X, x_0) \to (Y, y_0)$, then the diagram

$$\begin{tikzcd} \pi_k(X, x_0) \arrow[r, "\Phi"] \arrow[d, "\psi_*"] & H_k(X) \arrow[d, "\psi_*"] \\ \pi_k(Y, y_0) \arrow[r, "\Phi"] & H_k(Y) \end{tikzcd} \tag{2.1}$$

commutes. That is, Φ is a natural transformation of functors from π_k to H_k.

Recall now that if X is path connected, then Φ induces an isomorphism

$$\pi_1(X, x_0)_{\mathrm{ab}} \simeq H_1(X).$$

This fact has the following generalization, asserting that if X is also simply connected, then the first nontrivial higher homotopy group is isomorphic to the first nontrivial reduced homology group, and implying equation (1.1) for the first nontrivial homotopy groups of spheres.

Theorem 2.1 (Hurewicz isomorphism theorem). *Let $k \geq 2$. Suppose that X is path connected and that $\pi_i(X, x_0) = 0$ for all $i < k$. Then the Hurewicz map induces an isomorphism*

$$\pi_k(X, x_0) \simeq H_k(X).$$

To prove this theorem, we will need the following useful facts about homotopy groups. Below, $F_j I^k$ denotes the set of j-dimensional faces of the k-cube.

Lemma 2.2. *Let (X, x_0) be a pointed space.*

(a) For any $k \geq 1$, if $f : (S^k, p) \to (X, x_0)$ is homotopic, without fixing the base points, to a constant map, then $[f] = 0 \in \pi_k(X, x_0)$.

(b) For any $k \geq 2$, let $f : \partial I^{k+1} \to X$ be a map sending every $(k-1)$-dimensional face to x_0. Then

$$[f] = \sum_{\sigma \in F_k(I^{k+1})} [f|_\sigma] \in \pi_k(X, x_0).$$

Proof. We will just sketch the argument and leave the details as an exercise.

(a) Given a homotopy $\{f_t\}$ with $f_0 = f$ and f_1 constant, one can use the trajectory of p, namely the path $\gamma : I \to X$ sending $t \mapsto f_t(p)$, to modify this to a homotopy sending p to x_0 at all times. The endpoint of the homotopy will then be the composition of γ with a map $(S^k, p) \to (I, 0)$. Since I is contractible, we can further homotope this rel basepoints to a constant map. (Warning: More generally, if $f, g : (S^k, p) \to (X, x_0)$ are homotopic without fixing the base points, it does *not* follow that $[f] = [g] \in \pi_k(X, x_0)$. See §3.)

(b) We use a homotopy to shrink the restrictions of f to the k-dimensional faces of I^{k+1}, so that most of ∂I^{k+1} is mapped to x_0. Identifying $\partial I^{k+1} \simeq S^k$, and using the fact that $k \geq 3$, we can then move around these k-cubes until they are lined up as in the definition of composition in π_k. $\square$

Proof of the Hurewicz isomorphism theorem.[3] Using singular homology with cubes, we define a map

$$\Psi : C_k(X) \longrightarrow \pi_k(X, x_0)$$

as follows. The idea is that a generator of $C_k(X)$ is a cube whose boundary may map anywhere in X, and we have to modify it, via a chain homotopy, to obtain a cube whose boundary maps to x_0. To do so, we define a map

$$K : C_i(X) \longrightarrow C_{i+1}(X) \tag{2.2}$$

for $0 \le i \le k$ as follows.

Since X is path connected, for each 0-cube $p \in X$ we can choose a path $K(p)$ from x_0 to p. For each 1-cube $\sigma : I \to X$, there is a map $\partial I^2 \to X$ sending the four faces to x_0, σ, $K(\sigma(0))$, and $K(\sigma(1))$. Since X is simply connected, this can be extended to a map $K(\sigma) : I^2 \to X$ such that

$$\partial K(\sigma) = \sigma - K(\partial\sigma). \tag{2.3}$$

Continuing by induction on i, if $1 \le i < k$, then for each i-cube $\sigma : I^i \to X$, we can choose[4] an $(i+1)$-cube $K(\sigma) : I^{i+1} \to X$ which sends the faces to x_0, σ, and the faces of $K(\partial\sigma)$, and therefore satisfies equation (2.3). Finally, if $\sigma : I^k \to X$ is a k-cube, then there is a map $\partial I^{k+1} \to X$ sending the faces to x_0, σ, and the faces of $K(\partial\sigma)$. Let F denote the face sent to x_0. Identifying $(\partial I^{k+1}/F, F) \simeq (I^k, \partial I^k)$, this gives an element $\Psi(\sigma) \in \pi_k(X, x_0)$. Moreover, it is easy to see that the sum of the faces other than F is homologous to $\Psi(\sigma)$ regarded as a cube, i.e. there is a cube $K(\sigma)$ with

$$\partial K(\sigma) = \sigma - K\partial\sigma - \Phi(\Psi(\sigma)). \tag{2.4}$$

While Ψ may depend on the above choices, we claim that Ψ induces a map on homology which is inverse to Φ. To start, we claim that if $\sigma : I^{k+1} \to X$ is a $(k+1)$-cube, then

$$\Psi(\partial\sigma) = 0. \tag{2.5}$$

[3] Later in the course we will see a shorter and slicker proof using spectral sequences, but which I think ultimately has the same underlying geometric content.

[4] It is not really necessary to make infinitely many choices in this proof, but the argument is less awkward this way. Incidentally, when σ is degenerate, we can and should choose $K(\sigma)$ to be degenerate as well, so that we have a well-defined map (2.2) on the cubical singular chain complex.

To see this, first note that by Lemma 2.2(b), we have $\Psi(\partial\sigma) = [f]$, where $f : \partial I^{k+1} \to X$ sends each k-dimensional face of I^{k+1} to Ψ of the corresponding face of σ. By cancelling stuff along adjacent faces, f is homotopic, without fixing base points, to $\sigma|_{\partial I^{k+1}}$. This is homotopic to a constant map since it extends over I^{k+1}. So by Lemma 2.2(a) we conclude that $[f] = 0$. It follows from (2.5) that Ψ induces a map $\Psi_* : H_k(X) \to \pi_k(X, x_0)$.

It follows immediately from (2.4) that $\Phi \circ \Psi = \mathrm{id}_{H_k(X)}$.

Also, we can make the choices in the definition of K so that:

(*) If $i < k$ and if $\sigma : I^i \to X$ is a constant map to x_0, then $K(\sigma)$ is also a constant map to x_0.

It is then easy to see that $\Psi_* \circ \Phi = \mathrm{id}_{\pi_k(X,x_0)}$. □

3 Dependence of π_k on the base point

We will now show that if X is path connected, then the homotopy groups of X for different choices of base point are isomorphic, although not always equal. This is worth understanding properly, since analogous structures arise later in the course and also are common elsewhere in mathematics.

If $\gamma : [0, 1] \to X$ is a path from x_0 to x_1, we define a map

$$\Phi_\gamma : \pi_k(X, x_1) \longrightarrow \pi_k(X, x_0)$$

as follows. Let us temporarily reparametrize the k-cube as $I^k = [-1, 1]^k$. If $f : [-1, 1]^k \to X$ and $t = (t_1, \ldots, t_k) \in I^k$, let $m = \max\{|t_i|\}$ and define

$$\Phi_\gamma(f)(t) := \begin{cases} f(2t), & m \leq 1/2, \\ \gamma(2(1-m)), & m \geq 1/2. \end{cases}$$

[Draw picture.] Clearly this gives a well-defined function on homotopy groups. It is an exercise to show that Φ_γ is a group homomorphism.

The following additional facts are easy to see:

(i) If γ is homotopic to γ' (rel endpoints), then

$$\Phi_\gamma = \Phi_{\gamma'} : \pi_k(X, x_1) \longrightarrow \pi_k(X, x_0).$$

(ii) Suppose that γ_1 is a path from x_0 to x_1 and γ_2 is a path from x_1 to x_2, and let $\gamma_2\gamma_1$ denote the composite path. Then

$$\Phi_{\gamma_2\gamma_1} = \Phi_{\gamma_2}\Phi_{\gamma_1} : \pi_k(X, x_2) \longrightarrow \pi_k(X, x_0).$$

(iii) If γ is the constant path at x_0, then

$$\Phi_\gamma = \mathrm{id}_{\pi_k(X,x_0)}\,.$$

The above three properties[5] imply:

Proposition 3.1. *If X is path connected, then*

$$\pi_k(X, x_0) \simeq \pi_k(X, x_1) \tag{3.1}$$

for any two points $x_0, x_1 \in X$. Moreover, if X is simply connected, then this isomorphism is canonical, *and so $\pi_k(X)$ is a well-defined group without the choice of a base point.*

If X is not simply connected, then the isomorphism (3.1) might not be canonical. In particular, for a noncontractible loop γ based at x_0, the isomorphism Φ_γ of $\pi_k(X, x_0)$ might not be the identity. In general, by (i)–(iii) above, these isomorphisms define an action of $\pi_1(X, x_0)$ on $\pi_k(X, x_0)$. When $k = 1$, this action is just conjugation in $\pi_1(X, x_0)$, which of course is nontrivial whenever $\pi_1(X, x_0)$ is not abelian. If the action of $\pi_1(X, x_0)$ on $\pi_k(X, x_0)$ is nontrivial, then $\pi_k(X)$ is *not* a well-defined group without the choice of a base point. All one can say in this case is that $\pi_k(X)$ is a well-defined *isomorphism class* of groups. See also Exercise 3.5 below.

We now construct, for arbitrary $k > 1$, examples of spaces in which the action of π_1 on π_k is nontrivial. Let X be any space and let $f : X \to X$ be a homeomorphism. The *mapping torus* of f is the quotient space

$$Y_f := \frac{X \times [0,1]}{(x,1) \sim (f(x),0)}.$$

For example, if $X = [-1, 1]$ and f is multiplication by -1, then Y_f is the Möbius band. If f is the identity, then $Y_f = X \times S^1$.

There is a natural inclusion $\imath : X \to Y_f$ sending $x \mapsto (x, 0)$. Pick a base point $x_0 \in X$, and let $y_0 = \imath(x_0)$.

Lemma 3.2. *For $k \geq 2$, the inclusion induces an isomorphism*

$$\imath_* : \pi_k(X, x_0) \xrightarrow{\simeq} \pi_k(Y_f, y_0).$$

[5] A fancy way of describing these three properties is that $\pi_k(X, \cdot)$ is a functor from the *fundamental groupoid* of X to groups. Alternatively, $\pi_k(X\cdot)$ is a "twisted coefficient system" on X, see §12.

Proof. The mapping torus can equivalently be defined as

$$Y_f = \frac{X \times \mathbb{R}}{(x, t+1) \sim (f(x), t)}.$$

Hence $X \times \mathbb{R}$ is a covering space of Y_f. By the Lifting Criterion, since $k \geq 2$, a map $(S^k, p) \to (Y_f, y_0)$ lifts to a map $(S^k, p) \to (X \times \mathbb{R}, (x_0, 0))$. Since $\mathbb{R}$ is contractible, we can find a homotopy from this map to a map $(S^k, p) \to (X \times \{0\}, (x_0, 0)$. Projecting this homotopy back down to Y_f shows that $\imath_*$ is surjective. Applying the same argument to homotopies $S^k \times I \to Y_f$ shows that $\imath_*$ is injective. (Compare Example 1.2, and the long exact sequence of a fibration in §6.) $\square$

Now suppose that x_0 is a fixed point of f. This defines a loop γ in Y_f based at x_0, sending $t \mapsto (x_0, t)$. It is an exercise to show that the corresponding isomorphism of $\pi_k(Y_f, y_0)$ is given by (the inverse of) f_*, i.e.:

Proposition 3.3. *The following diagram commutes:*

$$\begin{array}{ccc} \pi_k(X, x_0) & \xrightarrow{f_*^{-1}} & \pi_k(X, x_0) \\ {\scriptstyle \imath_*}\big\downarrow & & {\scriptstyle \imath_*}\big\downarrow \\ \pi_k(Y, y_0) & \xrightarrow{\Phi_\gamma} & \pi_k(Y, y_0). \end{array}$$

Example 3.4. To get an explicit example where Φ_γ is nontrivial, let $X = S^k$, and let $f : S^k \to S^k$ be a degree -1 homomorphism with a fixed point x_0. By the naturality of the Hurewicz isomorphism (2.1), $f_* = -1$ on $\pi_k(S^k) = \mathbb{Z}$.

Exercise 3.5. If X is path connected, then there is a canonical bijection

$$[S^k, X] = \pi_k(X, x_0)/\pi_1(X, x_0).$$

Here the left hand side denotes the space of free (no basepoints) homotopy classes of maps from S^k to X, and the right hand side denotes the quotient of $\pi_k(X, x_0)$ by the action of $\pi_1(X, x_0)$.

Exercise 3.6. Show that if $\phi : X \to Y$ is a homotopy equivalence then $\phi_* : \pi_k(X, x_0) \to \pi_k(Y, \phi(x_0))$ is an isomorphism. *Hint:* First show that if $\{\phi_t\}_{t \in [0,1]}$ is a homotopy of maps from X to Y, and if $\gamma : [0, 1] \to Y$ is the path defined by $\gamma(t) = \phi_t(x_0)$, then

$$(\phi_0)_* = \Phi_\gamma \circ (\phi_1)_* : \pi_k(X, x_0) \to \pi_k(Y, \phi_0(x_0)).$$

4 Fiber bundles

A fiber bundle is a kind of "family" of topological spaces. These are important objects of study in topology and in this course, and also will help us compute homotopy groups.

Definition 4.1. A *fiber bundle* is a map of topological spaces $\pi : E \to B$ such that there exists a topological space F with the following property:

(Local triviality) For all $x \in B$, there is a neighborhood U of x in B, and a homeomorphism $\pi^{-1}(U) \simeq U \times F$, such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\simeq"] \arrow[d, "\pi"'] & U \times F \arrow[d] \\ U \arrow[r, "="] & U \end{tikzcd}$$

commutes, where the map $U \times F \to U$ is projection on to the first factor.

In particular, for each $x \in B$, we have $\pi^{-1}(x) \simeq F$. Thus the fiber bundle can be thought of as a family of topological spaces, each homeomorphic to F, and parametrized by B. One calls F the *fiber*, E the *total space*, and B the *base space*. A fiber bundle with fiber F and base B is an "F-bundle over B". Also, $\pi^{-1}(x)$ is the *fiber over* x, denoted by E_x. We denote the fiber bundle by the diagram

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B, \end{tikzcd}$$

or sometimes just by E.

Definition 4.2. If $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ are fiber bundles over the same base space, then a morphism[6] of fiber bundles from π_1 to π_2 is a map $f : E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$.

Example 4.3. For any B and F, we have the *trivial bundle* $E = B \times F$, with $\pi(b, f) = b$.

[6] Different definitions are possible. A less restrictive definition allows the base spaces to be different. A more restrictive definition requires the map to be a homeomorphism on each fiber.

The local triviality condition says that any fiber is locally isomorphic to a trivial bundle. Also, we will see in Corollary 5.8 below that any fiber bundle over a contractible CW complex is trivial. However, when B has some nontrivial topology, fiber bundles over B can have some global "twisting".

Example 4.4. A covering space is a fiber bundle in which the fiber F has the discrete topology. In this case an isomorphism of fiber bundles is the same as an isomorphism of covering spaces.

Example 4.5. Let $f : X \to X$ be a homeomorphism. Then the mapping torus Y_f is the total space of a fiber bundle

$$\begin{tikzcd} X \arrow[r] & Y_f \arrow[d, "\pi"] \\ & S^1 \end{tikzcd}$$

where $\pi(x,t) = t \mod 1$. Exercise: every fiber bundle over S^1 arises in this way.

The previous example has the following generalization, called the "clutching construction". Let $\operatorname{Homeo}(F)$ denote the space of homeomorphisms[7] of F, let $k > 1$, and let $\phi : S^{k-1} \to \operatorname{Homeo}(F)$. Identify

$$S^k = D^k \cup_{S^{k-1}} D^k,$$

and define a fiber bundle over S^k by

$$E = (D^k \times F) \cup_{S^{k-1}\times F} (D^k \times F),$$

where the two copies of $D^k \times F$ are glued together along $S^{k-1} \times F$ by the map

$$\begin{aligned} S^{k-1} \times F &\longrightarrow S^{k-1} \times F, \\ (x,f) &\longmapsto (x, \phi(x)(f)). \end{aligned}$$

Note that the fiber bundle depends only on the homotopy class of ϕ.

Conversely, it follows from Corollary 5.8, applied to $B = D^k$, that every fiber bundle over S^k arises this way.

[7] We topologize the space $\operatorname{Maps}(X,Y)$ of continuous maps from X to Y using the compact-open topology. For details on this see e.g. Hatcher. A key property of this topology is that if Y is locally compact, then a map $X \to \operatorname{Maps}(Y,Z)$ is continuous iff the corresponding map $X \times Y \to Z$ is continuous.

Example 4.6. S^1-bundles over S^2 are classified by nonnegative integers[8]. To see this, first observe that $\mathrm{Homeo}(S^1)$ deformation retracts[9]onto $O(2)$. Now given an S^1-bundle over S^2, one can arrange that ϕ maps to $SO(2)$, by changing the identification of one of the halves $\pi^{-1}(D^2)$ with $D^2 \times S^1$, via id_{D^2} cross an orientation-reversing homeomorphism of S^1. The nonnegative integer associated to the bundle is then $|\deg(\phi)|$; we will temporarily call this the "rotation number"[10]. Note that switching the identification of both halves will change the sign of $\deg(\phi)$.

Exercise 4.7. More generally, show that the clutching construction defines a bijection

$$\frac{\{F\text{-bundles over } S^k\}}{\text{isomorphism}} = \frac{\pi_{k-1}(\mathrm{Homeo}(F), \mathrm{id}_F)}{\mathrm{Homeo}(F)}$$

where $\mathrm{Homeo}(F)$ acts on $\pi_{k-1}(\mathrm{Homeo}(F), \mathrm{id}_F)$ by conjugation in $\mathrm{Homeo}(F)$.

Example 4.8. The *Hopf fibration*

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}$$

is given by the map

$$S^3 \to \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1 \simeq S^2.$$

Exercise 4.9. The Hopf fibration has "rotation number" 1. The unit tangent bundle of S^2 has "rotation number" 2.

Definition 4.10. A *section* of a fiber bundle $\pi : E \to B$ is a map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.

Intuitively, a section is a continuous choice of a point in each fiber.

[8]It is more common to consider *oriented* S^1-bundles over S^2, which are classified by integers. We will discuss these later.

[9]This is a good exercise. More generally, one can ask whether $\mathrm{Homeo}(S^k)$ deformation retracts to $O(k+1)$. This is known to be true for $k = 2$ and $k = 3$, the latter result being a consequence of the "Smale conjecture" proved by Hatcher (asserting that the inclusion $SO(4) \to \mathrm{Diff}(S^3)$ is a homotopy equivalence) and a result of Cerf. It is false for all $k \geq 4$.

[10]This is really the absolute value of the Euler class for a choice of orientation of the bundle; we will discuss this later.

Example 4.11. A section of a trivial bundle $E = B \times F$ is just a map $s : B \to F$.

Nontrivial bundles may or may not posess sections. We will study this question systematically a little later. For now you can try to convince yourself that an S^1-bundle over S^2 with nonzero rotation number does not have a section.

5 Homotopy properties of fiber bundles

We will now show, roughly, that the fiber bundles over a CW complex B depend only on the homotopy type of B. We will use the following terminology: if E is a fiber bundle over B, and if $U \subset B$, then a "trivialization of E over U" is a homeomorphism $\phi : \pi^{-1}(U) \simeq U \times F$ commuting with the projections.

Lemma 5.1. *Any fiber bundle E over $B = I^k$ is trivial.*

Proof. By the local triviality condition and compactness of I^k, we can subdivide I^k into N^k subcubes of size $1/N$, such that E is trivial over each subcube. We now construct a trivialization of E over the whole cube, one subcube at a time. By an induction argument which we leave as an exercise, it is enough to prove the following lemma. □

Lemma 5.2. *Let $X = I^k$ and let $A \subset I^k$ be the union of all but one of the $(k-1)$-dimensional faces of I^k. Suppose that E is a trivial fiber bundle over X. Then a trivialization of E over A extends to a trivialization of E over X.*

Proof. We can regard $E = X \times F$. The trivialization of E over A can then be regarded as a map $\phi : A \to \mathrm{Homeo}(F)$, and we have to extend this to a map $\overline{\phi} : X \to \mathrm{Homeo}(F)$. Such an extension exists because there is a retraction $r : X \to A$, so that we can define $\overline{\phi} = \phi \circ r$. □

Before continuing, we need to introduce the notion of pullback of a fiber bundle $\pi : E \to B$ by a map $f : B' \to B$. The *pullback* f^*E is a fiber bundle over B' defined by[11]

$$f^*E := \{(x,y) \mid x \in B',\ y \in E,\ f(x) = \pi(y)\} \subset B' \times E.$$

The map $f^*E \to B'$ is induced by the projection $B' \times E \to B'$. The definition implies that the fiber of f^*E over $x \in B'$ is equal to the fiber of E over $f(x) \in B$.

Example 5.3. In the special case when B' is a subset of B and f is the inclusion map, $f^*E = \pi^{-1}(B')$ is called the *restriction* of E to B' and is denoted by $E|_{B'}$. In particular, if $f = \mathrm{id}_B$, then $f^*E = E$.

Example 5.4. If f is a constant map to $x \in B$, then f^*E is the trivial bundle

$$f^*E = B' \times E_x.$$

Example 5.5. If E is the mapping torus of $\phi : X \to X$, and if $f : S^1 \to S^1$ has degree d, then f^*E is the mapping torus of ϕ^d.

Similarly, we will see that if E is an S^1-bundle over S^2 with rotation number r, and if $f : S^2 \to S^2$ has degree d, then f^*E is an S^1-bundle over S^2 with rotation number $|dr|$.

Lemma 5.6. *Let B be a CW complex and let E be a fiber bundle over $B \times [0,1]$. Then*

$$E|_{B\times\{0\}} \simeq E|_{B\times\{1\}},$$

where both sides are regarded as fiber bundles over B.

Proof. Let $f : B \times [0,1] \to B$ be the projection. We will construct an isomorphism

$$f^*(E|_{B\times\{0\}}) \simeq E. \tag{5.1}$$

Restricting this isomorphism to $B \times \{1\}$ then implies the lemma.

We define the isomorphism (5.1) to be the identity over $B \times \{0\}$. We now extend this isomorphism over $B \times [0,1]$, one cell (in B) at a time, by

[11] That is, f^*E is the *fiber product* of f and π over B'. There is a commutative diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B. \end{tikzcd}$$

induction on the dimension. (This is legitimate because a map from a CW complex is continuous if and only if it is continuous on each closed cell.) This means that without loss of generality, $B = D^k$, and the isomorphism (5.1) has already been defined over $D^k \times \{0\}$ and $\partial D^k \times [0,1]$. By Lemma 5.1, E and $f^*(E|_{B\times\{0\}})$ are trivial when $B = D^k \times [0,1]$, so after choosing a trivialization, the desired isomorphism (5.1) is equivalent to a map $D^k \times [0,1] \to \mathrm{Homeo}(F)$. As in the proof of Lemma 5.2, such a map from $D^k \times \{0\} \cup \partial D^k \times [0,1]$ can be extended over $D^k \times [0,1]$. □

Proposition 5.7. *Let B' be a CW complex, let E be a fiber bundle over B, and let $f_0, f_1 : B' \to B$ be homotopic maps. Then*

$$f_0^* E \simeq f_1^* E$$

as fiber bundles over B'.

Proof. The homotopy can be described by map

$$f : B' \times [0,1] \to B$$

with $f|_{B\times\{i\}} = f_i$. Then

$$f_0^* E = (f^*E)|_{B\times\{0\}} \simeq (f^*E)|_{B\times\{1\}} = f_1^* E,$$

where the middle isomorphism holds by Lemma 5.6. □

Corollary 5.8. *Any fiber bundle over a contractible CW complex is trivial.*

Proof. Let B be a contractible CW complex. Contractibility means that there is a point $x \in B$ and a homotopy between id_B and the constant map $f : B \to \{x\}$. So by Proposition 5.7, if E is a fiber bundle over B, then

$$E = \mathrm{id}_B^* E \simeq f^* E = B \times E_x,$$

which is a trivial bundle. □

More generally, a homotopy equivalence between B_1 and B_2 induces a bijection between isomorphism classes of F-bundles over B_1 and B_2. (We will see a nicer way to understand this when we discuss classifying spaces later in the course.)

Remark 5.9. Usually one considers fiber bundles with some additional structure. For example, a *vector bundle* is a fiber bundle in which each fiber has the structure of a vector space (and the local trivializations can be chosen to be fiberwise vector space isomorphisms). A key example is the tangent bundle of a smooth n-dimensional manifold M; this is a vector bundle over M whose fiber over each point is an n-dimensional real vector space. Vector bundles will be a major object of study later in the course.

6 The long exact sequence of a fibration

The following is a useful tool for computing some homotopy groups.

Theorem 6.1. *Let $F \to E \to B$ be a fiber bundle. Pick $y_0 \in E$, let $x_0 = \pi(y_0) \in B$, and identify $F = \pi^{-1}(x_0)$. Then there is a long exact sequence*

$$\cdots \to \pi_{k+1}(B, x_0) \to \pi_k(F, y_0) \to \pi_k(E, y_0) \to \pi_k(B, x_0) \to \pi_{k-1}(F, y_0) \to \cdots \tag{6.1}$$

The exact sequence terminates[12] at $\pi_1(B, x_0)$. The arrows $\pi_k(F, y_0) \to \pi_k(E, y_0)$ and $\pi_k(E, y_0) \to \pi_k(B, x_0)$ are induced by the maps $F \to E$ and $E \to B$ in the fiber bundle. The definition of the connecting homomorphism $\pi_k(B, x_0) \to \pi_{k-1}(F, y_0)$ and the proof of exactness will be given later. Let us first do some computations with this exact sequence.

Example 6.2. If $E \to B$ is a covering space, then it follows from the exact sequence that $\pi_k(E, y_0) \simeq \pi_k(B, x_0)$ for all $k \geq 2$. This generalizes Example 1.3.

Example 6.3. Applying the exact sequence to the Hopf fibration $S^1 \to S^3 \to S^2$, we get an exact sequence

$$\pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1).$$

We know that this is $0 \to \mathbb{Z} \to \pi_3(S^2) \to 0$, and therefore

$$\pi_3(S^2) \simeq \mathbb{Z}. \tag{6.2}$$

Furthermore, the Hopf fibration $S^3 \to S^2$ is a generator of this group.

[12] One can also continue the sequence with maps of sets $\pi_1(B, x_0) \to \pi_0(F) \to \pi_0(E)$ such that the "kernel" of each map, interpreted as the inverse image of the component containing y_0, is the image of the previous map.

Remark 6.4. There is a nice explicit description of the isomorphism (6.2). Any homotopy class in $\pi_3(S^2)$ can be represented by a smooth map $f : S^3 \to S^2$. By Sard's theorem, for a generic point $p \in S^2$, the map f is transverse to p, so that the inverse image $f^{-1}(p)$ is a smooth link in S^2. This link inherits an orientation from the orientations of S^3 and S^2. Let q be another generic point in S^2, and define the *Hopf invariant* of f to be the linking number[13] of $f^{-1}(p)$ and $f^{-1}(q)$:

$$H(f) := \ell(f^{-1}(p), f^{-1}(q)) \in \mathbb{Z}.$$

The integer H is a homotopy invariant of f and defines the isomorphism (6.2). This is a special case of the Thom-Pontrjagin construction, which we may discuss later in the course.

The long exact sequence can be used to compute a few homotopy groups of Lie groups, by the use of suitable fiber bundles. For example, let us try to compute some homotopy groups of $U(n)$. There is a map $U(n) \to S^{2n-1}$ which sends $A \mapsto Av$, where v is some fixed unit vector.

Exercise 6.5. This gives a fiber bundle $U(n-1) \to U(n) \to S^{2n-1}$.

We know that $U(1) \simeq S^1$. The $n = 2$ case is then a fiber bundle

$$S^1 \to U(2) \to S^3.$$

So by the long exact sequence, $\pi_1 U(2) = \mathbb{Z}$, $\pi_2 U(2) = 0$, and $\pi_3 U(2) = \mathbb{Z}$. In general, while the long exact sequence of this fiber bundle does not allow us to compute $\pi_k(U(n))$ in all cases (since we do not know all of the homotopy groups of spheres), it does show that $\pi_k(U(n))$ is independent of n when n is sufficiently large with respect to k. This group is denoted by $\pi_k(U)$. In fact, Bott Periodicity, which we may discuss later in the course, asserts that $\pi_k(U)$ is isomorphic to $\mathbb{Z}$ when k is odd and 0 when k is even.

We now construct the long exact sequence. It works in a more general context than that of fiber bundles, which we now define.

[13] Recall that the *linking number* of two disjoint oriented links L_1 and L_2 in S^3 is the intersection number of L_1 with a Seifert surface for L_2. Equivalently, if L_1 has k components, then $H_1(S^3 \setminus L_1) \simeq \mathbb{Z}^k$, with a natural basis given by little circles around each of the components of L_1 with appropriate orientations; and the linking number is obtained by expressing $[L_2] \in H_1(S^3 \setminus L_1)$ in this basis and taking the sum of the coefficients. The linking number is symmetric, and can be computed from a link diagram by counting the crossings of L_1 with L_2 with appropriate signs and dividing by 2.

Definition 6.6. A map $\pi : E \to B$ (not necessarily a fiber bundle) has the *homotopy lifting property* with respect to a pair (X, A) if given maps $f : X \times I \to B$ and $g : X \times \{0\} \cup A \times I \to E$ with $\pi \circ g = f|_{X\times\{0\}\cup A\times I}$, there exists $h : X \times I \to E$ extending g such that $\pi \circ h = f$.

Example 6.7. Recall that if $\pi : E \to B$ is a covering space, then any path in B lifts to a path in E, given a lift of its initial endpoint. This means that $\pi : E \to B$ has the homotopy lifting property with respect to the pair $(\mathrm{pt}, \emptyset)$.

Definition 6.8. A map $\pi : E \to B$ is a *Serre fibration* if it has the homotopy lifting property with respect to the pair $(I^k, \partial I^k)$ for all k.

Lemma 6.9. *Every fiber bundle is a Serre fibration.*

Proof. Let $(X, A) = (I^k \times I, I^k \times \{0\} \cup \partial I^k \times I)$, let $\pi : E \to B$ be a fiber bundle, and let $f : X \to B$. We are given a lift g to E of the restriction of f to A, and we must extend this to a lift of f over all of X. Now g is equivalent to a section of $f^*E|_A$, and the problem is to extend this section over X. We know that f^*E is trivial, so after choosing a trivialization, a section is equivalent to a map to F. But a map $A \to F$ extends over X because there is a retraction $r : X \to A$. $\square$

Exercise 6.10. Let $E = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le x \le 1\}$, let $B = I$, and let $\pi : E \to B$ send $(x, y) \mapsto x$. Then π is a Serre fibration, but not a fiber bundle.

Exercise 6.11. A map $\pi : E \to B$ is a *fibration* if it has the homotopy lifting property with respect to (X, ϕ) for all spaces X.

(a) Show that a fibration is a Serre fibration. *Hint:* The key fact is that $(I^k \times I, I^k \times \{0\} \cup \partial I^k \times I)$ is homeomorphic to $(I^k \times I, I^k \times \{0\})$.

(b) Show that all the fibers of a fibration over a path connected space B are homotopy equivalent. *Hint:* Use the homotopy lifting property where X is a fiber.

(c) Show that all the fibers of a Serre fibration over a path connected space B have isomorphic homotopy and homology groups.

By Lemma 6.9, Theorem 6.1 follows from:

Theorem 6.12. *If $\pi : E \to B$ is a Serre fibration, then there is a long exact sequence in homotopy groups* (6.1).

Proof. For $k \geq 1$ we need to define a connecting homorphism

$$\delta : \pi_{k+1}(B, x_0) \longrightarrow \pi_k(F, y_0).$$

Consider an element of $\pi_{k+1}(B, x_0)$ represented by a map $f : I^{k+1} \to B$ sending ∂I^{k+1} to x_0. We can lift f to E over $I^k \times \{0\} \cup \partial I^k \times I$, by mapping that set to y_0. By the homotopy lifting property, this lift extends to a lift $h : I^k \times I \to E$ of f. Then $h|_{I^k \times \{1\}}$, regarded as a map on I^k, sends $(I^k, \partial I^k) \to (F, y_0)$. We now define

$$\delta[f] := [h|_{I^k \times \{1\}}].$$

One can check that this is well-defined on homotopy classes, and that this is a homomorphism. With this definition, the proof of exactness is a more or less straightforward exercise. □

Exercise 6.13. Let E be an S^1-bundle over S^2 with rotation number r. Then the connecting homomorphism

$$\delta : \mathbb{Z} \simeq \pi_2(S^2) \to \pi_1(S^1) \simeq \mathbb{Z}$$

is multiplication by $\pm r$.

Example 6.14. Let X be a topological space and $x_0 \in X$. The *based loop space* $\Omega X = \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = x_0\}$. Define the space of paths starting at x_0 to be $PX = \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\}$. This fits into the *path fibration*

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ & & \Big\downarrow \pi \\ & & X \end{array}$$

where $\pi(f) = f(1)$. It is an easy exercise to check that this is a fibration. Now PX is contractible, so the long exact sequence gives

$$\pi_k(\Omega X, x_0) \simeq \pi_{k+1}(X, x_0)$$

for $k \geq 1$. This can also be seen directly from the definition of homotopy groups. But one can get a lot more information out of the path fibration as we will see later.

Although we have only taken some very first steps in computing homotopy groups, let us now turn to some of their applications.

7 First example of obstruction theory

The rough idea of obstruction theory is simple. Suppose we want to construct some kind of function on a CW complex X. We do this by induction: if the function is defined on the k-skeleton X^k, we try to extend it over the $(k+1)$-skeleton X^{k+1}. The obstruction to extending over a $(k+1)$-cell is an element of π_k of something. These obstructions fit together to give a cellular cochain $\mathfrak{o}$ on X with coefficients in this π_k. In fact this cochain is a cocycle, so it defines an "obstruction class" in $H^{k+1}(X;\pi_k(\text{something}))$. If this cohomology class is zero, i.e. if there is a cellular k-cochain η with $\mathfrak{o} = \delta\eta$, then η prescribes a way to modify our map over the k-skeleton so that it can be extended over the $(k+1)$-skeleton.

Let us now make this idea more precise by proving some theorems. For now, if X is a CW complex, $C_*(X)$ will denote the cellular chain complex. To simplify notation, we assume that each cell has an orientation chosen, so that $C_k(Z)$ is the free $\mathbb{Z}$-module generated by the k-cells. If e is a k-cell and if e' is a $(k-1)$-cell, we let $\langle \partial e, e' \rangle \in \mathbb{Z}$ denote the coefficient of e' in ∂e.

Theorem 7.1. *Let X be a CW complex. Then*

$$[X, S^1] = H^1(X;\mathbb{Z}).$$

Proof. Let α be the preferred generator of $H^1(S^1;\mathbb{Z}) = \mathbb{Z}$. Define a map

$$\begin{aligned} \Phi : [X, S^1] &\longrightarrow H^1(X;\mathbb{Z}), \\ [f : X \to S^1] &\longmapsto f^*\alpha. \end{aligned}$$

We want to show that Φ is a bijection.

Proof that Φ is surjective: let $\xi \in C^1(X;\mathbb{Z})$ with $\delta\xi = 0$. We need to find $f : X \to S^1$ with $f^*\alpha = [\xi]$. We construct f on the k skeleton by induction on k. First, we send the 0-skeleton to a base point $p \in S^1$. Next, if σ is a one-cell, we extend f over σ so that $f|_\sigma$ has winding number $\xi(\sigma)$ around S^1.

If $e : D^2 \to X$ is a 2-cell, then since ξ is a cocycle,

$$\sum_\sigma \langle \partial e, \sigma \rangle \xi(\sigma) = 0.$$

It is not hard to see (and we will prove a more general statement in Lemma 8.3(a) below) that the left hand side of the above equation is the winding number of $f \circ e|_{\partial D^2}$ around S^1. Hence, we can extend f over the 2-cell.

Now assume that f has been defined over the k-skeleton X^k for some $k \geq 2$. We can then extend f over the $(k+1)$-skeleton, because the obstruction to extending over any $(k+1)$-cell lives in $\pi_k(S^1) = 0$.

Proof that Φ is injective: Let $f_0, f_1 : X \to S^1$, and suppose that $f_0^*\alpha = f_1^*\alpha$. We can homotope[14] f_0 and f_1 so that they send the 0-skeleton of X to the base point $p \in S^1$. Then $f_i^*\alpha$ is represented by the cellular cochain β_i that sends each 1-cell σ to the winding number of $f_i|_\sigma$ around S^1. The assumption $f_0^*\alpha = f_1^*\alpha$ then means that there is a cellular 0-cochain $\eta \in C^0(X;\mathbb{Z})$ with $\beta_0 - \beta_1 = \delta\eta$. That is, if σ is a 1-cell with vertices $\sigma(0)$ and $\sigma(1)$, then

$$\beta_0(\sigma) - \beta_1(\sigma) = \eta(\sigma(1)) - \eta(\sigma(0)). \tag{7.1}$$

We now regard f_0 and f_1 as defining a map $X \times \{0,1\} \to S^1$, and we want to extend this to a homotopy $X \times [0,1] \to S^1$. We extend over $X^0 \times [0,1]$ such that if x is a 0-cell, then the restriction to $\{x\} \times [0,1]$ has winding number $-\eta(x)$ around S^1. Then equation (7.1) implies that there is no obstruction to extending the homotopy over $X^1 \times [0,1]$. Finally, for $k \geq 2$, if the homotopy has been extended over $X^{k-1} \times [0,1]$, then there is no obstruction to extending the homotopy over $X^k \times [0,1]$, since $\pi_k(S^1) = 0$. $\square$

Remark 7.2. The group structure on S^1 induces a group structure on $[X, S^1]$, and it is easy to see that this agrees with the group structure on $H^1(X;\mathbb{Z})$ under the above bijection.

8 Eilenberg-MacLane spaces

Theorem 7.1 can be regarded as giving a homotopy-theoretic interpretation of $H^1(\cdot;\mathbb{Z})$ for CW complexes. We now introduce an analogous interpretation of more general cohomology groups.

Consider a path connected space Y with only one nontrivial homotopy group, i.e.

$$\pi_k(Y) \simeq \begin{cases} G, & k = n, \\ 0, & k \neq n. \end{cases} \tag{8.1}$$

[14] We can do this because in general, any CW pair (X, A) has the homotopy extension property. That is, any map $(X \times \{0\} \cup (A \times I) \to Y$ extends to a map $X \times I \to Y$. One constructs the extension cell-by-cell, similarly to the proof of Lemma 5.6. In the present case, A is the 0-skeleton of X. A related argument proves the *cellular approximation theorem*, which asserts that any continuous map between CW complexes is homotopic to a cellular map.

Such a space is called an *Eilenberg-MacLane space*. We also say that "Y is a $K(G,n)$".

For example, S^1 is a $K(\mathbb{Z},1)$. Another example is that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$. Here one defines $\mathbb{C}P^\infty = \lim_{k\to\infty} \mathbb{C}P^k$ with the direct limit topology, where the inclusion $\mathbb{C}P^k \to \mathbb{C}P^{k+1}$ is induced by the inclusion $\mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{C}^{k+2} \setminus \{0\}$. This is a CW complex with one cell in each even dimension. One can prove that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ by using the cellular approximation theorem, as in the proof of Proposition 8.1 below, to show that $\pi_n(\mathbb{C}P^\infty) = \pi_n(\mathbb{C}P^k)$ whenever $2k \ge n+1$. One can then use the long exact sequence in homotopy groups associated to the fibration $S^1 \to S^{2k-1} \to \mathbb{C}P^k$ to show that $\pi_n(\mathbb{C}P^k) = 0$ if $n \ne 2$ and $k > 1$ and $2k-1 > n$. Similarly, $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$.

Explicit constructions of $K(G,n)$'s can be difficult in general, but there is still the following general existence result:

Proposition 8.1. *For any positive integer n and any group G (abelian if $n>1$):*

(a) There exists a CW complex Y satisfying (8.1).

(b) The CW complex Y in (a) is unique up to homotopy equivalence.

Proof. (a) Consider a presentation of G by generators and relations. Let Y^n be a wedge of n-spheres, one for each generator. For each relation, attach an $(n+1)$-cell whose boundary is the sum of the n-spheres in the relation. Now we have a CW complex Y^{n+1} with $\pi_n(Y^{n+1}) \simeq G$, and $\pi_i(Y^{n+1}) = 0$ for $i < n$. (When $n=1$ this follows from the Seifert-Van Kampen theorem, and when $n>1$ this follows from the Hurewicz theorem.) However $\pi_{n+1}(Y^{n+1})$ might not be zero. One can attach $(n+2)$-cells to kill $\pi_{n+1}(Y^{n+1})$, i.e. one can represent each element of $\pi_{n+1}(Y^{n+1})$ by a map $S^{n+1} \to Y^{n+1}$ and use this as the attaching map for an $(n+2)$-cell. Let Y^{n+2} denote the resulting complex. Then $\pi_{n+1}(Y^{n+2}) = 0$, because the cellular approximation theorem implies that an element of $\pi_{n+1}(Y^{n+2})$ can be represented by a map $S^{n+1} \to Y^{n+1}$. The cellular approximation similarly implies that attaching $(n+2)$-cells does not affect π_i for $i \le n$. One then attaches $(n+3)$-cells to kill π_{n+2}, and so on. (The resulting CW complex might be very complicated.)

(b) When G is abelian, this will be proved in Corollary 8.4 below. The case when $n=1$ and G is nonabelian follows from a related argument which we omit. $\square$

Theorem 8.2. *If G is abelian, and if Y is a $K(G,n)$, then an identification of $\pi_k(Y)$ with G determines, for any CW complex X, an isomorphism*

$$H^n(X;G) \simeq [X,Y].$$

Proof. This is similar to the proof of Theorem 7.1. We begin by defining a map

$$\Phi : [X,Y] \longrightarrow H^n(X;G)$$

as follows. Fix an identification of $\pi_n(Y)$ with G. (Note that our assumptions imply that $\pi_n(Y)$ does not depend on the choice of base point.) By the Hurewicz theorem and the universal coefficient theorem, we then obtain an identification

$$H^n(Y,G) = \operatorname{Hom}(G,G).$$

So there is a canonical element $\mathrm{id}_G \in H^n(Y,G)$. Now given $f : X \to Y$, we define

$$\Phi[f] := f^*(\mathrm{id}_G) \in H^n(X;G).$$

We will prove that Φ is a bijection.

To do so, we need the following lemma. Part (a) of the lemma is a useful "homotopy addition lemma" which generalizes Lemma 2.2(b). Part (b) of the lemma gives a more concrete description of the map Φ when f sends the $(n-1)$-skeleton X^{n-1} to a base point $y_0 \in Y$. (There is no obstruction to homotoping f to have this property.)

Lemma 8.3. *Suppose $f : X \to Y$ sends X^{n-1} to y_0. Then:*

(a) For any $(n+1)$-cell $e : D^{n+1} \to X$,

$$\sum_\sigma \langle \partial e, \sigma\rangle [f \circ \sigma] = [f \circ e|_{S^n}] \in \pi_n(Y, y_0).$$

(b) $\Phi[f] \in H^n(X;G)$ is represented by a cellular cocycle $\beta \in C^n_{\mathrm{cell}}(X;G)$ which sends an n-cell $\sigma : D^n \to X$ to

$$\beta(\sigma) = [f \circ \sigma] \in \pi_n(Y, y_0) = G. \tag{8.2}$$

Proof. Since f factors through the projection $X \to X/X^{n-1}$, for both (a) and (b) we may assume without loss of generality that X^{n-1} is a point x_0.

Proof of (a): By the Hurewicz theorem,

$$\pi_n(X^n, x_0) = H_n(X^n) = C_n^{\text{cell}}(X). \tag{8.3}$$

By equation (8.3) and the naturality of the Hurewicz isomorphism under the map $e|_{S^n} : S^n \to X^n$, we have

$$[e|_{S^n}] = \sum_{\sigma} \langle \partial e, \sigma \rangle [\sigma] \in \pi_n(X^n, x_0).$$

Applying f_* to this equation completes the proof.

Proof of (b): Note that β is a cellular cocycle by part (a), and so it determines a cohomology class in $H^n(X; G)$. To show that this cohomology class is $f^*(\text{id}_G)$ as desired, since the pullback on $H^n(\cdot; G)$ by the inclusion $X^n \to X$ is injective, we may assume without loss of generality that $X = X^n$ (in addition to $X^{n-1} = \{x_0\}$). Then $H^n(X; G) = C^n_{\text{cell}}(X; G)$, so we just have to check that $\Phi(f)(\sigma) = [f \circ \sigma]$ for $\sigma : D^n \to X$ an n-cell. This follows from naturality of the Hurewicz isomorphism under the map $f \circ \sigma : S^n \to Y$.

This completes the proof of Lemma 8.3. □

Continuing the proof of Theorem 8.2, we now show that Φ is a bijection.

To prove that Φ is surjective, let $\alpha \in H^n(X; G)$, and let $\beta \in C^n_{\text{cell}}(X; G)$ be a cellular cocycle representing α. Define $f : X \to Y$ cell-by-cell as follows. First define $f|_{X^{n-1}} = y_0$. Extend f over X^n so that (8.2) holds. Since β is a cocycle, it follows from Lemma 8.3(a) that there is no obstruction to extending f over X^{n+1}. There is then no obstruction to extending over higher skeleta since $\pi_i(Y) = 0$ for $i > n$. By Lemma 8.3(b), $\Phi(f) = \alpha$.

The proof that Φ is injective is completely analogous to the proof for the case $n = 1$, $G = \mathbb{Z}$ that we have already seen. □

Corollary 8.4. *If G is abelian, then a $K(G, n)$ that is a CW complex is unique up to homotopy equivalence.*

Proof. Suppose that Y and Y' are both CW complexes and $K(G, n)$'s. Fix identifications $\pi_n(Y) = \pi_n(Y') = G$. By Theorem 8.2,

$$[Y, Y] = [Y, Y'] = [Y', Y] = [Y', Y'] = \text{Hom}(G, G).$$

Furthermore, it follows from the definition of this correspondence that the composition of two maps between Y and/or Y' corresponds to the composition of the corresponding homomorphisms from G to G, and id_Y and $\text{id}_{Y'}$

correspond to id_G. Now let $f : Y \to Y'$ and $g : Y' \to Y$ correspond to id_G. Then $g \circ f$ also corresponds to id_G, so $g \circ f$ is homotopic to id_Y. Likewise $f \circ g$ is homotopic to $\mathrm{id}_{Y'}$. □

9 Whitehead's theorem

We now use another, simpler cell-by-cell construction to prove the following theorem, which shows that homotopy groups give a criterion for a map to be a homotopy equivalence.

Theorem 9.1 (Whitehead's theorem)**.** *Let X and Y be path connected CW complexes and let $f : X \to Y$ be a continuous map. Suppose that f induces isomorphisms on all homotopy groups. Then f is a homotopy equivalence.*

Proof. We first reduce to a special case. Define the *mapping cylinder*

$$C_f := \frac{(X \times I) \cup Y}{(x,1) \sim f(x)}.$$

Then f is the composition of two maps $X \to C_f \to Y$, where the first map is an inclusion sending $x \mapsto (x, 0)$, and the second map sends $(x,t) \mapsto f(x)$ and $y \mapsto y$. Furthermore, the map $C_f \to Y$ is a homotopy equivalence since it comes from a deformation retraction of C_f onto Y. So it is enough to show that the inclusion $X \to C_f$ is a homotopy equivalence. By the cellular approximation theorem, to prove Whitehead's theorem we may assume that f is cellular. Then C_f is a CW complex and $X \times \{0\}$ is a subcomplex. In conclusion, to prove Whitehead's theorem, we may assume that X is a subcomplex of Y and f is the inclusion.

So assume that X is a subcomplex of Y and that the inclusion induces isomorphisms on all homotopy groups. We now construct a (strong) deformation retraction of Y onto X. This consists of a homotopy $F : Y \times I \to Y$ such that $F(y, 0) = y$ for all $y \in Y$; $F(x,t) = x$ for all $x \in X$ and $t \in I$; and $F(y, 1) \in X$ for all $y \in Y$. We construct F one cell at a time. This reduces to the following problem: given

$$\begin{aligned} F : (D^k \times \{0\}) \cup (\partial D^k \times I) &\longrightarrow Y, \\ \partial D^k \times \{1\} &\longmapsto X, \end{aligned} \tag{9.1}$$

extend F to a map $D^k \times I \to Y$ sending $D^k \times \{1\} \to X$.

The restriction of F to $\partial D^k \times \{1\}$ defines an element $\alpha \in \pi_{k-1}(X)$ (mod the action of $\pi_1(X)$). Since F extends to a map (9.1), it follows that α maps to 0 in $\pi_{k-1}(Y)$. Since the inclusion induces an injection on π_{k-1}, it follows that we can extend α to a map $D^k \times \{1\} \to X$. Now F is defined on $\partial(D^k \times I) \simeq S^k$, and we need to extend F over all of $D^k \times I$. This may not be possible. The map we have so far represents an element $\beta \in \pi_k(Y)$ (mod the action of $\pi_1(Y)$), and we need this element to be zero. If $\beta \neq 0$, then since the map $\pi_k(X) \to \pi_k(Y)$ is surjective, we can change our choice of extension of F over $D^k \times \{1\}$ to arrange that $\beta = 0$. (Note that these choices can be made for all k-cells independently, so there is no homological issue as in our previous obstruction theory arguments.) □

Remark 9.2. If all we know is that X and Y have isomorphic homotopy groups, then X and Y need not be homotopy equivalent (which by Whitehead's theorem means that the isomorphisms on homotopy groups might not be induced by a map $f : X \to Y$).

In general, Whitehead's theorem can be hard to apply, because it may be hard to check that a map induces isomorphisms on all homotopy groups. Fortunately, there is an alternate version of Whitehead's theorem which can be more practical:

Theorem 9.3. *Let X and Y be simply connected CW complexes and let $f : X \to Y$ be a continuous map. Suppose that f induces isomorphisms on all homology groups. Then f is a homotopy equivalence.*

This can be proved using a relative version of the Hurewicz theorem, see e.g. Hatcher.

Corollary 9.4. *Let X be a closed, oriented, simply connected n-dimensional manifold with $H_i(X) = 0$ for $0 < i < n$. Then X is homotopy equivalent to S^n.*

Proof. We can find an embedding of the closed ball D^n into X. Define a map $f : X \to S^n$ by sending the interior of D^n homeomorphically to the complement of the north pole, and the rest of X to the north pole. Then Theorem 9.3 applies to show that f is a homotopy equivalence. □

10 Orientations of sphere bundles

We now use the ideas of obstruction theory to begin to analyze fiber bundles.

Definition 10.1. Let E be an S^k-bundle over B with $k \geq 0$. An *orientation* of E is a choice of generator of the reduced[15] homology $\widetilde{H}_k(E_x) \simeq \mathbb{Z}$ for each $x \in B$. This should depend continuously on x in the following sense. If E is trivial over a subset $U \subset B$, then a trivialization $E|_U \simeq U \times S^k$ determines identifications $E_x \simeq S^k$, and hence isomorphisms $\widetilde{H}_k(E|_x) \simeq \widetilde{H}_k(S^k)$, for each $x \in U$. We require that if U is connected, then the chosen generators of $\widetilde{H}_k(E|_x)$ all correspond to the same generator of $\widetilde{H}_k(S^k)$. Note that this continuity condition does not depend on the choice of trivialization over U.

The bundle E is *orientable* if it possesses an orientation. It is *oriented* if moreover an orientation has been chosen.

Example 10.2. If B is a smooth n-dimensional manifold, then choosing a metric on B gives rise to an S^{n-1}-bundle $STB \to B$ consisting of unit vectors in the tangent bundle, and an orientation of STB is equivalent to an orientation of B in the usual sense.

Example 10.3. The mapping torus of a homeomorphism $f : S^k \to S^k$, regarded as an S^k-bundle over S^1, is orientable if and only if f is orientation-preserving.

Exercise 10.4. Oriented S^1-bundles over S^2, up to orientation-preserving isomorphism, are classified by $\mathbb{Z}$. (We will prove a generalization of this in Theorem 11.8 below.)

An orientation is equivalent to a section of the *orientation bundle* $\mathscr{O}(E)$, a double cover of B defined as follows. An element of $\mathscr{O}(E)$ is a pair $(x, \mathfrak{o}_x)$ where $x \in B$ and $\mathfrak{o}_x$ is a generator of $\widetilde{H}_k(E_k)$. We topologize this as follows. If U is an open subset of B and $\mathfrak{o}$ is an orientation of $E|_U$, let $V(U, \mathfrak{o}) \subset \mathscr{O}(E)$ denote the set of pairs $(x, \mathfrak{o}_x)$ where $x \in B$ and $\mathfrak{o}_x$ is the generator of $\widetilde{H}_k(E_x)$ determined by $\mathfrak{o}$. The sets $V(U, \mathfrak{o})$ are a basis for a topology on $\mathscr{O}(E)$. With this topology, $\mathscr{O}(E) \to B$ is a 2:1 covering space, and an orientation of E is equivalent to a section of $\mathscr{O}(E)$.

We now want to give a criterion for orientability of a sphere bundle. By covering space theory, a path $\gamma : [0, 1] \to B$ induces a bijection

$$\Phi_\gamma : \mathscr{O}(E)_{\gamma(0)} \longrightarrow \mathscr{O}(E)_{\gamma(1)}.$$

[15] Of course the word "reduced" here only makes a difference when $k = 0$.

Moreover, if γ is homotopic to γ' rel endpoints, then $\Phi_\gamma = \Phi_{\gamma'}$. So if $x_0 \in B$ is a base point, we obtain a "monodromy" homomorphism

$$\Phi : \pi_1(B, x_0) \to \mathrm{Aut}(\mathscr{O}(E)_{x_0}) = \mathbb{Z}/2. \tag{10.1}$$

Since $\mathbb{Z}/2$ is abelian, this homomorphism descends to the abelianization of $\pi_1(B, x_0)$, so combining these homomorphisms for all path components of B gives a map $H_1(B) \to \mathbb{Z}/2$. By the universal coefficient theorem, this is equivalent to an element of $H^1(B; \mathbb{Z}/2)$, which we denote by

$$w_1(E) \in H^1(B; \mathbb{Z}/2).$$

Proposition 10.5. *Let $E \to B$ be an S^k-bundle with $k \geq 1$. Assume that the path components of B are connected (e.g. B is a CW complex). Then:*

(a) E is orientable if and only if $w_1(E) = 0 \in H^1(B; \mathbb{Z}/2)$.

(b) If E is orientable, then the set of orientations of E is an affine space[16] *over $H^0(B; \mathbb{Z}/2)$.*

Proof. (a) Without loss of generality, B is path connected. Since $\mathscr{O}(E) \to B$ is a 2:1 covering space, it has a section if and only if it is trivial. And the covering space is trivial if and only if the monodromy (10.1) is trivial.

(b) $H^0(B; \mathbb{Z}/2)$, regarded as the set of maps $B \to \mathbb{Z}/2$ that are constant on each path component, acts on the set of orientations of E in an obvious manner. This action is clearly free, and the continuity condition for orientations implies that it is transitive. □

When B is a CW complex, we can understand (a) in terms of obstruction theory as follows. We can arbitrarily choose an orientation over the 0-skeleton. The obstruction to extending this over the 1-skeleton is a 1-cocycle $\alpha \in C^1(B; \mathbb{Z}/2)$. This 1-cycle represents the class $w_1(E)$. If $\alpha = d\beta$, then β tells us how to switch the orientations over the 0-skeleton so that they extend over the 1-skeleton. There is then no further obstruction to extending over the higher skeleta.

[16] An *affine space* over an abelian group G is a set X with a free and transitive G-action. For $x, y \in X$, we can define the difference $x - y \in G$ to be the unique $g \in G$ such that $g \cdot y = x$. The choice of an "origin" $x_0 \in X$ determines a bijection $G \to X$ via $g \mapsto g \cdot x_0$. However this identification depends on the choice of x_0. For example, if A is an $m \times n$ real matrix and $\mathbf{b} \in \mathbb{R}^m$, then the set $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$, if nonempty, is naturally an affine space over $\mathrm{Ker}(A)$.

Exercise 10.6. w_1 is natural, in that if $f : B' \to B$, then

$$w_1(f^*E) = f^*w_1(E) \in H^1(B; \mathbb{Z}/2). \tag{10.2}$$

Equation (10.2) implies that w_1 is a "characteristic class" of sphere bundles. Specifically, it is equivalent to the *first Stiefel-Whitney class* (usually defined for real vector bundles rather than sphere bundles). We will study characteristic classes more systematically later in the course. Our next example of a characteristic class is the Euler class.

11 The Euler class of an oriented sphere bundle

Let $E \to B$ be an oriented S^k bundle with $k \geq 1$. Assume temporarily that B is a CW complex. We now define the *Euler class*

$$e(E) \in H^{k+1}(B; \mathbb{Z}). \tag{11.1}$$

This is the "primary obstruction" to the existence of a section of E, in the sense of Proposition 11.4 below.

First, we choose a section s_0 of E over the 0-skeleton B^0 (just pick any point in the fiber over each 0-cell). Now if we have a section s_{i-1} over the $(i-1)$-skeleton, and if $i \leq k$, then we can extend to a section s_i over the i-skeleton. The reason is that if $e : D^i \to B$ is an i-cell, then we know that the pullback[17] bundle e^*E over D^i is trivial, so we can identify $e^*E \simeq D^i \times S^k$. The section s_{i-1} induces a map $\partial D_i = S^{i-1} \to S^k$, and this extends over D_i because $\pi_{i-1}(S^k) = 0$. By induction we obtain a section s_k over the k-skeleton.

Now let us try to extend s_k over the $(k+1)$-skeleton. Let $e : D^{k+1} \to B$ be a $(k+1)$-cell. We choose an *orientation-preserving* trivialization $e^*E \simeq D^{k+1} \times S^k$. Then s_k defines a map $\partial D^{k+1} \to S^k$, which we can identify with an element of $\pi_k(S^k) = \mathbb{Z}$. This assignment of an integer to each $(k+1)$-cell defines a cochain

$$\mathfrak{o}(s_k) \in C^{k+1}(B; \mathbb{Z}).$$

Lemma 11.1. *$\mathfrak{o}(s_k)$ is a cocycle: $\delta\mathfrak{o}(s_k) = 0$.*

[17] The reason that we use pullback bundles here instead of restriction is that the map e might not be injective on ∂D^i.

Proof. Consider a $(k+2)$-cell $\xi : D^{k+2} \to B$. We need to show that

$$\sum_\sigma \langle \partial\xi, \sigma\rangle \mathfrak{o}(s_k)(\sigma) = 0. \tag{11.2}$$

To understand the following proof of (11.2), it helps to draw pictures in the case $k = 1$.

For each $(k+1)$-cell σ, let p_σ denote the center point of σ. Let P denote the set of points p_σ. Using smooth approximation and Sard's theorem, we can homotope[18] the attaching map $\xi|_{S^{k+1}}$ so that:

(i) The inverse image of p_σ is a finite set of points $x \in S^{k+1}$, to each of which is associated a sign $\epsilon(x)$.

(ii) The inverse image of a small $(k+1)$-disk D_σ around p_σ consists of one $(k+1)$-disk $D_x \subset S^{k+1}$ for each $x \in \xi^{-1}(p_\sigma)$, such that the restriction of ξ to D_x is a homeomorphism which is orientation-preserving iff $\epsilon(x) = +1$.

Given (i) and (ii), it follows from the definition of the cellular boundary map that

$$\langle \partial\xi, \sigma\rangle = \sum_{x \in \xi^{-1}(p_\sigma)} \epsilon(x). \tag{11.3}$$

Next, we can extend the section s_k over $B^{k+1} \setminus P$. Also, choose a trivialization $\xi^*E \simeq D^{k+2} \times S^k$. The extended section s_k then defines a map $f : S^{k+1} \setminus \xi^{-1}(P) \to S^k$. By (ii) above, for each $x \in \xi^{-1}(p_\sigma)$, we have

$$[f|_{\partial D_x}] = \epsilon(x)\mathfrak{o}(s_k)(\sigma) \in \pi_k(S^k).$$

Combining this with (11.3) gives

$$\sum_\sigma \langle \partial\xi, \sigma\rangle \mathfrak{o}(s_k)(\sigma) = \sum_{x \in \xi^{-1}(P)} [f|_{\partial D_x}] \in \pi_k(S^k).$$

Now let X denote the complement in S^{k+1} of the interiors of the balls D_x; this is a compact manifold with boundary. By naturality of the Hurewicz isomorphism, the right hand side of the above equation corresponds to $-f_*[\partial X] \in H_k(S^k)$. But this is zero, since f extends over X. □

[18]This homotopy may change B, but it does not change the left side of the equation (11.2) that we want to prove.

Definition 11.2. The Euler class (11.1) is the cohomology class of the cocycle $\mathfrak{o}(s_k)$.

Lemma 11.3. *The cohomology class $e(E)$ is well-defined.*

Proof. Let s_k and t_k be two sections over the k-skeleton. We will construct a cellular cochain $\eta \in C^k(B;\mathbb{Z})$ with

$$\delta\eta = \mathfrak{o}(s_k) - \mathfrak{o}(t_k). \tag{11.4}$$

The idea is to try to find a homotopy from s_k to t_k. Consider the pullback of E to $B \times I$ with the product CW structure; we can then regard s_k and t_k as sections defined over $B^k \times \{0\}$ and $B^k \times \{1\}$ respectively. There is no obstruction to extending this section over $B^{k-1} \times I$, so that we now have a section u_k over the k-skeleton of $B \times I$. By Lemma 11.1,

$$\delta\mathfrak{o}(u_k) = 0 \in C^{k+2}(B \times I;\mathbb{Z}). \tag{11.5}$$

Now define the required cellular cochain $\eta \in C^k(B;\mathbb{Z})$ as follows: If ρ is a k-cell in B, then

$$\eta(\rho) := \mathfrak{o}(u_k)(\rho \times I) \in \mathbb{Z}.$$

(That is, $\eta(\rho)$ is the obstruction to obstruction to extending the homotopy over ρ.) Then equation (11.4), evaluated on a $(k+1)$-cell σ in B, follows from equation (11.5) evaluated on the $(k+2)$-cell $\sigma \times I$ in $B \times I$. $\square$

To put this all together, we have:

Proposition 11.4. *Let E be an oriented S^k-bundle over a CW complex B. Then there exists a section of E over the $(k+1)$-skeleton B^k if and only if $e(E) = 0 \in H^k(B;\mathbb{Z})$.*

Proof. ($\Rightarrow$) If there exists a section s_k over the k-skeleton that extends over the $(k+1)$-skeleton, then by construction the cocycle $\mathfrak{o}(s_k) = 0$.

($\Leftarrow$) Suppose that $e(E) = 0$. Let s_k be a section over the k-skeleton; then we know that $\mathfrak{o}(s_k) = \delta\eta$ for some $\eta \in C^k(B;\mathbb{Z})$. Keeping s_k fixed over the $(k-1)$-skeleton, we can modify it to a new section t_k over the k-skeleton, so that over each k-cell, s_k and t_k differ by η; see Exercise 11.5 below. That is, equation (11.4) holds. Then $\mathfrak{o}(t_k) = 0$, which means that t_k extends over the $(k+1)$-skeleton. $\square$

Exercise 11.5. Let X be a path connected space such that $\pi_k(X)$ is abelian and the action of π_1 on π_k is trivial (so that $\pi_k(X)$ is just the set of homotopy classes of maps $S^k \to X$ with no base point). Let $f : S^{k-1} \to X$. Then the set of homotopy classes of extensions of f to a map $D^k \to X$ is an affine space over $\pi_k(X)$. The difference between two extensions g_+ and g_- is obtained by regarding g_+ and g_- as maps defined on the northern and southern hemispheres respectively of S^k, and gluing them together along the equator (where they are both equal to f) to obtain a map $S^k \to X$.

One can show, using the cellular approximation theorem, that the Euler class as defined above is natural and does not depend on the CW structure on B. We omit the details. In fact, later in the course we will see alternate definitions of the Euler class which do not use a CW structure at all.

Example 11.6. For an oriented S^1-bundle over S^2, the Euler class agrees with the rotation number. This follows from the definitions if we use a cell decomposition of S^2 with two 2-cells corresponding to the northern and southern hemispheres.

Remark 11.7. The Euler class is called the "primary obstruction" to the existence of a section. There can also be "secondary obstructions" and higher obstructions, involving higher homotopy groups of spheres. That is, when $k > 1$ and $\dim(B) > k + 1$, it is possible that $e(E) = 0$ and yet no section over all of B exists. An example of this (which requires a bit of proof) is given by an S^2-bundle over S^4 in which a nonzero element of $\pi_3(SO(3)) = \mathbb{Z}$ is used in the clutching construction.

On the other hand, when $k = 1$ the Euler class is the only obstruction to the existence of a section, because the higher homotopy groups of S^1 vanish. More generally, we have the following very satisfying result, which can be regarded as a higher-dimensional analogue of Theorem 7.1.

Theorem 11.8. *Let B be a CW complex. Then the Euler class defines a bijection from the set of oriented S^1 bundles over B, up to orientation-preserving isomorphism, to $H^2(B; \mathbb{Z})$.*

Proof. To prove that the Euler class is injective, we use the fact that the space of orientation-preserving homeomorphisms of S^1 deformation retracts onto S^1. Given two oriented S^1 bundles over B, there is then no obstruction to finding an isomorphism between them over the 1-skeleton. It is an exercise to show that the obstruction to finding an isomorphism over the 2-skeleton is the difference between the Euler classes. If this obstruction vanishes, then there is no obstruction to extending the isomorphism over higher skeleta since the higher homotopy groups of S^1 are trivial.

Now let us prove that the Euler class is surjective. Let $\mathfrak{o} \in C^2(B;\mathbb{Z})$ be a cellular cocycle. We will construct an oriented S^1 bundle $E \to B$ and a section s_1 of E over the 1-skeleton such that $\mathfrak{o}$ is the obstruction to extending s_1 over the 2-skeleton.

Over the 1-skeleton, we define $E|_{B^1} = B^1 \times S^1$, and let s_1 be a constant section.

We now inductively extend E over the k-skeleton by gluing in one copy of $D^k \times S^1$ for each k-cell $\sigma : D^k \to E$. To do so we need a gluing map $S^{k-1} \times S^1 \to E|_{B^{k-1}}$ which projects to the attaching map $\sigma|_{S^{k-1}} : S^{k-1} \to B^{k-1}$ and which restricts to an orientation-preserving homeomorphism on each fiber. That is, we need to specify an orientation-preserving bundle isomorphism

$$S^{k-1} \times S^1 \longrightarrow (\sigma|_{S^{k-1}})^*(E|_{B^{k-1}}). \tag{11.6}$$

When $k = 2$, the map (11.6) is an orientation-preserving bundle isomorphism $S^1 \times S^1 \to S^1 \times S^1$, i.e. a map $S^1 \to \mathrm{Homeo}^+(S^1)$, where the superscript '+' indicates orientation-preserving. We choose this to be a map $S^1 \to S^1$ with degree $-\mathfrak{o}(\sigma) \in \mathbb{Z}$. This ensures that $\mathfrak{o}$ is the obstruction to extending s_1 over the 2-skeleton.

When $k \geq 3$, we can choose any bundle isomorphism (11.6); we just need to show check that such an isomorphism exists, i.e. that $(\sigma|_{S^{k-1}})^*(E|_{B^{k-1}})$ is trivial. By the injectivity of the Euler class, it is enough to check that

$$e((\sigma|_{S^{k-1}})^*(E|_{B^{k-1}})) = 0 \in H^2(S^{k-1};\mathbb{Z}).$$

When $k = 3$, this follows from the cocycle condition $\delta\mathfrak{o} = 0$, similarly to the proof of Lemma 11.1. When $k > 3$, this is automatic. □

Remark 11.9. In §8 we saw another geometric interpretation of $H^2(B;\mathbb{Z})$, namely as $[B;\mathbb{C}P^\infty]$. The resulting bijection from $[B;\mathbb{C}P^\infty]$ to the set of isomorphism classes of oriented S^1-bundles over B can be described directly as follows. There is a "universal" oriented S^1-bundle $E \to \mathbb{C}P^\infty$, namely the direct limit of the bundles $S^1 \to S^{2k+1} \to \mathbb{C}P^k$ (which are defined as in the Hopf fibration). One then associates, to $f : B \to \mathbb{C}P^\infty$, the pullback bundle $f^*E \to B$.

To describe the situation that oriented S^1 bundles are classified by homotopy classes of maps to $\mathbb{C}P^\infty$, one says that $\mathbb{C}P^\infty$ is a "classifying space" for oriented S^1-bundles. We will discuss the more general theory of classifying spaces later in the course.

12 Homology with twisted coefficients

A natural context in which to generalize some of the previous discussion is provided by homology with "twisted" or "local" coefficients.

Definition 12.1. A *local coefficient system* on a space X consists of the following:

(a) For each $x \in X$, an abelian group G_x,

(b) for each path $\gamma : [0,1] \to X$, a homomorphism $\Phi_\gamma : G_{\gamma(0)} \longrightarrow G_{\gamma(1)}$, such that:

   (i) Φ_γ depends only on the homotopy class of γ rel endpoints,

   (ii) If γ_1 and γ_2 are composable paths then $\Phi_{\gamma_1\gamma_2} = \Phi_{\gamma_1}\Phi_{\gamma_2}$,

   (iii) If γ is a constant path then Φ_γ is the identity.

   Note that properties (i)–(iii) imply that Φ_γ is in fact an isomorphism.

We sometimes denote a local coefficient system by $\mathscr{G}$ or by $\{G_x\}$, leaving the isomorphisms Φ_γ implicit.

Example 12.2. A *constant* local coefficient system is obtained by setting $G_x = G$ for some fixed group G, and $\Phi_\gamma = \mathrm{id}_G$ for all γ.

Example 12.3. If $n > 1$, or $n = 1$ and $\pi_1(X, x_0)$ is abelian for all $x_0 \in X$, then $\{\pi_n(X, x)\}$ is a local coefficient system on X.

Example 12.4. If $E \to B$ is a Serre fibration, then $\{H_*(E_x)\}$ is a local coefficient system on B.

Example 12.5. A *bundle of groups* is a covering space $\widetilde{X} \to X$ such that each fiber has the structure of an abelian group, which depends continuously on $x \in X$ (in the sense that there are local trivializations which are group isomorphisms on each fiber). Any bundle of groups gives rise to a local coefficient system on X. If X is a "reasonable" space, then the converse is true.

Remark 12.6. If X is path connected and $x_0 \in X$ is a base point, then a local coefficient system on X determines a monodromy homomorphism

$$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(G_{x_0}).$$

Conversely, if G is any abelian group, then any homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(G)$ is the monodromy of a local coefficient system on X with $G_x = G$.

Now let $\mathscr{G} = \{G_x\}$ be a local coefficient system on X. We want to define the homology with local coefficients $H_*(X, \mathscr{G})$, as well as the cohomology $H^*(X, \mathscr{G})$. When $\mathscr{G}$ is a constant local coefficient system with $G_x = G$, this will agree with the usual singular (co)homology with coefficients in G.

Let $\sigma : I^k \to X$ be a singular cube. For every path $\gamma : I \to I^k$, the isomorphism $\Phi_{\sigma\circ\gamma}$ defines an isomorphism $G_{\sigma(\gamma(0))} \simeq G_{\sigma(\gamma(1))}$. Since I^k is contractible, these isomorphisms are canonical, so that the groups $G_{\sigma(t)}$ for $t \in I^k$ are all isomorphic to a single group, which we denote by G_σ. Also, if σ' is a face of σ, then $G_{\sigma'} = G_\sigma$.

Now define $C_*(X, \{G_x\})$ to be the free $\mathbb{Z}$-module generated by pairs (σ, g) where $g \in G_\sigma$, modulo degenerate cubes as usual, and modulo the relation

$$(\sigma, g_1) + (\sigma, g_2) = (\sigma, g_1 + g_2).$$

(One can also describe this as finite linear combinations of distinct nondegenerate cubes, where the coefficient of each cube σ is an element of G_σ.) Define the differential $\partial(\sigma, g)$ by the usual formula (0.1), with g inserted in each term. We have $\partial^2 = 0$ as usual. The homology of this complex is then $H_*(X, \{G_x\})$. The cohomology $H^*(X, \{G_x\})$ is the homology of the chain complex consisting of functions that assign to each singular cube σ an element of the group G_σ, with the differential given by the dual of (0.1).

If X is a CW complex, then we can analogously define cellular homology and cohomology with local coefficients.

Example 12.7. A local coefficient system $\mathscr{G}$ on S^1 is specified by a group G and a monodromy map $\Phi : G \to G$. If we choose a cell structure with one 0-cell e_0 and one 1-cell e_1, then under appropriate identifications, the differential in the cellular chain complex $C_*(S^1; \mathscr{G})$ is given by

$$\partial(e_1, g) = (e_0, g - \Phi g).$$

Hence $H_0(S^1; \mathscr{G}) = G/\operatorname{Im}(1 - \Phi)$ and $H_1(S^1; \mathscr{G}) = \operatorname{Ker}(1 - \Phi)$. Also $H^k(S^1; \mathscr{G}) = H_{1-k}(S^1; \mathscr{G})$.

Example 12.8. Let B be a CW complex, let $k \geq 1$, and let $E \to B$ be an S^k-bundle without an orientation (and possibly nonorientable). Then our previous construction of the Euler class generalizes to give a cohomology class

$$e(E) \in H^{k+1}(B; \{H_k(E_x)\}),$$

which vanishes if and only if E has a section over the $(k+1)$-skeleton. Note here that the local coefficient system $\{H_k(E_x)\}$ is isomorphic to the constant local coefficient system $\mathbb{Z}$ exactly when E is orientable.

Example 12.9. Let E be a fiber bundle over a CW-complex B with fiber F. Suppose that F is path connected. Let k be the smallest positive integer such that $\pi_k(F) \neq 0$; suppose that $\pi_k(F)$ is abelian and that $\pi_1(F)$ acts trivially on $\pi_k(F)$. Then the previous example generalizes to give a cohomology class

$$\alpha \in H^{k+1}(B; \{\pi_k(E_x)\})$$

which vanishes if and only if E has a section over the $(k+1)$-skeleton.

Example 12.10. One can use local coefficients to generalize Poincaré duality to manifolds which are not necessarily oriented or even orientable. Any n-dimensional manifold X has a local coefficient system $\mathscr{O}$, such that $\mathscr{O}_x = H_n(X, X\backslash\{x\}) \simeq \mathbb{Z}$. An orientation of X, if one exists, is a section of $\mathscr{O}$ which restricts to a generator of each fiber. In any case, regardless of whether or not an orientation exists, if X is a compact n-dimensional manifold without boundary, and if $\mathscr{G}$ is any local coefficient system on X, then there is a canonical isomorphism

$$H^k(X; \mathscr{G}) = H_{n-k}(X; \mathscr{G} \otimes \mathscr{O}). \tag{12.1}$$

This isomorphism is given by cap product with a canonical fundamental class $[X] \in H_n(X; \mathscr{O})$. (If X has a triangulation, e.g. if X is smooth, then this fundamental class is represented in the cellular chain complex by the sum of all the n-simplices. This sum is well-defined without any orientation choices if one uses coefficients in $\mathscr{O}$.) To prove this, take your favorite proof of Poincaré duality[19] and insert local coefficients everywhere.

An instructive example is $X = \mathbb{C}P^2$. Consider the usual CW structure which has one i-cell e_i for $i = 0, 1, 2$. The usual cellular chain complex

[19] My favorite proof, for smooth manifolds, is the one using Morse homology. We will see this later in the course.

$C_*^{\mathrm{cell}}(\mathbb{C}P^2)$ is then by $\partial e_2 = \pm 2e_1$ and $\partial e_1 = 0$, where the sign depends on choices of orientations of the cells. Now fix an orientation of $\mathbb{C}P^2$ at the center point of each of the cells. This allows us to also regard $C_*^{\mathrm{cell}}(\mathbb{C}P^2; \mathscr{O})$ as generated by e_0, e_1, e_2. However since the orientation of $\mathbb{C}P^2$ gets reversed if one goes from one end of e_1 to the other or from one side of e_2 to the other, one now has $\partial e_2 = 0$ and $\partial e_1 = \pm 2e_0$, where the sign depends on the orientation choices. It is then easy to check that Poincaré duality (12.1) holds in this example.