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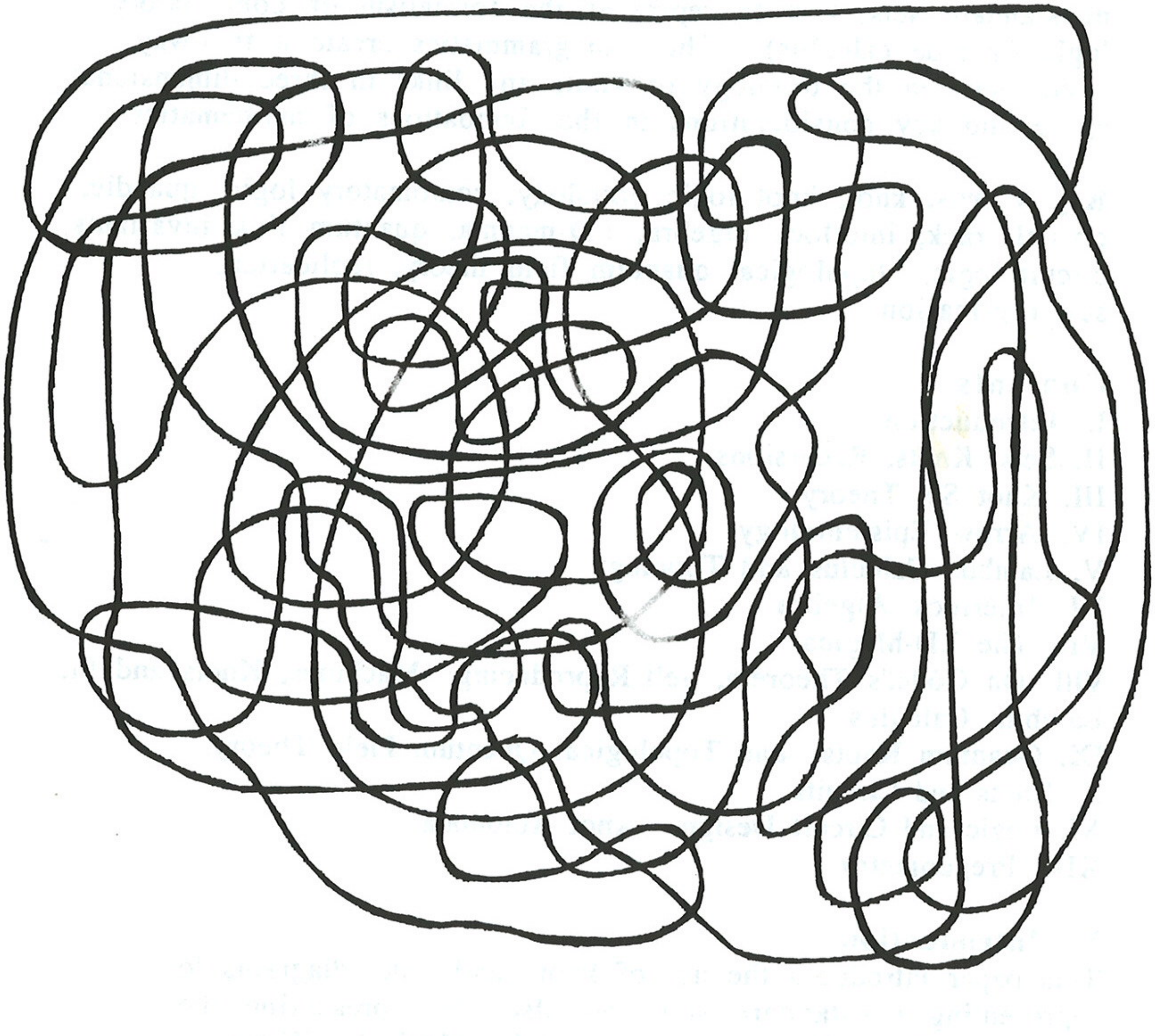
{ In "Knots and Applications" K&E
Series on Knots and Everything – Vol. 6.
ed. by L. Kauffman, World. Sci. 1995, pp. 1–110. }



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by Louis H. Kauffman

Department of Mathematics, Statistics and Computer Science
851 South Morgan Street
University of Illinois at Chicago ,Chicago, Illinois 60607-7045
e-mail: U10451@UICVM.BITNET

Abstract. Knot and link diagrams are used to represent nonstandard sets, and to represent the formalism of combinatory logic (lambda calculus). These diagrammatics create a two-way street between the topology of knots and links in three dimensional space and key considerations in the foundations of mathematics.

Key Words. knot, knot logic, topology, combinatory logic, quandle, crystal, rack, interlock algebra, LD-magma, quantum link invariants, circuit logic, topological quantum field theory, replication, self-replication.

Contents

I. Introduction

This paper introduces the use of knot and link diagrams for representing nonstandard sets and also for representing the formalism of combinatory logic (lambda calculus). These diagrammatics create a two-way street between the topology of knots and links in three dimensional space and key considerations in the foundations of mathematics. The paper explores the relationship of this foundational study with the structure of quantum link invariants and with applications of knot theory to biological structure.

Section II reviews concepts of set theory from an original point of view, emphasizing the relative consistency of sets that do not satisfy the axiom of foundation - by constructing models in terms of notations, graphs and subsets of the plane. Section II also introduces ideas from knot theory and shows how to prove that you cannot cancel knots , just skirting paradox in the process. Section II includes a discussion of reentry and recursive forms in relation to knots, wild embeddings and fractals. An example is given of a sequence of graph embeddings whose complexity increases linearly, while an associated unlinking number is conjectured to increase exponentially. The section ends with a discussion of indicational calculi, non-standard logic, quantum logic and boundary logic.

Section III introduces knot set theory, a set theory whose membership relation is represented by one arc underpassing another. Knot set theory accomodates sets that are members of themselves and sets whose members are defined mutually. The diagrammatic representation of knot sets is so constructed that topologically equivalent diagrams represent the same set. One of the consequences in involving the topology in this way is that knot sets use a "fermionic" convention for the treatment of lists of identicals. The fermionic convention is that identicals cancel in pairs. Thus in the fermionic convention the set {a,a} is equivalent to the empty set. Ordinary set theory uses the "bosonic" convention that identicals condense in pairs (so that {a,a} = {a} in standard sets).

Section IV, discusses concepts of reference in relation to knot set theory.

Section V gives a construction that translates between knot diagrams and combinatory logic. In this formalism the broken arcs of the diagram are used to represent different elements in a lambda calculus, and the diagrams themselves naturally represent non-associative compositions of these elements. We show how to write key constructions in the lambda calculus such as the Church-Curry fixed point theorem in terms on these diagrams. We then investigate the relationship of this formalism with the topology and show how it is intimately related to the algebraic concepts of quandle, crystal and rack (See [J], [K6], [RF]) as used by knot theorists. The quandle, crystal and rack are non-associative algebras that derive from a diagram of the knot and are topological invariants of it. In section VI we take this correspondence further by defining an extension of the crystal, the *interlock algebra* of a knot.

The interlock algebra is an algebra of lambda operators associated with the knot diagram. It is a topological invariant of the diagram and it contains complete information about the topology of the knot. Two knots are isotopic in three space if and only if their interlock algebras are isomorphic. The interlock algebra of a knot contains two types of lambda elements - those with no free variable and those with one free variable (multiple variables will occur in the case of a link). This presence of operators with free variables in the interlock algebra allows an intrinsic identification of subalgebras that are needed for the topology. The construction of the interlock algebra is an application of combinatory logic to topology. Section VI ends with a brief discussion of the classical Alexander polynomial.

Section VII discusses a problem in universal algebra - the structure of non-associative systems with a single non-commutative binary operation that admits a left-distributive law over itself: a(bc) = (ab)(ac). These algebras are called *LD-magmas.* We have already met this condition in studying quandles in section 4. Here the left-distributive law is studied for its own sake. The word problem for free magmas was solved by Patrick DeHornoy in a beautiful and startling application of the Artin braid group. We sketch his method.

Section VIII sketches how the fixed point theorem for the lambda calculus is related to recursive forms, self-reference and Gödel's incompleteness theorem. This section contains a digression on forms of self-replication, including DNA, the Building Machine, the Mighty Simple Self-Rep and the Knot Logical Self-Rep (which turns out to be a picture of the syntax of the Building Machine). The self-replication of a knot is accomplished by a slide equivalence more drastic than the handle-sliding of Kirby calculus. The section ends with a description of Kirby calculus in this context.

Section IX is an introduction to the logic of Dirac brackets in the context of topological invariants. Section X discusses relations between knot theory, electricity and switching circuit theory. Section XI, on asynchronous automata, is a description of a domain in circuit design that has analogies with knot theory. In this context we see that quandles, crystals and racks (Sections 5 and 6) implicate a concept of knot automata.

Section XII explores pregeometry in the sense of John Wheeler. We make the case that knot and link diagrammatics are central to an appropriate conception of pregeometry. An appendix discusses the bracket model for the Jones polynomial.

The author would like to express his thanks to Louis Crane, Lee Smolin, Carlo Rovelli, Julian Barbour and John Wheeler for helpful conversations. Research for this paper was partially supported by the Program for Mathematics and Molecular Biology, University of California at Berkeley and by NSF Grant No. DMS 9205277.

II. Sets, Knots, Recursions

It is customary either to build the theory of sets axiomatically, or to construct it from the intuitive concepts of membership and collection. It is well-known that a naive approach leads to paradoxes.

For example, the Russell set R is defined to be the set of all sets that are not members of themselves. X is a member of R exactly when X is not a member of X. On substitution of R for X, we find that R is a member of R exactly when R is not a member of R.

Initially, it is not clear whether the difficulty with the Russell set is in the notion of set formation, the idea of self-membership, the use of the word "not" , the use of the word "all" or elsewhere. The Theory of Types [WhR] due to Russell and Whitehead placed the difficulty in the use of self-membership, and solved the paradox by prohibiting this and other ways of mixing different levels of discourse.

The Gödel-Bernays set theory (See [K], Appendix on Elementary Set Theory.) creates a different solution to the Russell paradox by making one large distinction between *set* and *class* . Of two sets A and B it can be said without ambiguity that A is a member of B, or B is a member of A, or neither is a member of the other. A class is a *set* if it is a member of another class. Classes are determined by their members, and classes can be defined in terms of properties: Given a property P, there exists a class C(P) equal to the class of all x such that *P(x) is true and x is a set.*

In this system, the Russell class is
R = {x | x is not a member of x and x is a set}.
Thus R is a class, but R is not a set.

In a system of the Gödel-Bernays type, there is nothing inherently wrong with self-membership. In fact, self-membership and other forms of contradiction of the "axiom of foundation" (which disallows infinite descending chains of membership.) are very interesting to explore using geometry, topology and diagrams. To this end, let us start from the beginning and construct some sets.

The empty set is commonly denoted by empty brackets: { }. Notationally, sets indicated only through brackets are a subcollection of all the ways of making well-formed brackets:

A finite expression E in brackets is well-formed if
1. E is empty.
or
2. E = { F } G where F and G are well-formed.

These two rules give a complete characterization of the well-formed bracket expressions. A *finite ordered multi-set S* is an expression in the form
S = {T} where T is any well-formed expression. It follows that $T = A_1\ A_2 \ldots A_n$ where n is a positive integer, and each A_i is a finite ordered multi-set. The A_i's are the *members* of S.

We write the members of S without commas between them.

For example, if S= { { } { { } } }, then the members of S are

{ } and {{ }}.

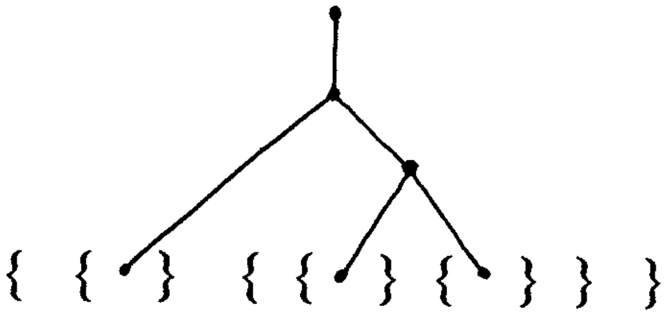
A multi-set may have a multiplicity of identical members as in

X={ { } { } { } }.

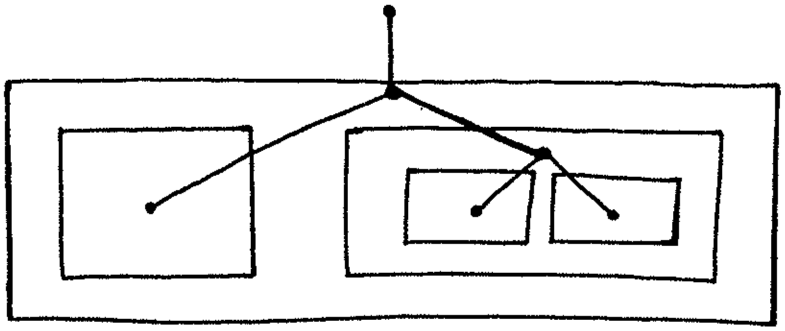
Ordered multi-sets are equal exactly when they have identical sequences of brackets. To emphasize this point, let L denote a left bracket , { , and R denote a right bracket , } . Then the set X above is encoded by the sequence LLRLRLRR.

To obtain the usual category of finite sets, factor the ordered multi-sets by the equivalence relation generated by XY = YX and XX = X where X and Y are well-formed expressions. It then follows from our definitions that two finite sets are equal exactly when they have the same members.

It is easy to see that the class of ordered finite multi-sets is isomorphic to the class of rooted planar trees - by graphical duality as indicated below.



Another way to think of these sets is to replace each pair of brackets by a rectangle in the plane. Then any set is a collection of disjoint rectangles, with a single outermost rectangle - the set boundary. The members of the set are delineated by the rectangles inside this outermost rectangle that are outermost or at the same level as all other rectangles in the pattern. The tree is still obtained by graphical duality as shown below.



In both cases there is a natural notion of depth obtained by counting crossings inward from the outermost rectangle, or by counting nodes from the root of the tree. The equivalence relation on rectangles that generates finite sets is: *take the collections of rectangles up to homeomorphisms of the plane.* Here we use a sophisticated concept to define an elementary one. The use of this will become apparent at once when we enlarge the category and obtain a model of non-standard sets.

Let FIST (First Infinite Sets) denote the class of (not necessarily finite) disjoint collections of rectangles in the plane such that each collection S has a single outermost rectangle, and the collection of rectangles inside that outermost rectangle is a disjoint union of elements of FIST (These are the members of S.) If A and B are in FIST, then we shall say that A=B if there is a homeomorphism of the plane that carries A to B.

Call a collection of rectangles in the plane, taken up to homeomorphism of the plane, a *form*. Thus, finite (and some infinite) sets can be interpreted as forms, but not all forms are sets. In any form we can say unambiguously of two rectangles whether one is inside or outside of the other.

Forms can be framed and juxtaposed.

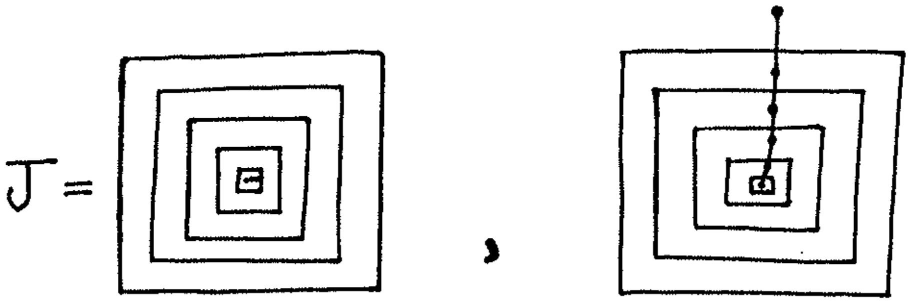
Let {X} denote the result of putting a rectangle around the form X. Call this operation the *framing* of the form X. Let XY denote the *juxtaposition* of the forms X and Y. To get multi-sets from forms, consider forms that are framed.
For example,

0. { }
1. { { } }
2. { { } { } }
3. { { } { } { } }

...

can be regarded as a list of multisets, with 0,1,2,3,... members.
No commas are needed in the internal list of a set represented in this fashion. One simply searches for the different frames at depth 1, to get the list of members. (The depth counts the number of crossings made inward from the outermost region in the form.)

In FIST the simplest element that is a member of itself is shown below and denoted by the letter J. J is an infinite nest of rectangles, or an infinite linear tree.



Note that J = { J } where we interpret the brackets as the addition of the outer rectangle. There is nothing inherently infinite about the description J= { J }, but its recursive unfolding leads to an infinite construction corresponding to an infinity of nested brackets:

$$J=\{\ \ \{\ \ \{\ \ \{\ \{\{\{\ldots\}\}\}\ \}\ \ \}\ \ \}\ \ \}$$

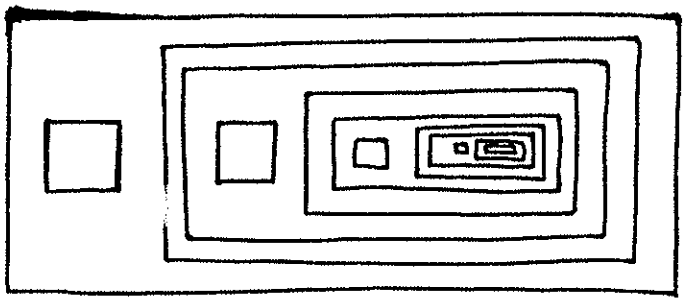
With this rectangle model in mind, consider elements of FIST that are defined by systems of equations. For example, A = {{} B}, B = {A} yields

$$A=\{\{\}\,B\}=\{\{\}\,\{A\}\}$$

$$=\{\{\}\,\{\{\{\}\,\{A\}\}\}\}$$

$$=\{\ \{\}\ \ \{\{\ \ \{\}\ \ \{\{\ \ \{\}\ \ \{\{\ \ \{\}\ \ \{\ldots\}\ \ \}\}\ \ \}\}\ \ \}\}\ \ \}.$$

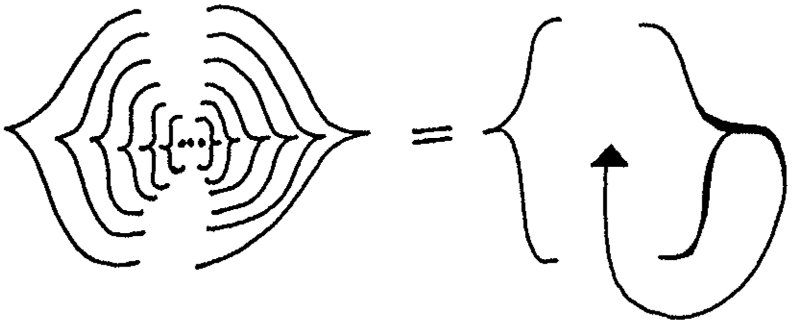
A and B correspond to non-homeomorphic systems of rectangles, and so give a pair of distinct but entangled sets in FIST.



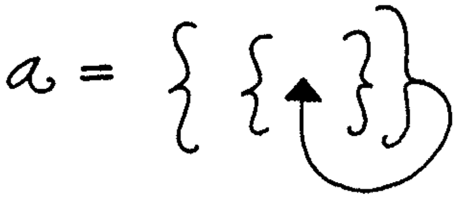
Reentry Notation, Recursive Forms and Infinite Regress. A set that is a member of itself can be diagrammed as a set with an arrow pointing into the inside of the set where the self inclusion occurs (Compare [K16].)

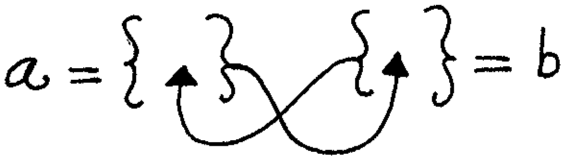
M = { M } = { 

In this form, one tends to take a model of infinite regress or recursion as in

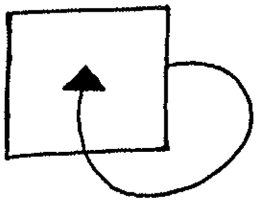


Similarly, in the case of interlock (a={b}, b={a}) we have a= {{a}} and the reentry description

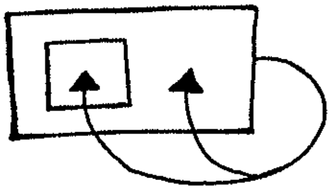


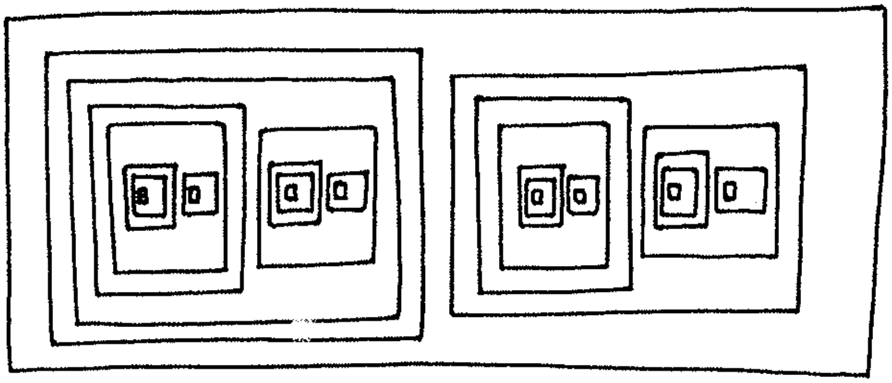


The reentry concept goes beyond set formation to a domain of recursive forms. To indicate recursive forms that are not necessarily interpreted as sets it is convenient to use a rectangular box notation. Thus we write

$J =$ 

and

$F = \boxed{\boxed{F}\; F} =$ 



The second recursive form, F, can be called the *Fibonacci Form* since the number of divisions of this form at depth n is the nth Fibonacci number. (The form divides the plane into disjoint connected regions. These are the *divisions* of the form. A division is said to have *depth n* if it requires n inward crossings of rectangle boundaries to reach that region from the outermost region in the plane. Each rectangle divides the plane into a bounded region and an unbounded region. A crossing of the boundary of a given rectangle is said to be an *inward crossing* if it goes from the unbounded region to the bounded region.)

To see this and other facts about the divisions of a form, *let* F_n *denote the number of divisions of an arbitrary form* F *at depth* n.

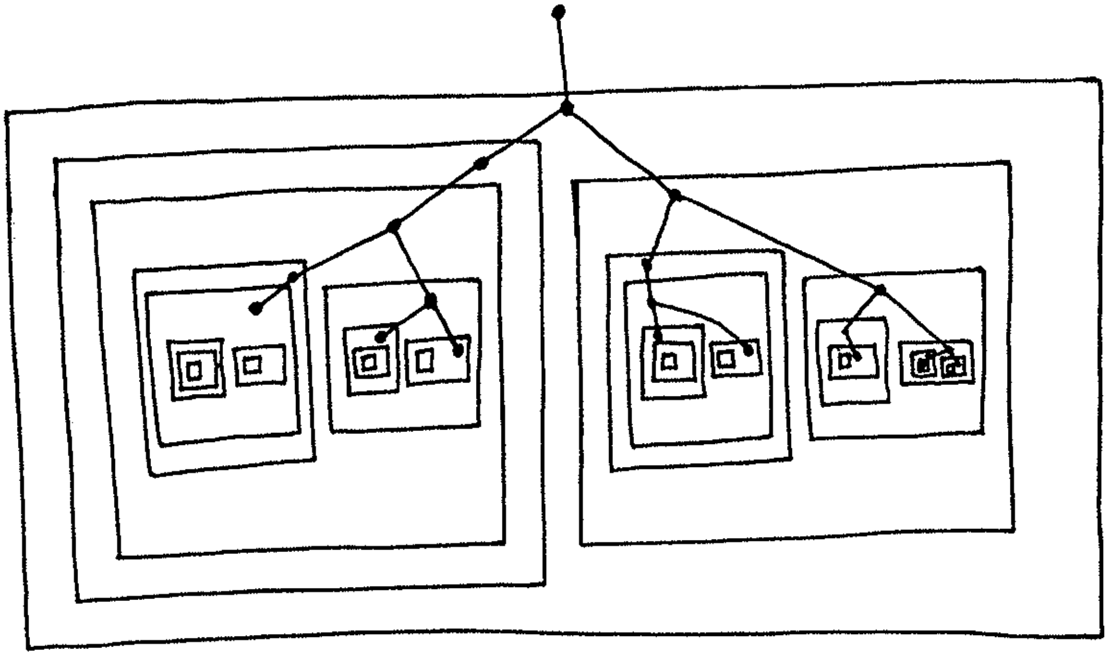
Then, for any forms X and Y,
1. $(XY)_n = X_n + Y_n$
2. $\{X\}_n = X_{n-1}$.

In the case of the Fibonnaci form, we have $F=\{\{F\}F\}$. Hence $F_n = F_{n-2} + F_{n-1}$. Since $F_0 = F_1 = 1$, this proves our assertion about the Fibonacci series as the depth counts of the Fibonacci form.

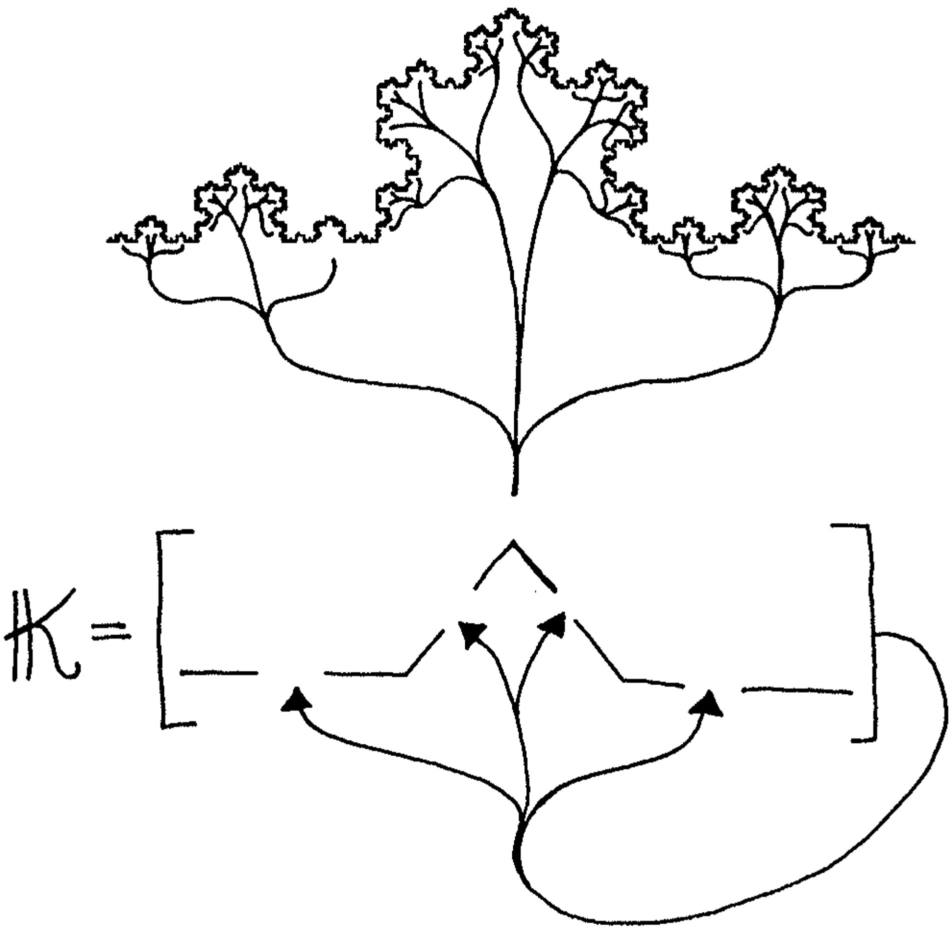
From here it is quite natural to define the *growth rate*, $\mu(F)$, of a form F as the limit of F_{n+1}/F_n as n goes to infinity.

$$\mu(F) = \lim_{n \to \infty} (F_{n+1}/F_n).$$

The growth rate of the Fibonacci form is the golden ratio, $(1+\sqrt{5})/2$.



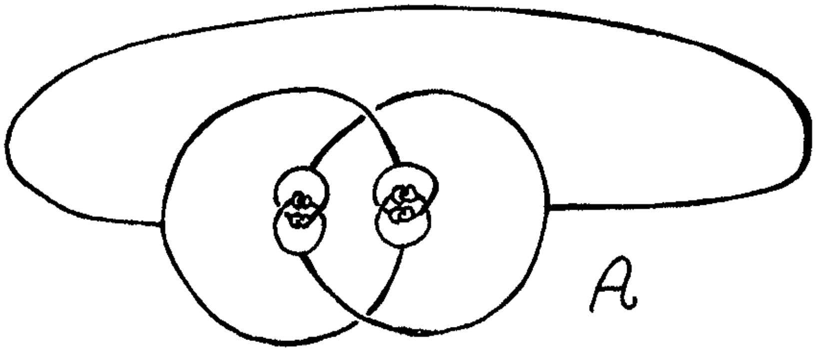
Recursive forms and their growth rates are intimately related to fractals. For example, the Koch fractal reenters its own indicational space in four major places as shown below.



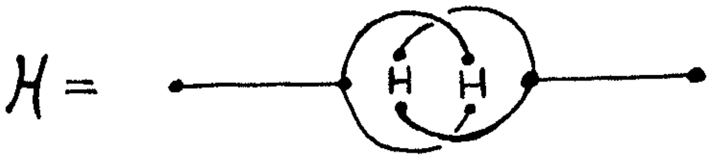
We schematically indicate the structure of the Koch fractal by the recursive form K = {K {KK} K} . Extra brackets have been placed inside this form to indicate the special grouping of the middle two copies of the Koch fractal. These copies are the triangular pushout in the fractal itself. This recursive form can be regarded as the *pregeometry* of the fractal. It contains skeletal information about the fractal, but does not describe the geometry of its actual construction. The fractal dimension of the Koch fractal is encoded in its recursive form. The fractal dimension of the Koch is Log(4)/Log(3). Four (4) is the growth rate of the form A={AAAA} and three (3) is the growth rate of the form B={BBB}. K itself can be viewed as an A by seeing it as a repetition of 4 copies (this is the duplication rate). K can also be viewed as a B by seeing it as an internal group of three (this is the shrink rate in the geometry). The fractal dimension is the ratio of the logarithms of these two growth rates related to the recursive form.

Alexander's Horned Sphere

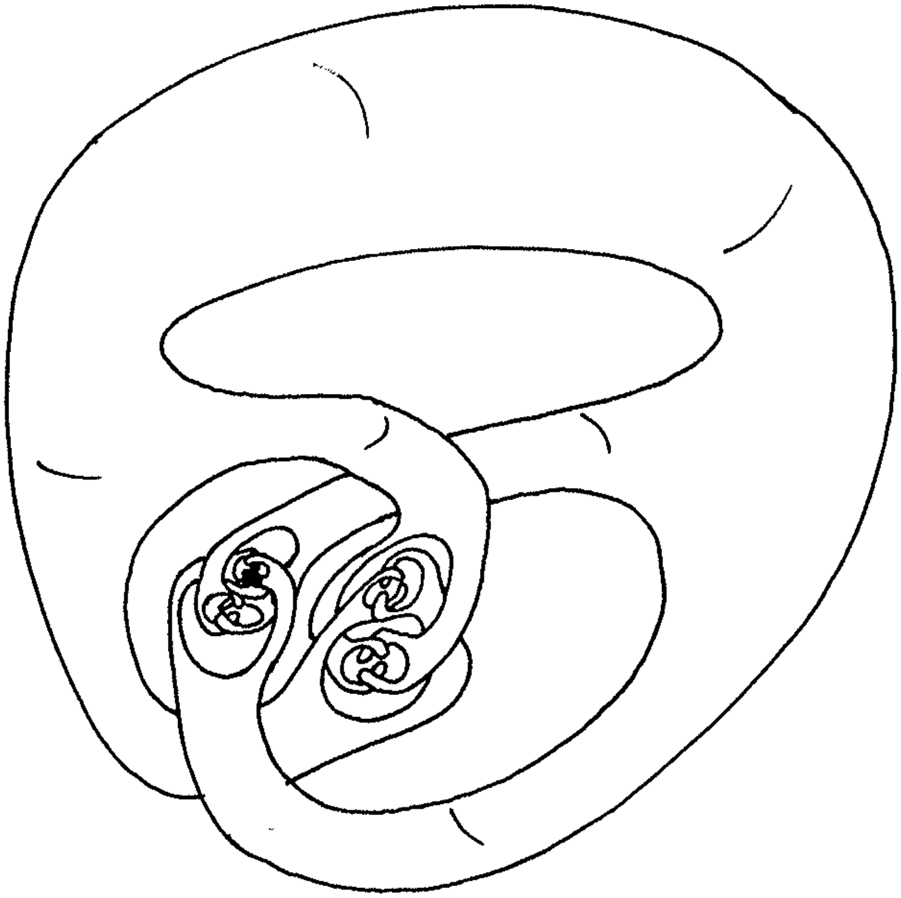
Now we go to topology and look at the reentry form associated with the famous Alexander Horned Sphere [HY]. The schematic for this construction is illustrated below.



This infinite graph can be described as a reentry form as shown below.

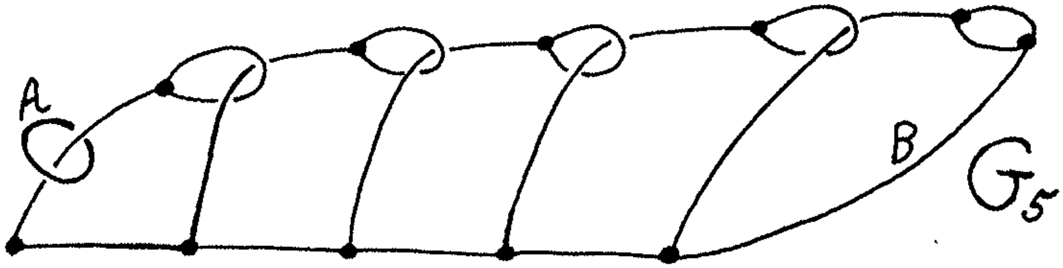


The limit of this construction produces a wildly embedded tree in three-space that is self-linked (i.e. the fundamental group of the complement of this tree is non-trivial). Any finite stage of the construction produces an unlinked embedding of a tree. The Alexander Horned Sphere is obtained by taking a limit of the boundaries of tubular neighborhoods of the finite trees in this construction. It is an embedding of a two dimensional sphere into three dimensional space such that the inside of the sphere is simply connected, but the outside is not simply connected.



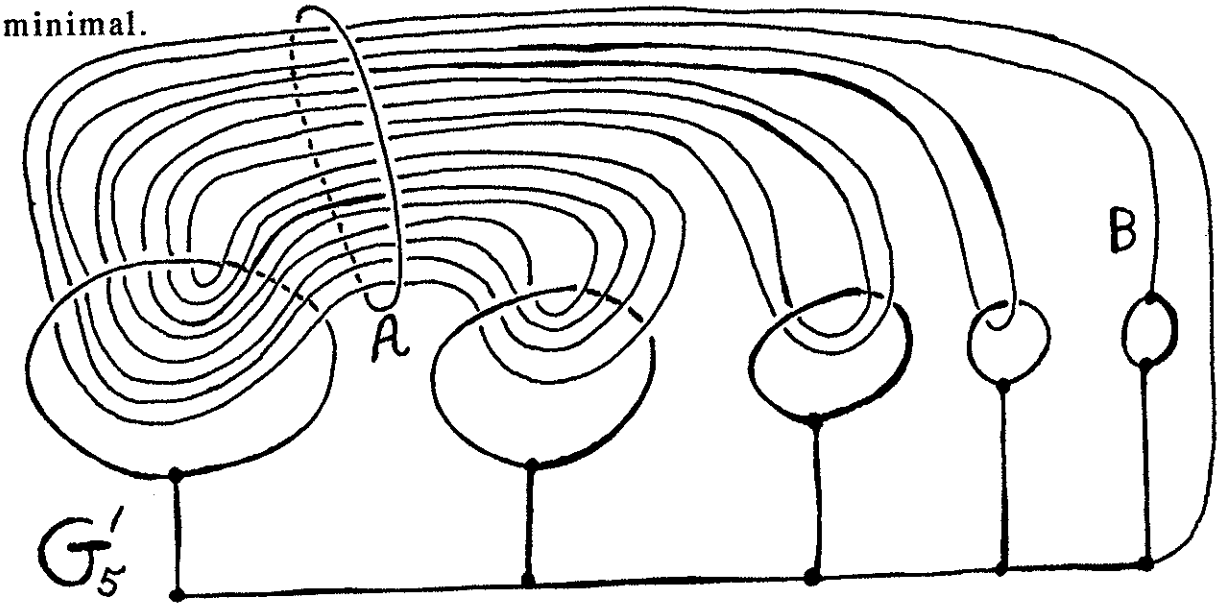
The most remarkable thing about the horned sphere is that it is a sphere. The limit construction does not touch itself anywhere. There is a Cantor's set worth of wild points on this embedded sphere such that any neighborhood of a wild point contains infinitely many branches of the structure.

An example of recursive unlinking.
Consider the graph embedding shown below.



This is a special case of the graph embedding G_n where n is equal to 5. In G_n there is a series of n hoops, each one successively slipped through the previous one, all tied together at their bases, and so that the arc B is attached from the last hoop to its own base. Suppose that it is desired to unlink the circle labelled A from this graph under the stipulation that A is allowed to make crossing exchanges only with the arc labelled B. One can perform any isotopy of the embedding coupled with these allowed crossing changes. Then *I conjecture that G_n requires at least 2^{n-1} crossing exchanges with B in order to become unlinked.* If this conjecture is true, then we have an unlinking problem whose complexity goes up exponentially, while the complexity of the underlying graph embeddings that support it goes up linearly. This example shows how the sort of recursive construction associated with an object like the horned sphere can pose an actual complexity problem in topology for the finite stages of the recursion.

The isotopy shown below of G_5 to a graph G' with the hoops unentangled, should give the reader a glimpse of evidence for this conjecture. It is clear that A can be unlinked from G' by 2^4 exchanges. Hence, up to isotopy, A can be unlinked in G_5 by 2^4 exchanges. A similar construction shows that A can be unlinked in G_n with 2^{n-1} exchanges. We conjecture that this procedure is minimal.



The Method of Infinite Repetition

There is a technique in topology called *the method of infinite repetition* . It begins with the paradox:

$$\begin{aligned} 0 &= (1-1)+ (1-1)+ (1-1)+ (1-1)+ \ldots \\ &= 1-1+1-1+1-1+1-1+\ldots \\ &= 1 + (-1+1) + (-1+1) + (-1 +1) + \ldots \\ &= 1 + 0 \\ &= 1. \end{aligned}$$

Theorem. Let it be assumed that infinite sums make sense and that $a+b = b+a$ and $x+(y+z) = (x+y)+z$, $0+x =0$ for all a,b,x,y,z. Then $a+b = 0$ implies that $a=0$ and $b=0$.

Proof:

$0 = 0 + 0 + 0 + 0 + \ldots$

$= (a+b) + (a+b) + (a+b) + \ldots$

$= a + b + a + b + a + b + \ldots$

$= a + (b+a) + (b+a) + (b+a) + \ldots$

$= a + 0 + 0 + 0 + \ldots$

$=a.$

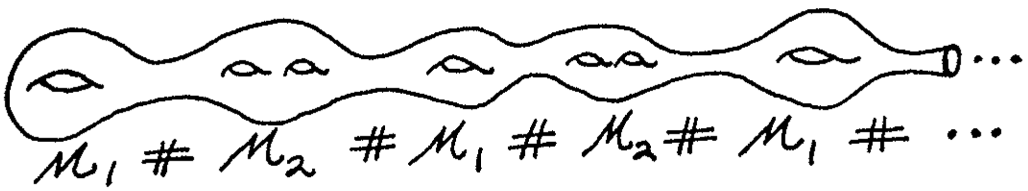
Similarly, $b=0$. This completes the proof.//

Of course, for numbers, infinite sums do not necessarily make sense, and so we have not proved that zero equals one. There are, however, topological applications to this formalism. Here is an example: Let M and M' be (compact orientable) surfaces. The connected sum of M and M' , M#M' , is obtained by excising a disk from each surface and connecting them to each other by a tube whose ends are glued to the circular boundaries of the two regions left by the excision in each surface.

We shall prove, by infinite repetition, the

Theorem. $M\#M' = S^2$ implies that $M = S^2$ and $M' = S^2$.

Here S^2 denotes the surface of a two dimensional sphere (the boundary of a three dimensional ball.) and $=$ denotes homeomorphism of surfaces. It is easy to check that $M\#S^2 = M$ for any surface M and that the connected sum operation is well-defined for finite sums, and that it is commutative and associative. Can we make sense of an infinite sum? The answer is yes, but one leaves the category of compact surfaces: Put the surfaces M_1, M_2, M_3, ... in a row extending to the (viewer's) right. Form $M_\infty = M_1\#M_2\#M_3\#\ldots$ by connecting them together by straight tubes between adjacent surfaces. The resulting surface M_∞ is well-defined but no longer compact. For example $S_\infty = S^2\#S^2\#S^2\#\ldots$ is homeomorphic to the plane R^2.



In this case an infinite sum of "zeroes" is not zero! However, for any surface M, $M\#S_\infty = M-\{pt\}$, since removing a point is equivalent to the connected sum with R^2. Thus:

If $M\#M' = S^2$, then

$$
\begin{aligned}
&S_\infty\\
&= (M\#M')\#(M\#M')\#\ldots\\
&= M\#(M'\#M)\#(M'\#M)\#\ldots\\
&= M\#(S_\infty)\\
&= M-\{pt\}.
\end{aligned}
$$

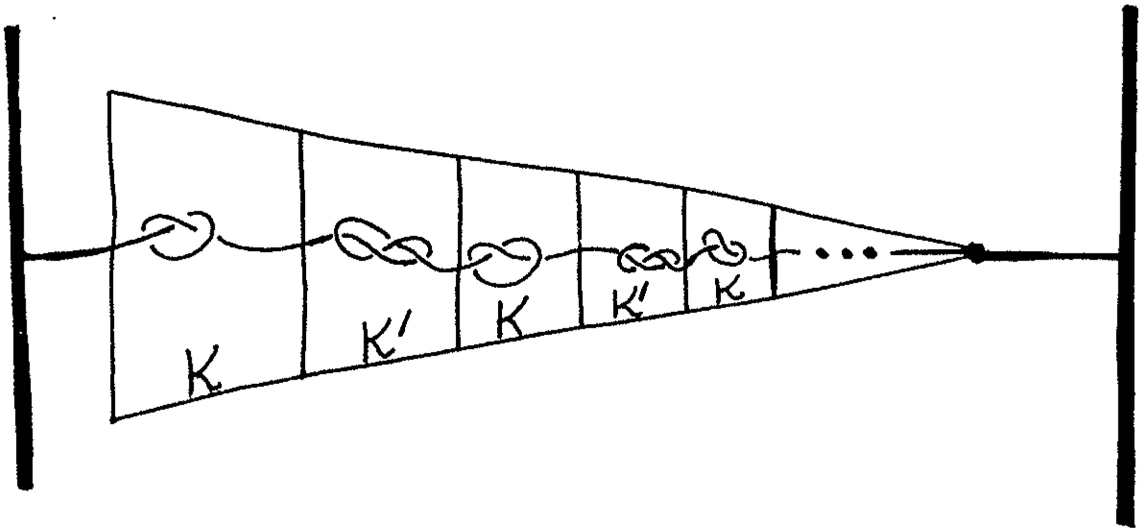
Now form the one-point compactification of both sides and conclude that $S^2 = M$.

Because S_∞ is not the 2-sphere, we cannot use this argument to conclude that if $M\#M'$ is smoothly homeomorphic to the 2-sphere, then M is smoothly homeomorphic to the 2-sphere. Differentiability may fail in the neighborhood of the missing point. In fact, for surfaces the theorem still holds in the smooth category, but the same argument transposed to higher dimensions has this limitation. For example in dimension 7, there are manifolds M and M' homeomorphic to spheres but not diffeomorphic to spheres such that M#M' is diffeomorphic to the standard 7 sphere (See [KM]).

You Can't Cancel Knots

Tie a knot in a piece of rope and then tie another knot adjacent to it. (In this picture of knots, one is *not allowed* to move any rope past the end points. Think of the end-points as attached to opposite walls of a room. With the ends attached to the wall, the rope can be moved so long as it is not removed from the wall or torn apart.)

Is it possible that the two knots taken together can undo one another even though they are individually knotted? The answer is NO. The proof is by infinite repetition [F]: Let O denote the unknot. Let K#K' denote the connected sum of knots obtained by adjacent tying. Instantiate K_∞ = K#K'#K#K'#K#... as an infinite weave in a compact space by introducing a limit point as shown below.



Then K_∞ is, by the method of infinite repetition, equal to both K and to O. Hence K must be unknotted.

This argument goes into the larger category knots with infinite amounts of weave to make its conclusions. In order to show that the conclusion holds in the usual category of finite weaves, a topological theorem is needed stating that if finitely woven knots are equivalent in the larger category of infinite weaves then they are equivalent in the category of finite weaves. The result that supports this conclusion is found in [MO].

The Conway Proof

There is a very beautiful proof of the impossibility of knot cancellation due to John Conway (See [G].). His proof does not go off into infinite weave. Here is a sketch of Conway's proof:

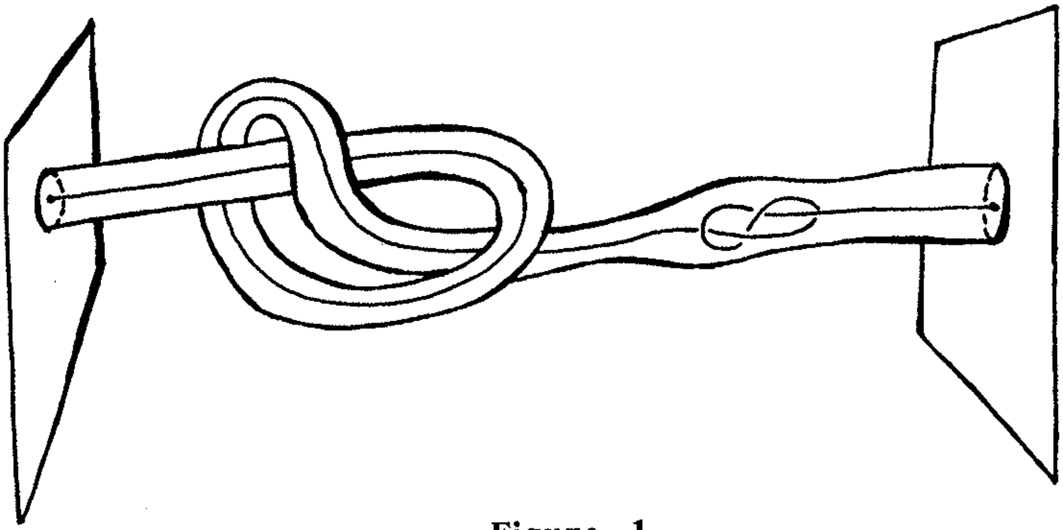


Figure 1

Put a tube T around K#K' (as shown in Figure 1 above) so that the tube is a tubular neighborhood of K and so that the tube engulfs K'. If K#K' = O, then there is a homeomorphism of the room to whose walls K#K' is attached that leaves the walls of the room fixed, and straightens K#K' to a straight line L extending from the left wall to the right wall. The tube T will be deformed by this homeomorphism to a new tube T' that does not intersect the line L. Let P be plane in the room containing L. Then P intersects the left and right walls of the room in the endpoints of L and in four points of the tube T' (two on each wall). Let a and b denote the intersection of P with T' on the left wall and let c and d denote the intersection of P with the right wall. Then P intersects T' in arcs that emanate from a,b,c,d and some closed curves in P. The arc from a cannot reach either b or d because it is separated from these points by the line L in the plane P. Therefore the arc from a must extend to c. This arc AC from a to c is necessarily unknotted in the room, since it is a non-self-intersecting arc in the plane P. However the arc AC is the image under the homeomorphism of an arc extending from one end of the tube T to the other, and by construction, this means that the arc AC must be equivalent to the knot K (since the tube is knotted in the pattern of K). Therefore we have shown that in the course of unknotting K#K' we have necessarily unknotted K itself! Therefore you cannot cancel knots.//

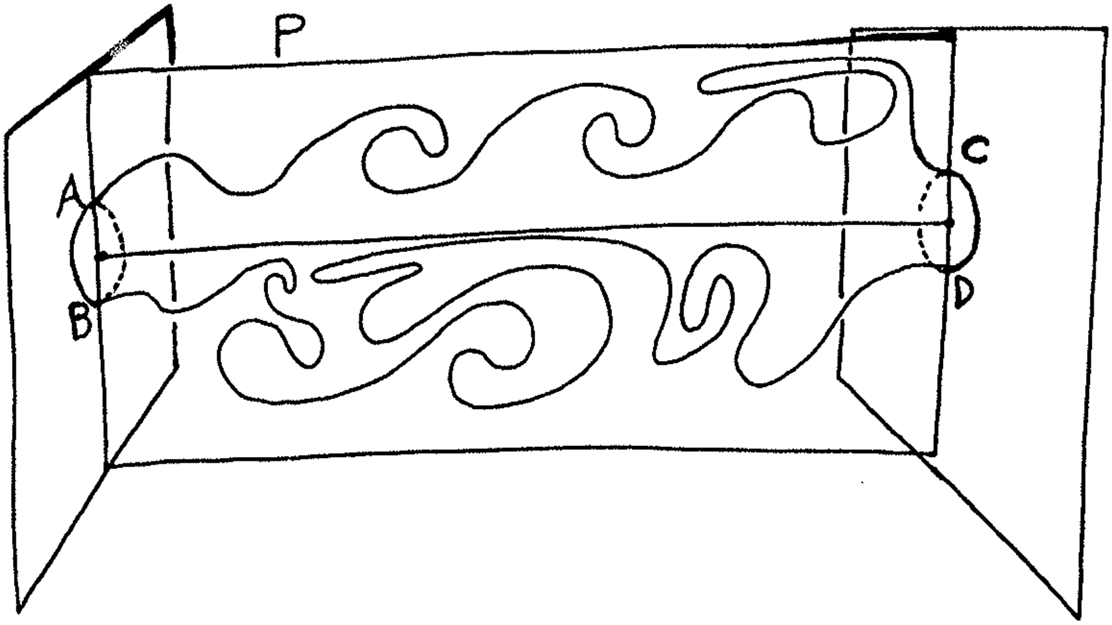
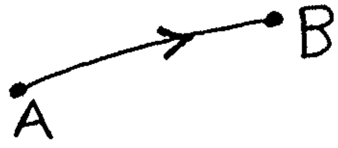


Figure 2

Graphs that Encapsulate Infinity

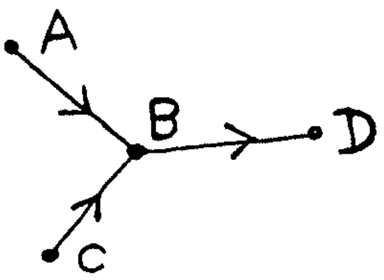
There is a very elegant way to represent sets in FIST that are described by systems of equations: *Any* directed graph represents such a set.

Each node in the graph represents a set. An edge directed from node A to node B encodes the relation that *B is a member of A.*



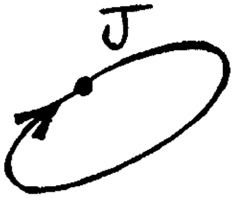
$$B \in A$$

(This method of representation is used by Aczel [AC].)
A single finite set is a rooted tree where all the edges are directed away from the root as in the examples preceding this discussion. Nevertheless, any directed graph yields a set, or sets. For example,

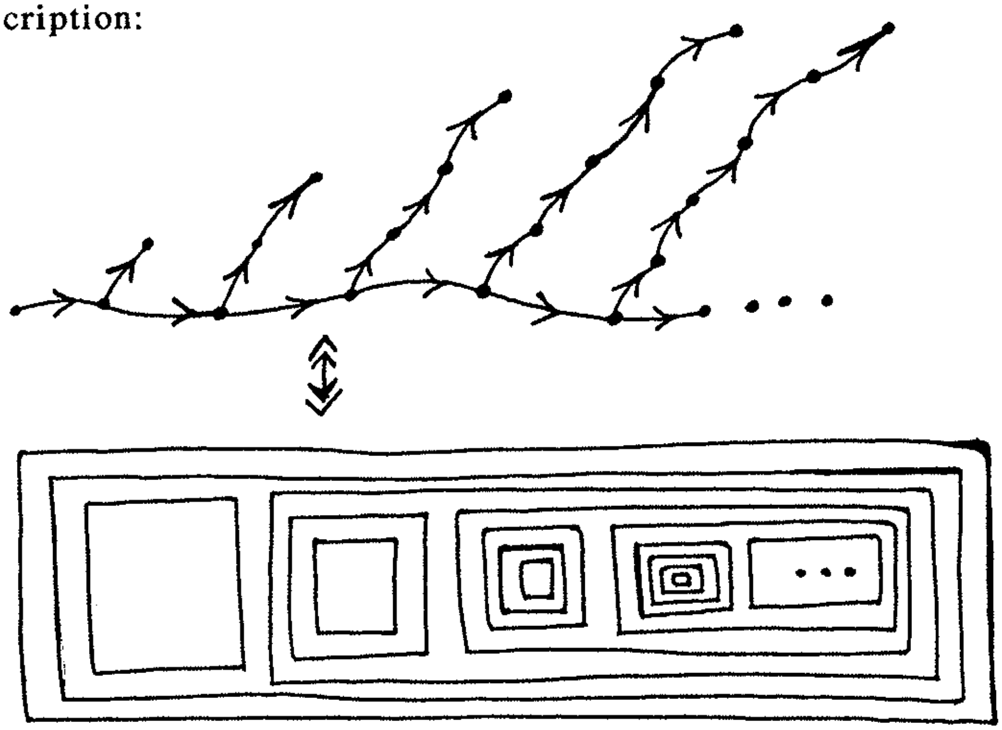


Here $A = \{ B \}$, $B = \{ D \}$, $C = \{ B \}$, $D = \{ \ \}$. (A node with no outwardly directed edges connotes the empty set.) In this case, we see that $A = \{\{\{ \ \}\}\}$, $B = \{\{ \ \}\}$, $C = A$, $D = \{ \ \}$. The symmetry of the graph with respect to the nodes A and C corresponds to the equality of the corresponding sets.

The set $J = \{J\}$ is represented as a node with a self-directed edge.



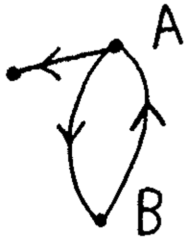
The category of sets in FIST that are represented by finite directed graphs is pleasant to contemplate, but it only scratches the surface of FIST. For example, the following infinite tree has no finite graph description:



Here are a few more examples:

1. A = {B} and B={A}.

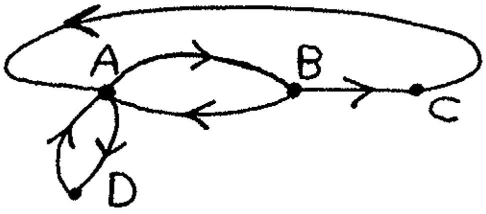
Here the corresponding sets in FIST are identical since we obtain A= {{{{....}}}} and B = {{{{...}}}}. We may wish that this graph represented two distinct sets A and B that mutually create one another. This end can be achieved by taking the graphs at face value, rather than accepting the model involving these recursive limits as the end of the story. In the next section we shall do just this in the context of knot sets. In the FIST context, one can obtain the effect of distinguishing A and B by giving one a different membership structure from the other via a "label" as in A = {B, { } } and B = {A}.



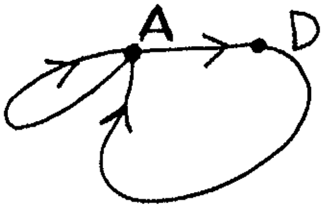
2. F= {{F} F}.

The solution in FIST is F = { { {{ ... } ... } } { { ... } ... } }. This is the Fibonacci form (considered earlier in this section).

4. Consider the set in FIST specified by the graph shown below



The corresponding system of equations is
A={B,D}
B={A,C}
C={A}
D={A}
The last two equations force C=D, and these then force A=B. Thus the system is equivalent to the system A={A,D} and D= {A} or to the graph



This example shows how different graphs can lead to the same elements of FIST. It is an interesting question to determine the minimal graphs that represent a given system of mutually defined sets in FIST. The nodes of such a directed graph are mutually distinguished from one another in terms of the mutual membership relations. An analogy to this situation for undirected graphs is found in the extremal variety graphs of Barbour and Smolin [BaS]. In an extremal variety graph, all points are distinct due to the presence of distinguishing neighborhood structures in the unoriented graph. Thus, the extremal variety graph represents a space in which the points are distinguished from one another due entirely to their mutual relationships. Minimal directed graphs for sets in FIST are an oriented analog of the extremal varieties.

Pregeometry

These remarks look forward to the discussion of pregeometry in section 10. A minimal directed graph or a maximal variety graph can be regarded as a miniature world in which the nodes are the observers. Each observer obtains its identity from its relations with the other observers. In the case of directed graphs, each observer's immediate perception is of its members (the nodes that are one directed edge away). Further reports yield the members of members and eventually the full system of relationships that constitute this world. The problem of pregeometry is how it can come to pass that such worlds acquire geometry and topology that is natural with respect to the structure of relations, and giving rise to known physical law. It is our contention (see Section 10) that knot theory gives a new way to consider the question of pregeometry.

In the next section, we discuss a representation of sets that interfaces with knots and links in three dimensional space. We conclude the present section with two general remarks about the models with which this section began.

Remark1. Indicational Calculus, Boolean logic and the Calculus of Indications.

We have seen that the full set of well-formed parenthesis structures is a background of the theory of finite sets. Let us denote these structures modulo the relations XY=YX and XX=X by parentheses written in angle-bracket form. Thus < <> <<>> <> > = < <> <> <<>> > = < <> <<>> > denotes the set whose members are an empty set and a set consisting of an empty set. The expression <<>><> is a *form* but not a set in the terminology used earlier in this section. Now consider the quotient of the class of forms generated by the extra relation <<>> = e where e denotes the empty word. Let = continue to denote this equivalence relation. Then <<>><> = <> and <<>> =
where the blank space is the empty word. All finite forms fall into the two distinct equivalence classes corresponding to the empty word and the mark <> . We represent these classes by <<>> and <>.

The collection of forms up to this new equivalence satisfy many equations. For example, <<X>> = X for any X and <X>X = <> for any X. By interpreting
<X> as the *negation* of X,
XY as **X or Y,**
<<X><Y>> as **X and Y,**
<<>> as **False,** <> as **True,**
one recovers the full structure of Boolean algebra. This is the calculus of indications of G. Spencer-Brown [S-B] expressed in parenthesis notation. Boolean algebra arises from the boundary structure of finite set theory. The calculus of indications begins with well-formed parenthetical expressions modulo the equivalence generated by

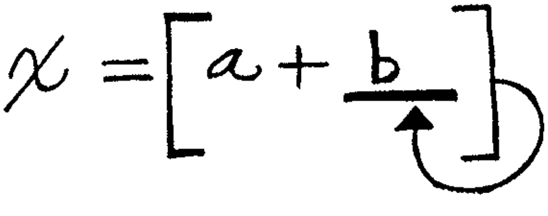
<><> = <> and <<>> = .

These equivalences can be performed within otherwise identical larger expressions.

Imaginary Boolean Values

Infinite expressions in the context of the calculus of indications, give non-Boolean values. For example, if $P = <<<\ldots>>>$, then $P = <P>$. Infinite expressions are not necessarily reducible to one of the two states $<<>>$ or $<>$. It is an interesting problem to enlarge the context of Boolean algebra to handle such values. See [K15], [K16], [K17], [KV] for a discussion of solutions to this problem. Spencer-Brown [S-B] makes the perspicuous observation that there is a direct analogy between the imaginary Boolean value $P = <P>$ and **i**, the square root of minus one: **i** is the solution to $\mathbf{i} = -1/\mathbf{i}$. If we ask to solve $x = F(x)$ with $F(x) = -1/x$, then $x=1$ implies $x = -1$ and $x=-1$ implies that $x = 1$. The problem of finding a square root of minus one is analogous to the liar paradox. Complex numbers provide a solution to this paradox in the numerical domain. Just so one can consider imaginary values in logical domains.

The solution $P=<<<\ldots>>>$ to $P = <P>$ is the analog of the solution $x = a + b/(a + b/(a + b/(a + \ldots)))$ for $x = a + b/x$. In the case where $a= 0$, $b= -1$ there is no real numerical value for this continued fraction. When $x^2 = ax + b$ has a real root, then the continued fraction converges and gives a real answer. When $x^2 = ax + b$ does not have a real root then the continued fraction does not converge, but the recursion $x \longrightarrow a + b/x$ is quite interesting to study in its own right, producing an intriguing class of oscillations of the form $x_{n+1} = a + b/x_n$. (*Exercise*: show that these oscillations all take the form $x_n = \tan(n\theta + \Phi)$ for appropriate choice of theta and phi depending upon a and b.) In Figure 3 we show a typical plot of x_n (vertical axis) against n (horizontal axis) in the case where $x^2 = ax + b$ has no real root. (Here the starting value for x is 1 and a=1, b=-6.)



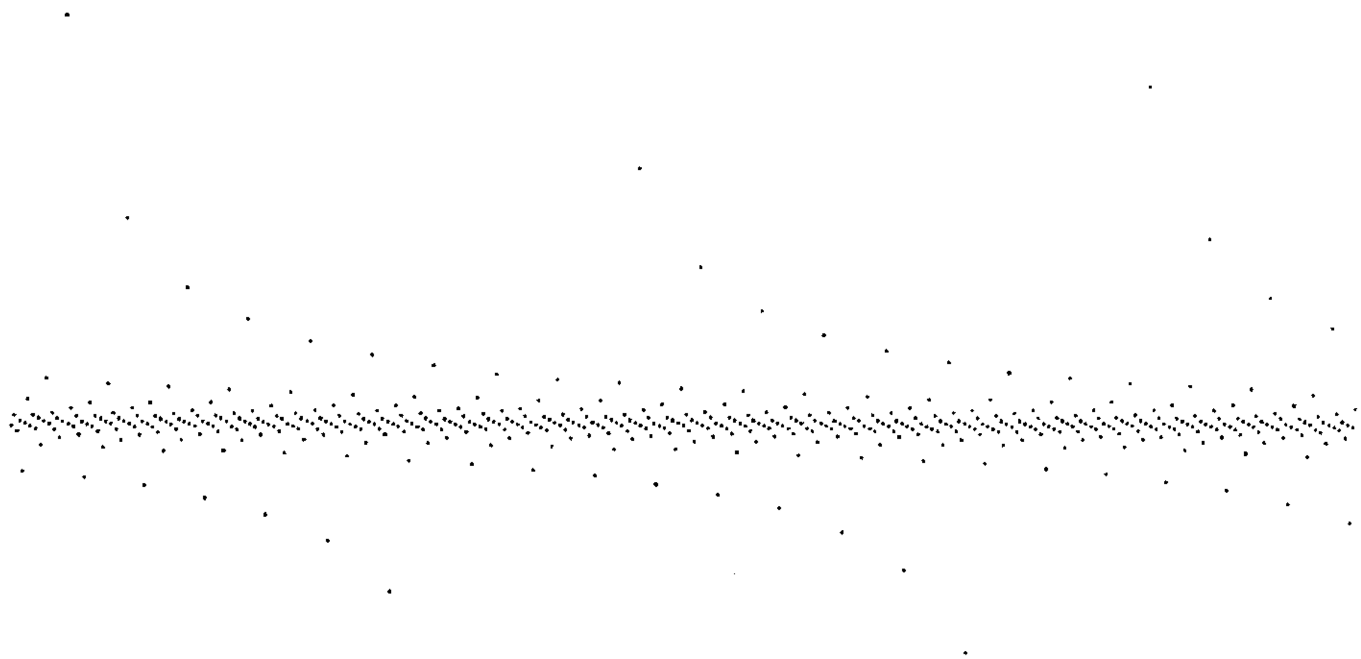


Figure 3

Paradox can be studied through the recursive process inherent in its syntactic form. (See [K16], [K17], [KV], [H1], [H2].) In the case of the complex numbers it is interesting to point out that the view of the square roots of minus one as oscillations between 1 and -1 is mirrored in the matrix representation of these roots by the matrices whose squares are minus the identity.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

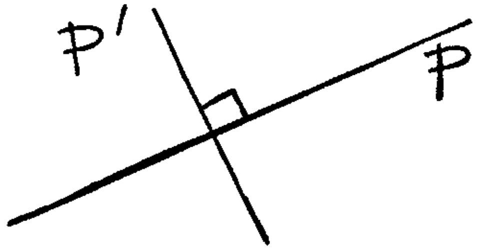
In thinking about the square root of minus one, one must ask which one (i or -i)? Similarly, in regarding the imaginary value **P** = <P>, one encounters *two* oscillations. There are two corresponding sequences, depending on whether the starting value is 0 or 1. These solutions can be formalized as ordered pairs of Boolean values [a,b] with [a,b]' = [b',a'], and [a,b][c,d] = [ac,bd]. Let I=[0,1] and J=[1,0]. Then I and J are the two views of the alternation ...0101010101... with I'=I, J'=J and IJ=[0,0]=0. This construction gives a DeMorgan Algebra [K15],[K16], [KV]. As we shall see later in this essay (section 10) an entirely different world opens up if we ask for the same conditions, but IJ=0.

Remark2. Quantum Logic

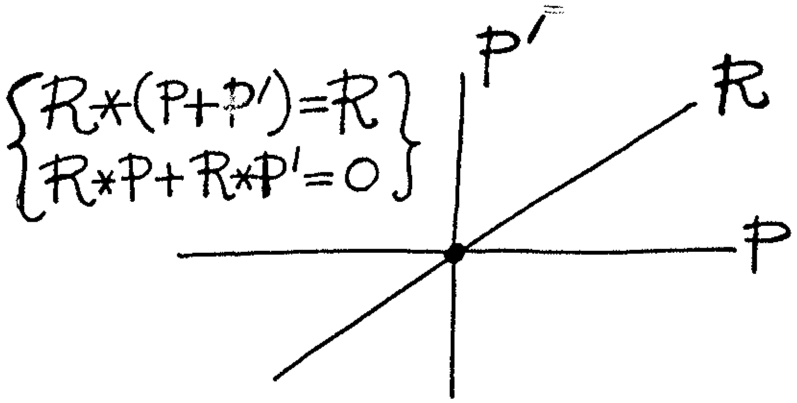
Recall the simplest form of quantum logic (See [F1] ,[F2],[F3], [O]) based on a vector space V with a notion of orthogonal complement for subspaces (W' is the orthocomplement of W). Elements in the algebra of this logic are subspaces of V. The negation of W is its orthocomplement W'. The sum of subspaces A and B (A+B) is the subspace spanned by A and B in V. The product of A and B (A*B) is their set theoretic intersection. Let 1 denote V and 0 denote the zero subspace.

In this logic, we have $A+A' = 1$, $A*A' = 0$ for any A. The law of the excluded middle still holds, and there is no element J in the logic such that $J'=J$. On the other hand, if V is two dimensional, and P and Q represent perpendicular lines in V, while R represents a line independent from both P and Q then we have
$1= 1*R = (P + Q)*R$ while $P*R + Q*R = 0 + 0 = 0$.
The distributive law does not hold in the quantum logic.

Such a non-Boolean logic is called a quantum logic because it models the operations of states and projections in a quantum mechanical system. Addition of vectors corresponds to the superposition of states. Here we are concerned not with the naturality of this structure with respect to quantum mechanics, but rather with its naturality in respect to mathematical foundational ideas. Vector spaces are a rather late development in the hierarchy of mathematical constructions. Can one encounter quantum logic nearer to the bottom? One answer is an appeal to geometry. If we describe in notation this move to quantum logic it becomes: Let (for three dimensions) the whole space, a plane, a line or a point indicate a given proposition. Let the negation of this proposition be indicated by a linear space that is perpendicular to the indicator for a given proposition. Thus, in a plane, if we diagram P by a line



then P' is a line perpendicular to P.



At once there arises the infinite multiplicity of lines in between P and P'. If the plane itself is all (1) and a point the void (0), then we can only save the law of the excluded middle by letting $P+P'$ indicate the plane spanned by these two lines. It is nevertheless this very existence of intermediates that makes the logic non-distributive. For we take R to be a line going straight between P and P', and we find that $R*(P+P')$ is not equal to $R*P + R*P'$. The quantum logic is the logic of the first movement of notation into geometry.

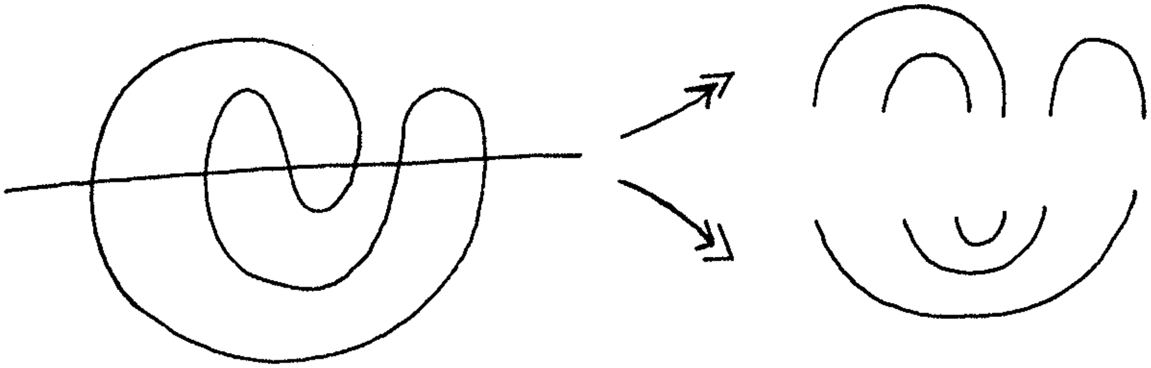
Quantum logic is the pre-geometry of notation. Boolean logic is obtained *in notation* by ignoring the existence of intermediate states.

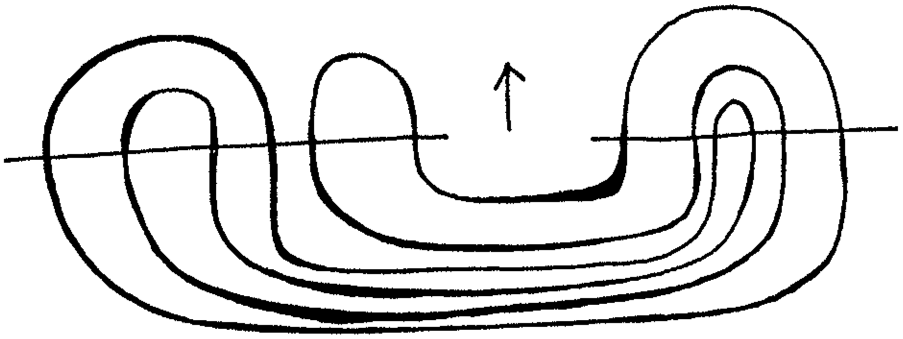
This discussion makes no claim that its remarks about notation and quantum logic have a direct bearing on quantum mechanics. Such issues deserve more exploration.

Remark3. Ordered Parentheses, Boundary Logic and the Temperley Lieb Algebra

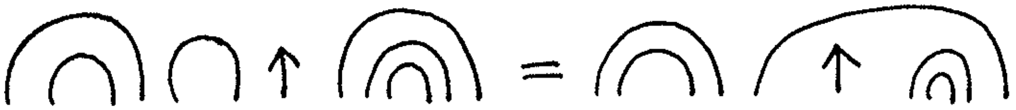
In this section we have taken the point of view that ordered parenthetical expressions in brackets (finite ordered multi-sets) are precursors to finite set theory. In examining the structure of such expressions it is useful to tie left and right ends of the parenthesis into a single form that shall be called a *cap.* This notational device is indicated below.

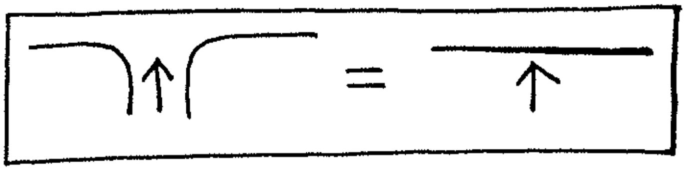
Call parenthetical expressions written in this notation *capforms.* The capforms are intimately related to a number of topological problems. One way to see this is to draw a simple closed curve (i.e. a curve with no self-intersections) in the plane and slice it with a straight line. the line cuts the curve into two capforms such that the feet of each cap are on the line.

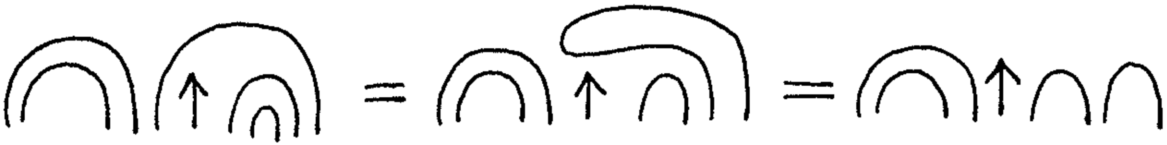


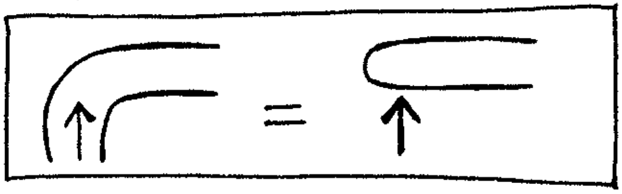


The interaction of these two capforms produces the single simple closed curve. In fact, we formalize the interaction of the two capforms as a cancellation (or connection) of nearby boundaries. We indicate nearby interacting boundaries by an arrow.

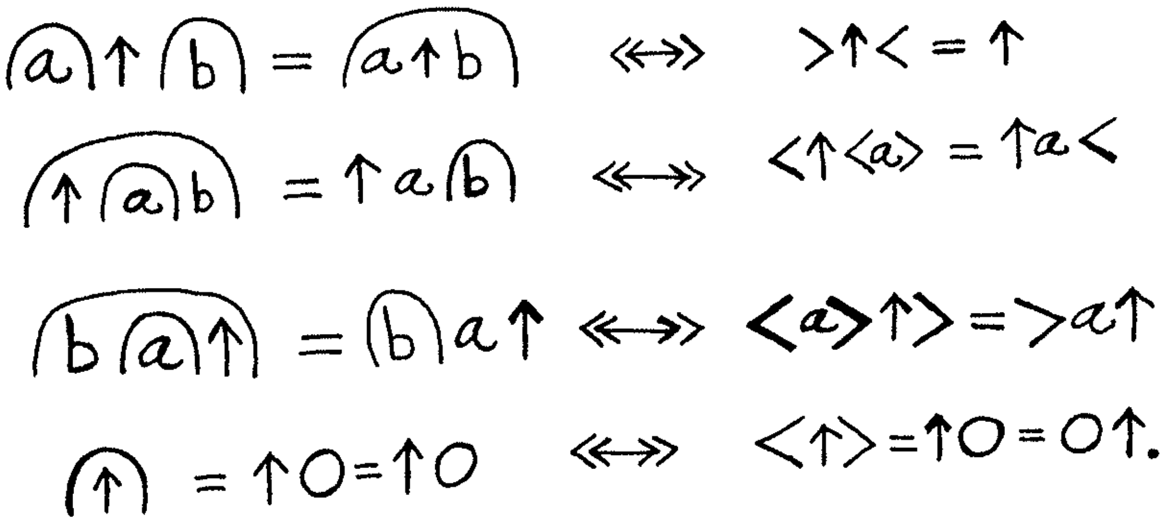








This gives rise to the following rules in a calculus of capform boundaries that we call *boundary logic* (See [BRI] for a distinct but related use of this term.).

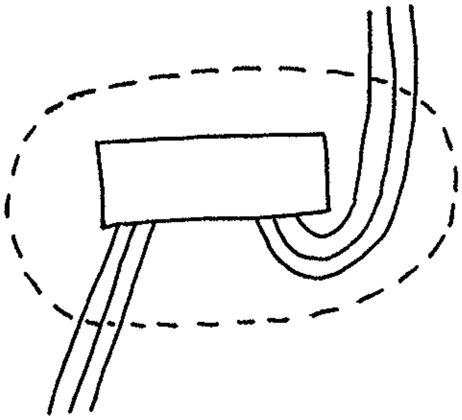


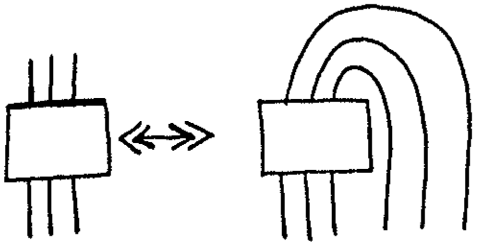
To determine whether two capforms interact to produce a single simple closed curve, one can either calculate in boundary logic or draw geometric connections and trace the resulting plane curves:

Remark. By using the boundary logic in parenthetical form, we can formalize it with rules for string replacements. Then the equivalent of the above graphical calculations can be performed by a digital computer. (See [K6], Appendix to Second Edition, pp. 605-608.)

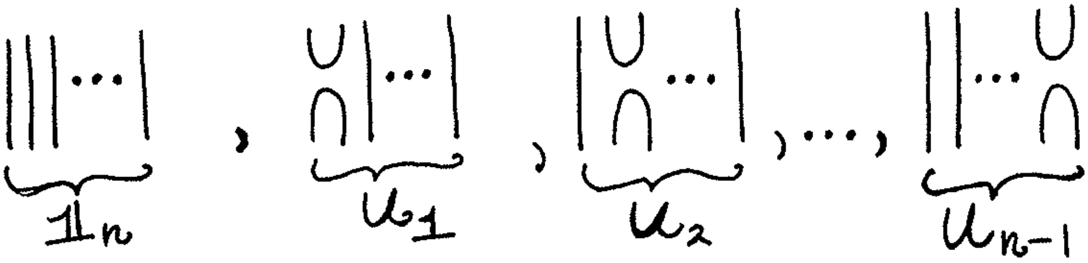
If C_n denotes the capforms with n caps, define a binary operation $C_n \times C_n \longrightarrow C_n$ by $X\#Y = X \mid^n Y$ where $\mid^n$ denotes the n-fold iteration of the boundary joining operation. This product operation can be described quite explicitly by regarding a capform in C_n as having n left legs and n right legs. X#Y is the result of joining the right legs of X to the left legs of Y as shown below.

The structure of this product on C_n is better understood by rewriting the elements of C_n so that the left legs appear at the top of a box, and the right legs appear at the bottom.





Then one can verify that every capform is a product of the elementary capforms shown below. These forms are the generators of the (diagrammatic) Temperley-Lieb algebra [K3], [K6].

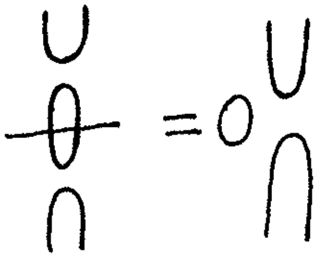


The following relations describe this algebra

$$
\begin{gathered}
U_iU_{i+1}U_i = U_i, \\
U_iU_{i-1}U_i = U_i, \\
U_iU_j = U_jU_i \text{ if } |i-j|>2, \\
(U_i)^2 = dU_i.
\end{gathered}
$$

Here d denotes the value assigned to a single free loop (the loop is taken to commute with other elements of the algebra.)

The last relation is illustrated below.



$$: U_i^2 = d U_i$$

The Temperley-Lieb algebra originated in certain problems in statistical mechanics (See [BX].), and it has a very strong influence on many problems in the theory of knots and links.

The fundamentals of set theory are intimately connected, through combinatorial structures and the theme of boundaries, with logic, topology and mathematical physics.
All this from framing nothing!

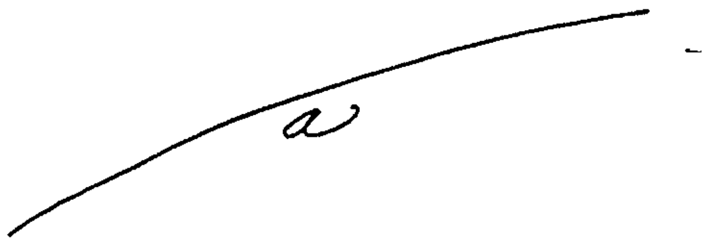
III. Knot Set Theory

A diagrammatic alternative to Venn diagrams can model a non-standard set theory.

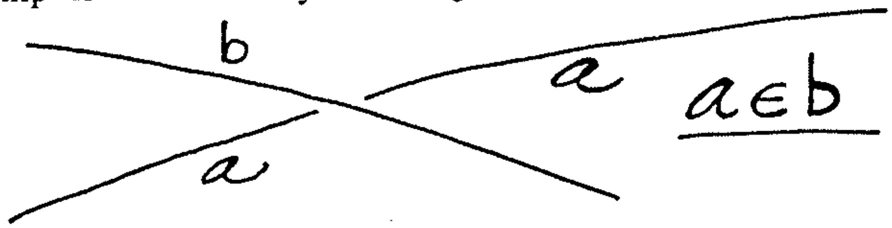
This section describes such a diagrammatic model and explains its relationship with the theory of knots and links in three dimensional space.

We begin with undefined objects denoted by letters a,b,c,... and a notion of membership denoted $a \varepsilon b$ (a "belongs" to b). It will be possible for a to belong to itself ($a \varepsilon a$) or for a to belong to b while b belongs to a. In the model there is no infinite regress and the system, a formal diagrammatic theory, is consistent relative to standard discrete mathematics.

Here is a description of the model. Objects will be indicated by non-self intersecting arcs in the plane. A given object may correspond to a multiplicity of arcs. This is indicated by labelling the arcs with the label corresponding to the object. Thus the arc below corresponds to the label a.

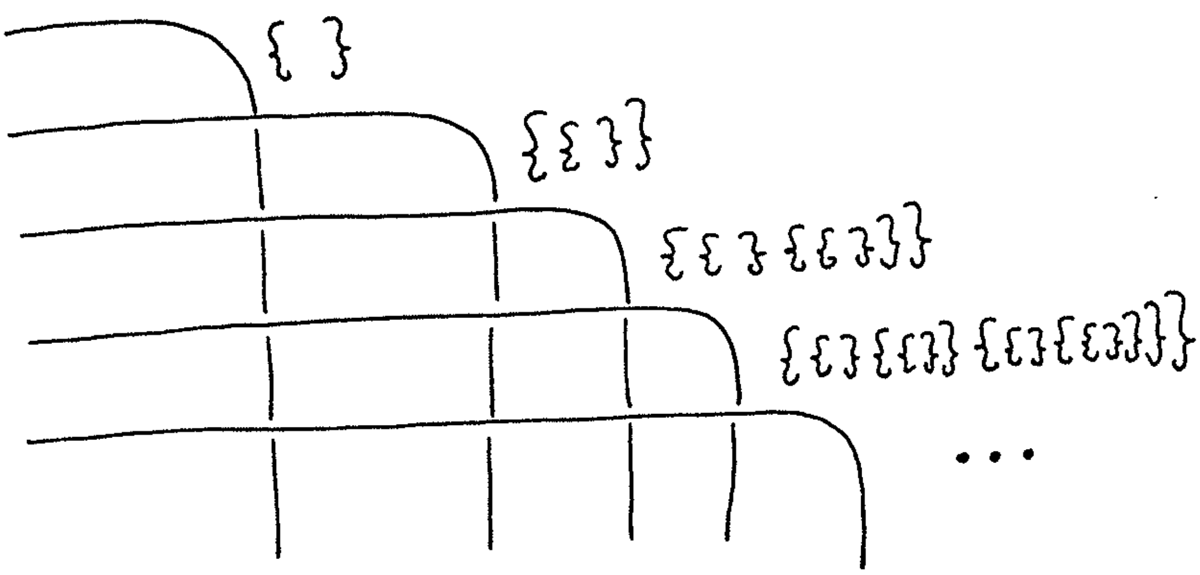


Membership is indicated by the diagram shown below.



Here we have shown $a \varepsilon b$. The arc b is unbroken, while a labels two arcs that meet on opposite sides of b. Following the pictorial convention of illustrating one arc passing behind another by putting a break in the arc that passes behind, one says that *a passes under b.* The pictorial convention is important both for the logic and for the deeper relationship with three dimensional space that we shall elucidate shortly.

It is an easy matter to illustrate certain basic constructions in set theory. For example, the von Neumann construction of sets of arbitrary finite cardinality is traditionally done by starting with the empty set $\phi = \{ \}$, and building a sequence of sets X_n with
$X_0 = \{ \}, X_1 = \{\{ \}\}, X_2 = \{\{ \}, \{\{ \}\}\}$.
Here $X_{n+1} = X_n \cup \{ X_n \}$ where U denotes the operation of union. The diagrams below show how to implement this construction using the overcrossing convention for membership.



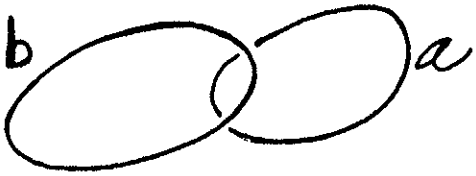
With these same diagrams it is possible to indicate sets that are members of themselves

$$a = \{a\}$$

and sets that are members of each other

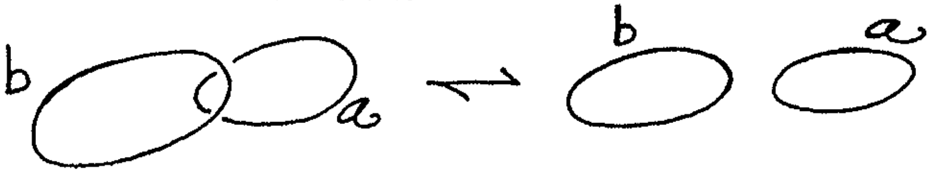
$$a = \{b\}$$
$$b = \{a\}.$$

As they stand, these diagrams indicate sets that may have a multiplicity of identical members. Thus

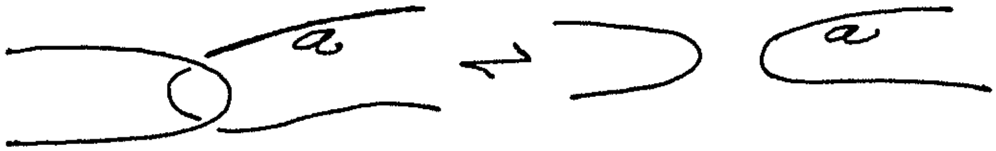


Here $b = \{a,a\}$ and $a = \{ \}$.

The traditional way to condense multiplicities of identicals is to regard them as all equivalent to one another. This amounts to the condensation rule $\{ \ldots a,a \ldots\} = \{ \ldots a \ldots \}$. In the case of our diagrams another solution is suggested. In this solution, *identicals cancel in pairs* and we have $\{ \ldots a,a \ldots\} = \{ \ldots \ \ldots \}$. Thus $\{a, a\} = \{ \}$. This is diagrammed as shown below:



It is easy to remember this diagrammatic transformation, since it can be interpreted as a drawing of one strand of rope being slipped out from under another. We shall accordingly adopt the rule of cancellation of identicals as fundamental to knot set theory.



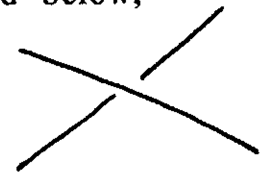
$$\{aa\cdots\} \rightleftharpoons \{\cdots\}$$

Digression on Knots.

The diagrams that we are drawing have a well-known interpretation as diagrams of knots, links and tangles in three dimensional space. By convention, a knot consists in a single closed curve, a link may have many closed curves and a tangle has arcs with free ends. Also by convention, topological changes in a tangle do not involve moving the free ends or in passing strands over the free ends.

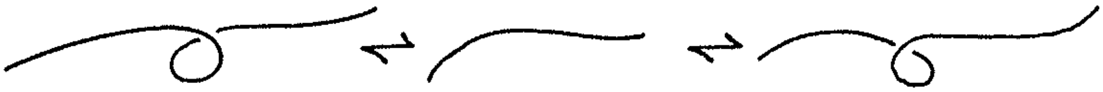
There is a direct relationship between the topology of these knots, links and tangles and the properties of the knot set theory.

Reidemeister [R] proved that any knot or link in three dimensional space can be represented by a diagram containing only crossings of the type indicated below,

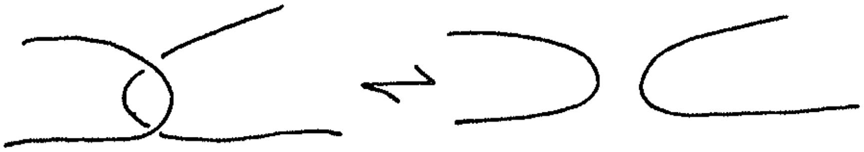


and that two knots or links (A knot is an embedding of a single closed curve into three space. A link is an embedding of a collection of curves into three space.) are isotopic in three space if and only if their diagrams are equivalent to one another under a finite sequence of transformations of the types I, II, and III as indicated below. (Isotopy corresponds directly to the physical picture of transforming one rope to another by pushing, pulling , stretching but no tearing.)

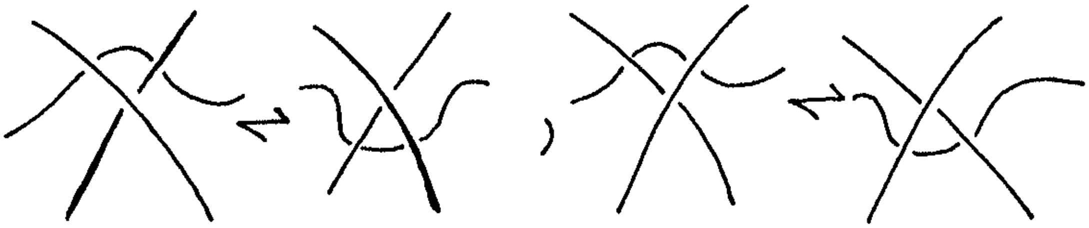
I.



II.

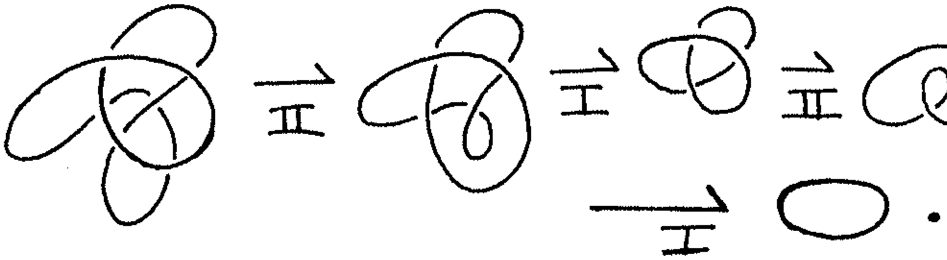


III.

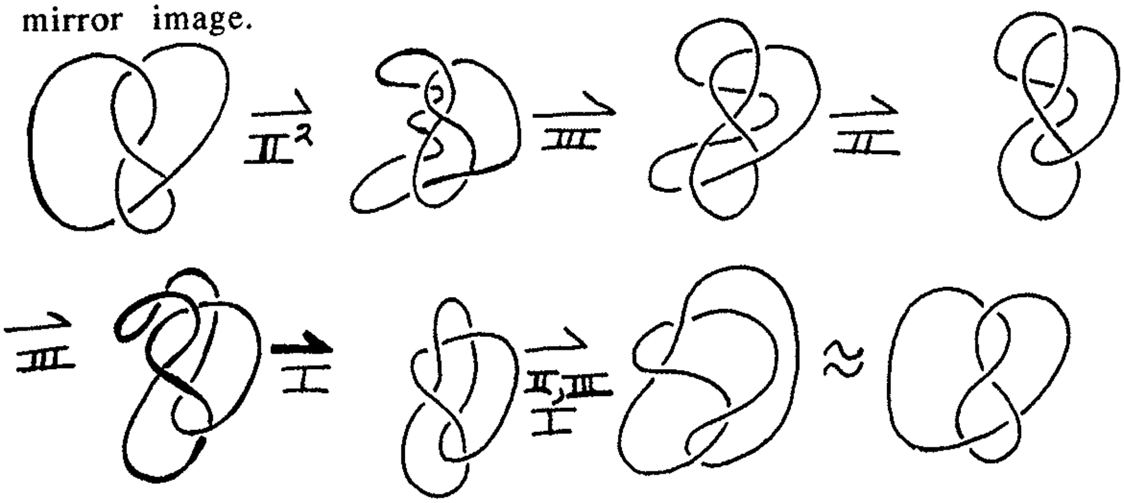


The same theorems apply to tangles, with the caveat that the free ends of the tangles remain fixed during the applications of the moves, and that strands are not allowed to pass over the ends of the tangles.

Here is a simple example of unknotting via the Reidemeister moves.



Here is a subtler example, turning the figure eight knot into its mirror image.



It is a very tricky matter to extract topological data about knots and links from their diagrams. We shall have more to say about this later.

The Triangle Move. The Reidemeister moves derive from properties of the projection of a curve from three-dimensional space to a plane or to the surface of a sphere. In fact Reidemeister had a *single* move for knots and links in three space. This single move, the *triangle move* , generates the three Reidemeister moves. The triangle move is defined for piecewise linear knots and links in three-space. A piecewise linear link is made up from finitely many straight line segments. Any link represented by a differentiable embedding, or any link that can be drawn by hand in a finite amount of time, can be approximated by a piecewise linear link. Given a pl (short for piecewise linear) link, a triangle move is performed by the following prescription:

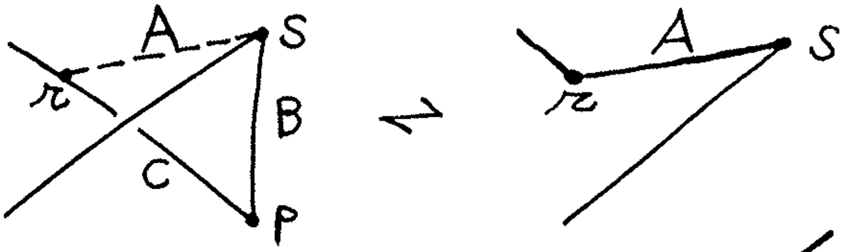
Perform one of the following two types of operations.

1. Mark a straight segment A on the link K. Let r and s denote the endpoints of A. This segment A can be a proper subsegment or an entire segment of K. Let p be a point in the complement of the link K such that the triangle with vertices r,s,p intersects K only along A. Let B denote the segment rp and C the segment sp. Cut the segment A from the link and replace it by the union of the segments B and C.
2. Let B and C be consecutive segments marked on the link K. (By consecutive I mean that they share a single endpoint.) Let A be the segment determined by the endpoints of B and C that are not shared between them. Let ABC denote the triangle (surface) determined by

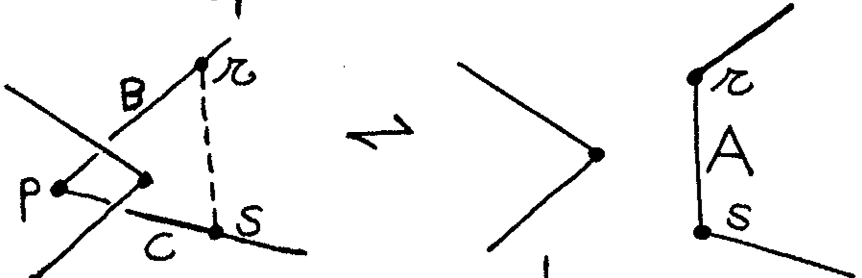
the segments A, B and C. Assume that ABC intersects K in exactly B and C. Then cut A and B from K and paste in C.

The diagrams below illustrate how projections of triangle moves generate the three Reidemeister moves. Two pl links in three dimensional space are ambient isotopic if and only if they can be related by a finite sequence of triangle moves. Careful consideration of the projections shows that sequences of Reidemeister moves on diagrams captures the content of an ambient isotopy.

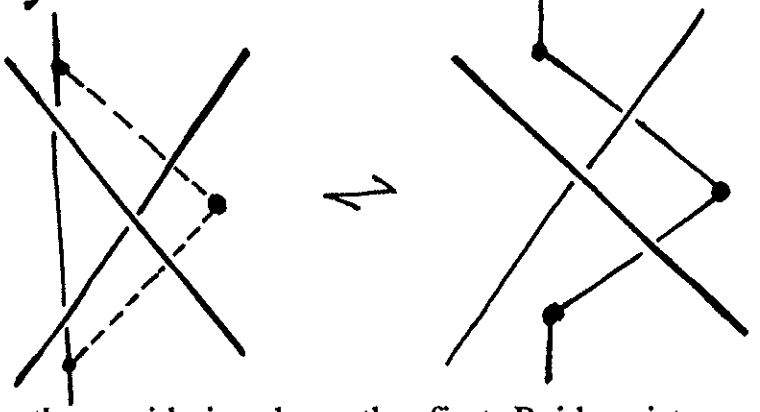
I.



II.

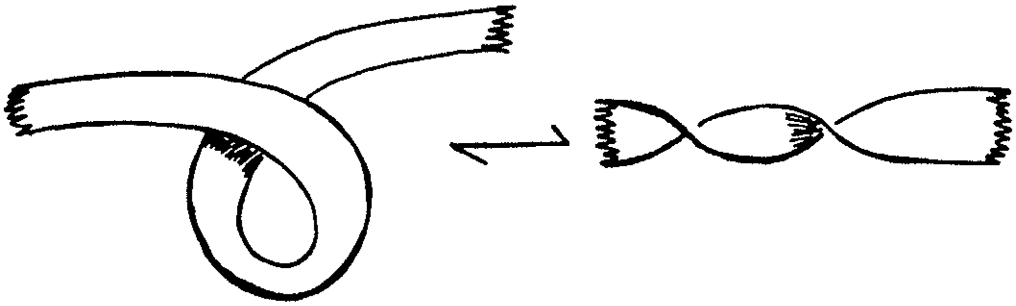


III.



It is worth considering how the first Reidemeister move is generated by a simple triangle move. This shows clearly the illusory nature of self-membership from the point of view of three dimensional space *if* we stick to pure topology.

On the other hand, if the loop is actually a physical loop in a rope, then the cancellation of the loop shown in the the first move must be paid for by a corresponding twist in the rope. This is most easily illustrated by replacing the line drawing by a drawing of a twisted band as shown below.



This band picture of the first Reidemeister move shows that we can regard it as an exchange rather than an elimination or creation of the loop.

The reason for dwelling on the first Reidemeister move in our context is that this move allows the creation or cancellation of self-membership in the corresponding knot set. If we take the point of view that the diagrams represent twisted bands (called framed knots and links), then the self-membership is not lost as we go to the topology. A corresponding equivalence relation on links is called *regular isotopy*. Regular isotopy is generated by the second and third Reidemeister moves. We shall return to this idea later in the discussion.

End of Digression.

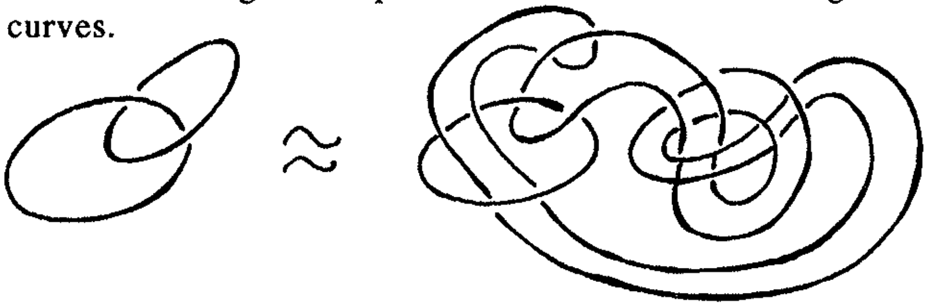
Note that by the cancellation of identicals, diagrams related by the second Reidemeister move represent the same knot set. The third move does not change any membership relations. Finally, invariance of a knot set under the first Reidemeister move would entail quotienting the theory by self membership. As we have remarked above, it is natural to consider only equivalence of knots and links up to regular isotopy - the equivalence relation generated by the second and third Reidemeister moves - or to regard the diagrams as representative of embedded bands in space. In the latter case, self membership is catalogued by the twists in a thickened arc, as well as loops in that arc.

If we maintain the distinction of self-membership by using only regular isotopy on the diagrams, then the Russell paradox becomes meaningful in the knot set domain, but there is still a strange twist about self-membership. By the convention of cancellation of identicals we have the equivalence, X = {X,X, C} where X={C} is the reduced form of the knot set X, and C denotes the contents of X. *Any knot set has a representative that is a member of itself. It is only of the reduced forms for the knot sets that we can speak of a set that is or is not a member of itself.*

The most radical interpretation is: Use diagrams with free ends (tangles) and allow the first Reidemeister move on knot sets. This means that any knot set has representatives that are members of themselves and it has representatives that are not members of themselves. The states of self-membership and non-self membership are equivalent. Up to representation, a (radical) knot set is a member of itself if and only if it is not a member of itself!

We have resolved the Russell paradox in this domain by having every set a member of itself and not a member of itself. The topological interpretation of knot sets shows that self-membership can be quotiented from the set theory (so that a given set has representatives that are members of themselves and representatives that are not members of themselves). The quotient theory is as consistent as the theory of knots and links in 3-space. Since this theory can be expressed in terms of ordinary set theory, this provides a relative consistency proof for radical knot sets.

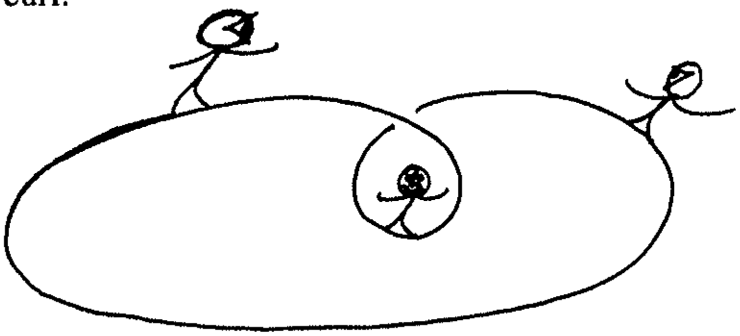
Mutuality such as a = {b} , b={a} is another matter. Here there is no reduction to anything simpler, and topologically, mutuality corresponds to nothing more paradoxical than the linking of two space curves.



In this version of knot sets one can make a diagram of a given knot set , and then use this diagram as a weaving pattern for a physical weave. Throw that weave into three space. Flatten the weave back onto a plane. The result is an equivalent knot set. The information in a knot set is encoded into the topology.

Knot Sets Avoid Infinite Regress

The knot set gives a way to conceptualize nonstandard sets without recourse to infinite regress. Infinity has been transposed into topology where inside and outside can equivocate through a twist in the boundary. In knot sets we obtain the multiple levels of ordinary set theory without the seemingly necessary hierarchy. This is nowhere more evident than in the self membering set represented by a curl.



Here an observer on the curl itself will go continuously from being container to being member as he walks along the ramp. Membership becomes topological relationship.

Remark. The reader may be familiar with other non-standard models for set theory such as those in the book by Peter Aczel [AC]. The constructions given here are very close in spirit to those of Aczel. There are two major differences. The first difference is in our choice to handle identicals via cancellation rather than condensation. The second is in the background use of reentering forms to indicate recursively defined constructions. We do not utilize the same demand for uniqueness of labelling as in [AC]. This is a technical matter and will be discussed elsewhere. The surprise in our construction is that the theory has a topological interpretation.

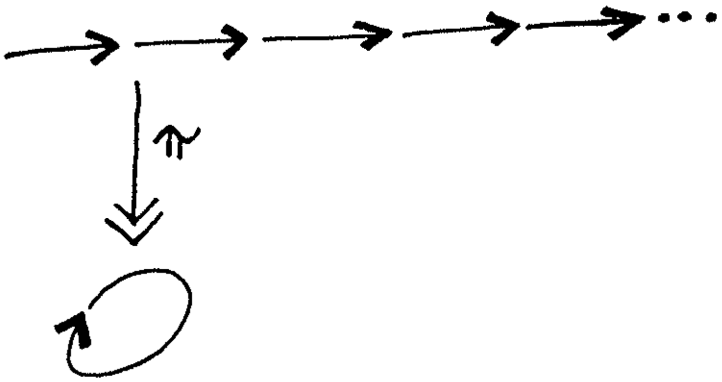
The version of knot sets discussed herein has a precursor in the work of the Swedish logician Stig Kanger in the early 1940's ([P], pp. 13-14.). Kanger represented sets as cords - with a cord tied around another cord representing a set with the other cord as a member. A cord tied around itself becomes a set that is a member of itself. Our knot sets, based on the diagrams for knots, turn out to have a deeper relationship to the topology of knots than the Kanger system. Kanger's idea is very significant, and it is interesting to compare it to the earlier systems of numeration (Quipu) that are based upon tying knots in a rope.

IV. Arrow Epistemology

An arrow points.

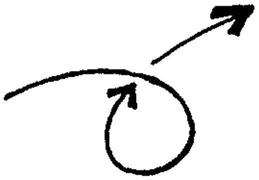
The arrow accomplishes its pointing via the distinction between inside and outside (convex side versus concave side) made by the arrow head. The body of the arrow extends the domain of the concave side into a flexible arm that can reach outward from the base of the arrow.

Once the body of the arrow becomes flexible, then elementary notational topology makes it possible for the arrow to point to itself. At this point, two forms of self pointing arise - pointing to the base (origin) of the body of the arrow and pointing to the interior of the body of the arrow. The former is the simplest form of self-reference, and leads via the unfolding shown below to a direct relationship with fixed points and recursion. The unfolding corresponds to describing all the trips that one can make around the circle formed by the self-pointing arrow. Thus a = ----> denotes one trip while **aa** = ---->----> denotes two trips and **A** = **aaa...** = ----->---->----> **....** denotes infinitely many trips. Note that **A** = **aA**. In this way the unfolding **A** of the self-pointing arrow is a a fixed point for the operation of "affixing an arrow on the left".



In the second alternative, the arrow points to its own body.

We have seen that this alternative can be extended to a notation for self-membership or reference of the body of the arrow to itself in the form in which an undercrossing points to (is a member of) the overcrossing line.

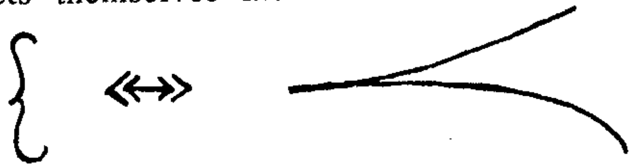


Self-pointing of an arrow or a line bifurcates into two interpretations depending upon whether the end of segment is seen as a pointer or whether an interior point of a segment is seen in relation to another interior point. In projection these two points of view come together through the convention of the cut segment at a crossing.

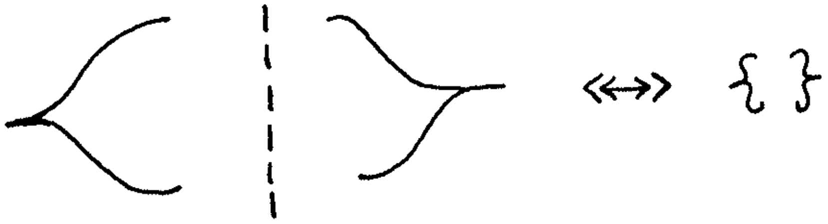
Any reference is a distinction. The notation adopted for a fundamental distinction has a remarkable influence on the way we think about it. In standard set theory a set is indicated by a pair of curly brackets: { } (This is the empty set.).

A Story

The brackets themselves indicate bifurcation from a point.



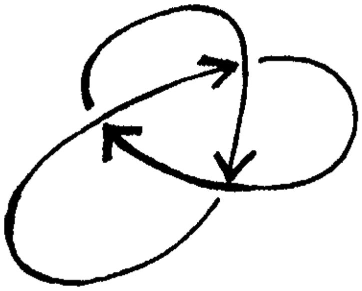
Each bracket instantiates the growth of a distinction from a state of unity (the point of the cusp).



A further operation beyond bifurcation is necessary for the formation of a set. The bifurcation that is the (left) bracket is copied and mirrored to form the (right) bracket. A left and right bracket taken together become a container. Once we have reached the level of being able to make a distinction, and to make a copy of that distinction intrinsically distinct (the mirror imaging) from the original, then we are prepared to form a new distinction (the container). The new distinction occurs at a different level from the original distinctions. This allows the hierarchy that is set theory.

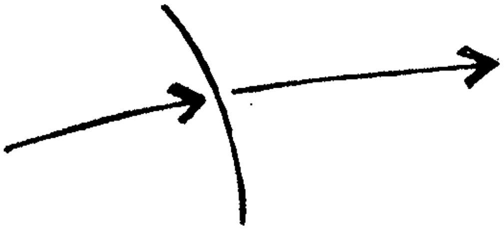
Knot Structure

The self-pointing arrow is not a knot. The circle diagram for an unknot does not point to itself at all, but is simply a closed circular form. Examine the trefoil.



The diagram consists in three arrows, each one pointing to the body of the next.

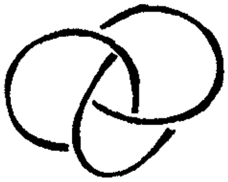
The extra convention that the base of one arrow is always correlated with the tip of another is special to the knot theory.



It allows the interpretation of the two arrows taken together as part of an undercrossing line, and hence the set theoretic and geometric interpretations that we have already discussed. If we contravene this convention, then we obtain diagrams such as the one below, where the base of an arrow simply begins from some point on an arc.

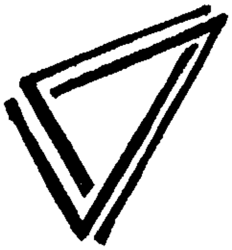
This gives us a set of planar diagrams that can be studied on their own terms. Self-pointing can take the form shown below.

The triplet structure of the trefoil is still present in diagrams such as this one.



One reason for considering such a wider class of diagrams it that it enables us to draw connections with the kind of diagrammatics that occurs in artistic, linguistic, physical and philosophical contexts. For example, the irreducible tripartite relation of sign, signifier and signified occurs in the work of Charles Sanders Pierce [PI] and is ubiquitous in semiotics and linguistics.

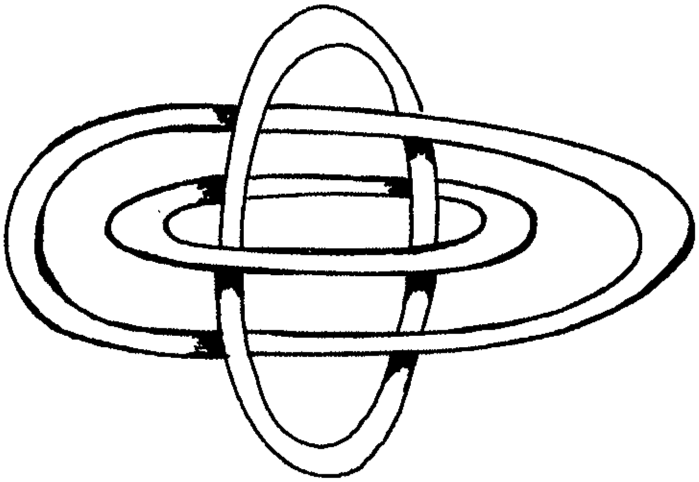
A symbol almost identical to the trefoil structure



occurs in the stylized form shown above in the the work of Annetta Pedretti [P] on language. Here the three parts represent the distinguishing subject, the that which is distinguished and the background binding the distinguished and the one who distinguishes.

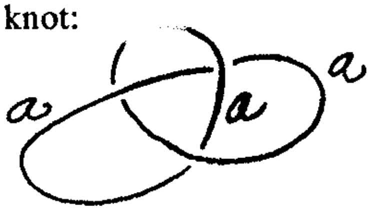
A similar tripartite structure arises as soon as one includes the boundary in any distinction. Two sides and the boundary joining them form a tripartite structure where each part is determined by the other two parts. No boundary exists without the two sides. No side exists without the potential to cross the boundary from the other side. Frederick Joseph Staley [STA] calls such triplets triadic dualisms.

A triadic dualism need not have the appearance of either a trefoil or a distinction. The most striking topological example of a triadic relation is the link shown below. This link, the Borommean rings, is topologically linked, consisting of three unknotted circles. The rings fall apart upon the removal of any one of the triplet.



V. Lambda Calculus and Topology

It is natural to enquire whether the knot sets shed light on the topological structure of knots and links themselves.
Consider a trefoil knot:

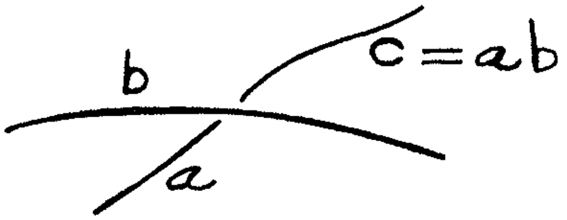


The set is just the self-membering a={a,a,a}, and hence equivalent to the empty set in the radical theory and to one twist (a={a}) in the regular theory (regular theory uses regular isotopy). Many topologically distinct diagrams correspond to a given knot set.

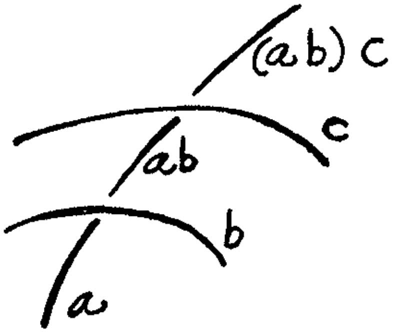
It is tempting to consider the possibility that the knots and links can be viewed in terms of a subtle kind of logic. This is in fact the case.

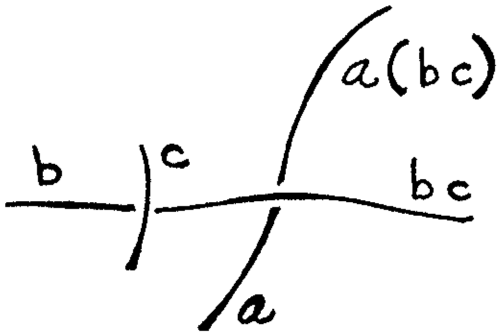
Non-Associative Formalism in Knot Diagrams

Label the arcs in a link diagram. Regard the label on the arc c obtained by underpassing b from a as a product of a and b : c= ab.



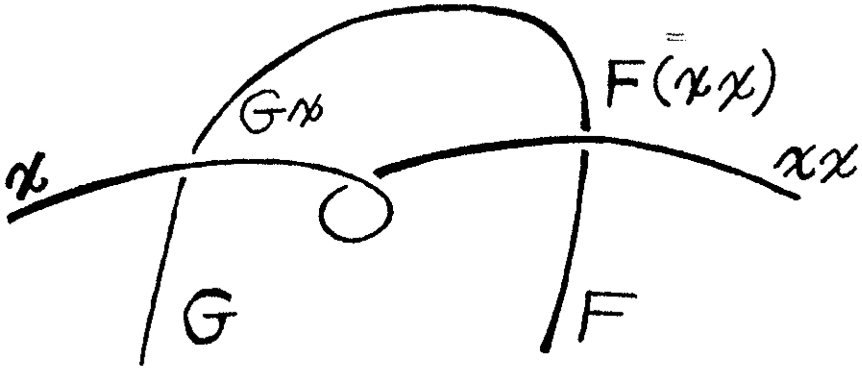
Here we abandon the notion of membership at a crossing and replace it with an algebraic product. Think of the overcrossing line as acting on the undercrossing line to produce the label for the continuation of the undercrossing. This is an inherently non- associative formalism, as the diagrams below demonstrate.



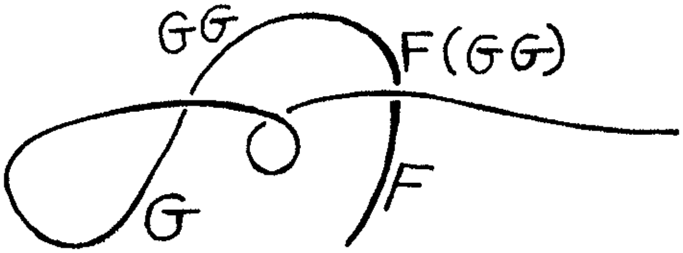


In this mode we can diagram the constructions of the lambda calculus of Church and Curry [B]. No direct knowledge of the lambda calculus is needed for the discussion to follow. However, the last part of this section is a discussion of the lambda calculus in relation to knots.

Consider Gx = F(xx). If we substitute G for x, we obtain GG = F(GG). At this level of formalism, every F has a fixed point GG where Gx = F(xx). Diagramming the nonassociative algebra inherent in this discussion we have:

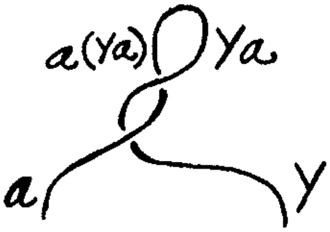


Taking G=x in the above diagram, by tying together the lines, we obtain GG = F(GG):

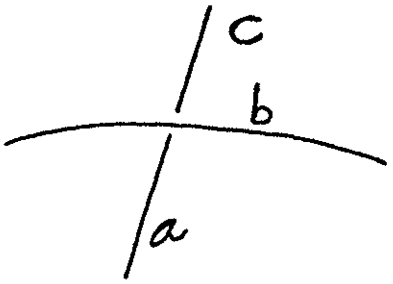


In this way, we obtain a knot diagrammatic interpretation of the basic fixed point construction of the lambda calculus. The analogy with our previous construction of self membering knot sets is striking, but these lambda calculus constructions use much more of the structure of the knot and link diagrams.

Here is a knot diagrammatic interpretation of the equation Ya = a(Ya). It is a double leveled twist.

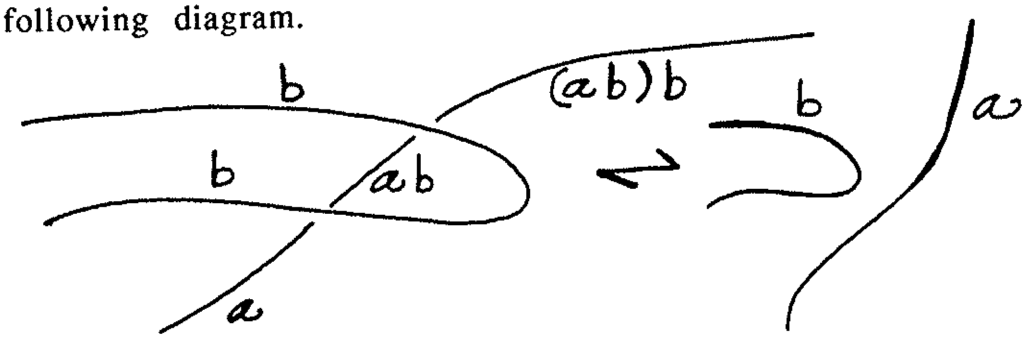


At this point we must take a more careful look at our conventions for handling diagrammatic non-associative products. If we take the convention for multiplication literally, then it can be read in two ways at a given crossing as shown below.

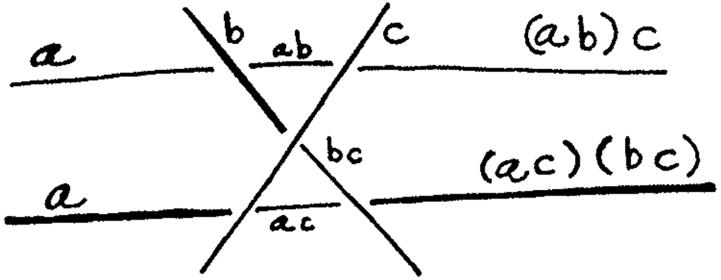


$$\left\{\begin{array}{l} c = ab \\ a = cb \end{array}\right\} \Rightarrow c = (cb)b.$$

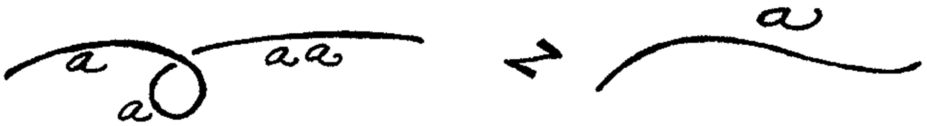
Thus c=ab and a=cb. Hence c = (cb)b. For consistency, we demand that c=(cb)b for all elements b and c. Look at the diagrammatic consequences of taking the axiom c = (cb)b. We have the following diagram.



Under the axiom c = (cb)b, the algebra cannot see the second Reidemeister move. The demand for invariance under the third Reidemeister move leads to yet another axiom: (ab)c = (ac)(bc). This states that the algebra is right-distributive over itself.



Finally for the type I move we need aa=a for all a.



Thus we need an algebraic system with one binary operation and satisfying the axioms:

1. aa = a

2. c = (cb)b

3. (ab)c = (ac)(bc)

An algebra satisfying these axioms is called an *involuntary quandle* [J]. If we eliminate the first axiom it is called a *light crystal* [K2], [K6].