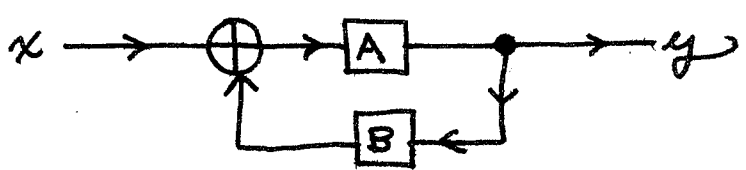


Linear Equations and Feedback

by Louis H. Kauffman

I. Introduction

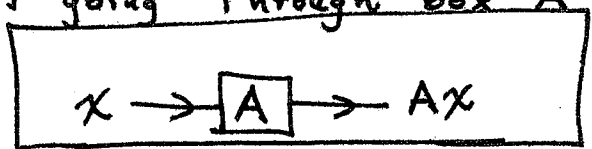
One may not ordinarily think of a simple system of linear equations as a model for a feed-back process. Yet this can be the case! Consider the following diagram:



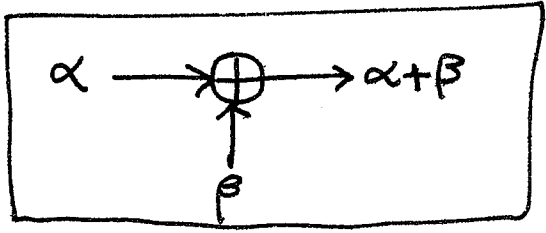
This diagram indicates the signal or information flow in a system whose input is x , whose output is y .

The signal x is transformed by the black box A and fed-back through the box B to be recombined \oplus with the original input.

Suppose we represent the result of going through box A by Ax :



and combination \oplus by $+$:



Then we can understand the behaviour of the circuit: First x goes through and is fed-back. (We assume that the input x is held fixed in time.) Then the "new input" is $x + BAx$. This results in further new input of $x + BAx + BA(x + BAx)$, and so on...

The system is said to be a linear system if the transfer functions A and B are linear so that $A(x + \beta) = Ax + A\beta$ etc. Then we get the sequence of inputs and outputs:

input	output
x	Ax
$x + BAx$	$Ax + ABax$
$x + BAx + BABax$	$Ax + ABax + ABABax$
$x + BAx + BABax + BABABax$	$Ax + ABax + ABABax + ABABABax$
...	...

If we let the system run for a very long time then we expect it to settle into a stable state. At the stable state, the input and output are no longer perturbed by the feedback. This means that

$$A(x + By) = y$$

since the feed-back is By and the new output is $A(x + \langle \text{feedback} \rangle)$.

Thus $A(x + By) = y$

$$Ax + ABY = y$$

$$Ax = y(1 - AB)$$

$$\therefore y = \left(\frac{A}{1 - AB} \right) x$$

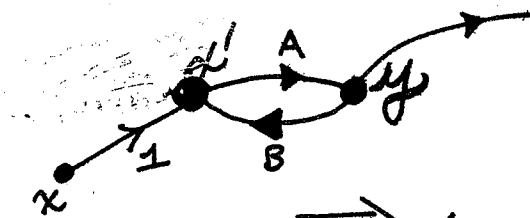
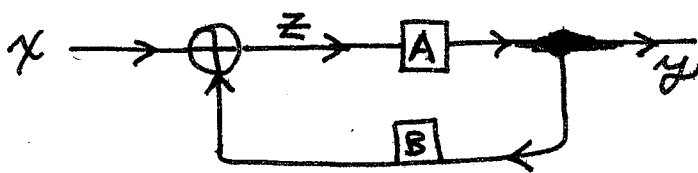
and formally,

$$y = A(1 + AB + ABAB + ABABAB + \dots)x$$

$$y = Ax + ABx + ABABx + \dots$$

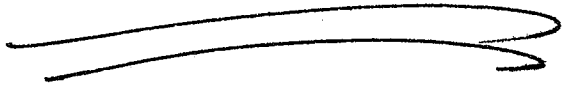
This result coincides with our recursive analysis of the situation.

And we obtained it by solving a simple linear equation. Lets look at this again:

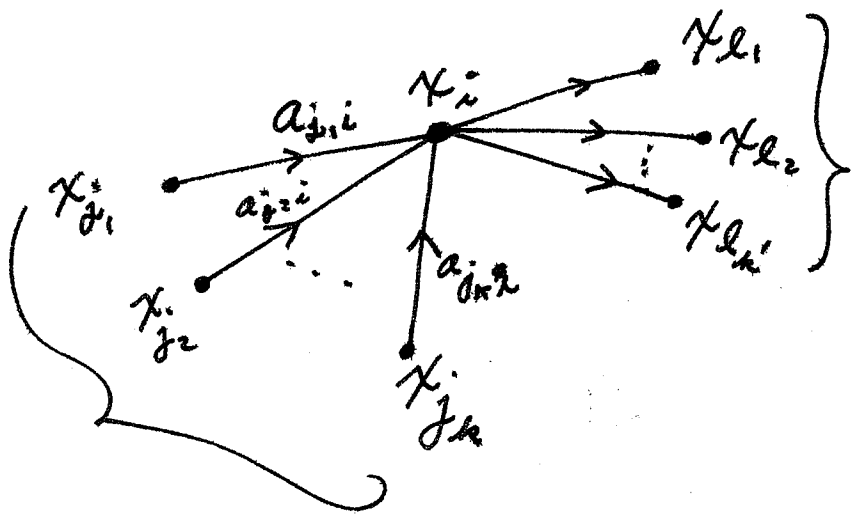


$$\begin{aligned} x' &= x + By \\ y &= Ax' \end{aligned}$$

$$\Rightarrow y = A(x + By)$$



Here we abbreviate the linear system with a graph with nodes (•) that represent the summing (\oplus) of the earlier notation. Each node is labeled by a letter X_i . Each arc (edge) is labelled by a letter A, B, C, ... representing a transfer function. The nodal equation expresses X_i as a sum of transfer functions applied to X_j 's that are directed into the node labelled X_i :

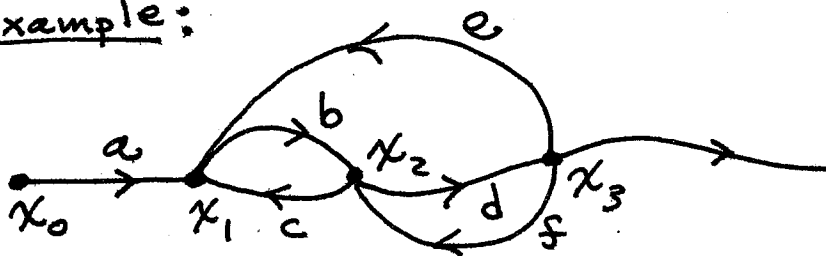


Nodal Equation

$$X_i = a_{j_1 i} X_{j_1} + \dots + a_{j_k i} X_{j_k}$$

Solving this system of equations will give the stable state of the system.

Example:



Here the input is x_0 and the output is x_3 . There are feed-backs from x_2 and x_3 back to x_1 . The nodal equations are:

$$x_1 = a x_0 + c x_2 + e x_3$$

$$x_2 = b x_1 + f x_3$$

$$x_3 = d x_2$$

Since x_0 is regarded as input, we re-write in the form

$$-a x_0 = -x_1 + c x_2 + e x_3$$

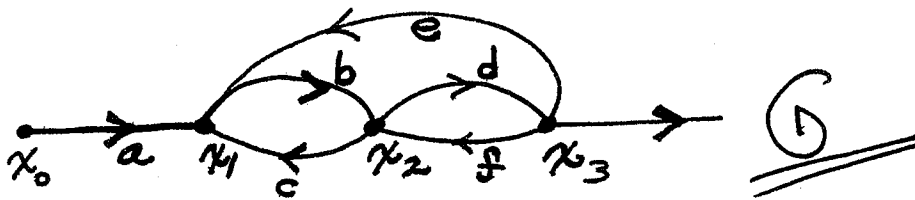
$$0 = b x_1 - x_2 + f x_3$$

$$0 = \quad \quad d x_2 - x_3$$

$$\begin{bmatrix} -a x_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & c & e \\ b & -1 & f \\ 0 & d & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The linear algebra problem is, of course, to solve for x_1, x_2, x_3 in terms of x_0 . (Particularly x_3 , since we regard it as output)

Note the form of the matrix for this system.



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$$\begin{bmatrix} +ax_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} +1 & -c & -e \\ -b & +1 & -f \\ 0 & -d & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$M = \begin{bmatrix} +1 & -c & -e \\ -b & +1 & -f \\ 0 & -d & +1 \end{bmatrix}$$



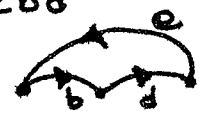
Each diagonal entry of M equals $+1$.

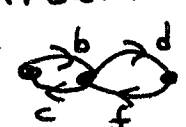
The columns correspond to the nodes that are not inputs.

Thus $M_{ii} = +1$
 $M_{ij} = \left(\begin{array}{l} \text{label on a line} \\ \text{directed from } x_j \text{ to } x_i \\ \text{(for } i \neq j) \end{array} \right)$

Now to solve the system, we can use Cramer's rule. This requires computing the determinant $|M|$:

$$\begin{aligned} |M| &= + \begin{vmatrix} +1 & -f \\ -d & +1 \end{vmatrix} + (e) \begin{vmatrix} -b & -f \\ 0 & +1 \end{vmatrix} - e \begin{vmatrix} -b & +1 \\ 0 & -d \end{vmatrix} \\ &= +(1-df) + (e)(-b) - e(bd) \\ &= +1 - df - bc - ebd \end{aligned}$$

Thus directed circuits in the graph (i.e. feed-back loops) appear in the determinant expansion. Note that each circuit has no repeated vertices (thus  does not appear). There is a general result in back of this!

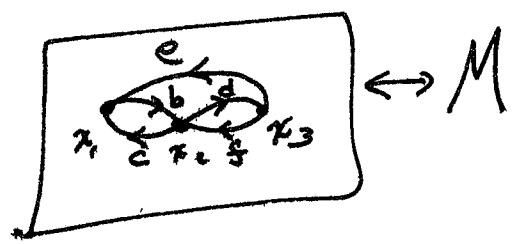
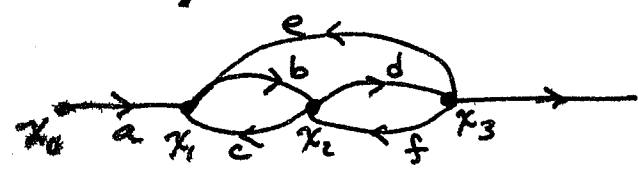
Before discussing the general result lets complete the computation:

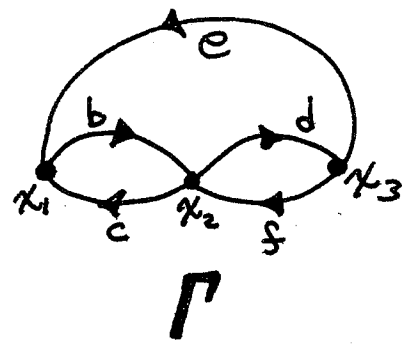
$$X_3 = \frac{\begin{vmatrix} 1 & -c & aX_0 \\ -b & 1 & 0 \\ 0 & -d & 0 \end{vmatrix}}{|M|} \quad (\text{By Cramer's Rule.})$$

$$\therefore X_3 = \frac{aX_0 \begin{bmatrix} b & d \end{bmatrix}}{1 - df - bc - ebd}$$

$$\boxed{\frac{X_3}{X_0} = \frac{abd}{1 - df - bc - ebd}}$$

abd is a directed path from X_0 to X_3 .





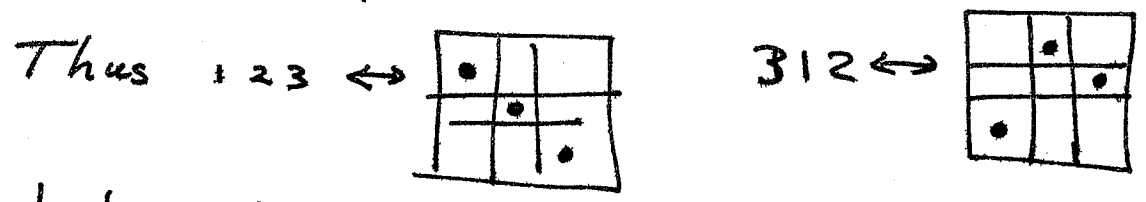
	x_1	x_2	x_3
x_1	1	-c	-e
x_2	-b	1	-f
x_3	0	-d	1

$M(\Gamma)$

This is how we associate a matrix $M(\Gamma)$ to a directed graph Γ .

Now consider what is involved in computing $|M(\Gamma)|$.

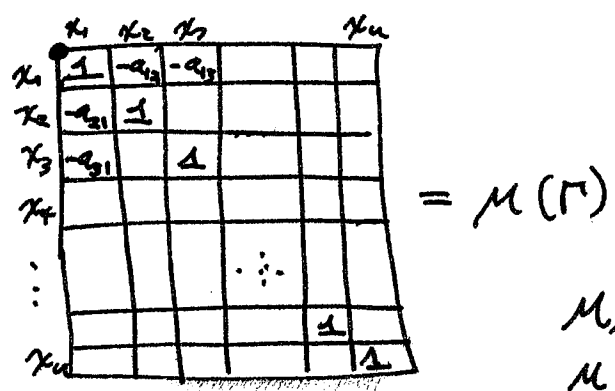
$|M(\Gamma)| =$ a sum of products of elements of $M(\Gamma)$ corresponding to a permutation of 1, 2, 3 times the sign of the permutation.



Let $\epsilon(\sigma)$ denote the sign of a permutation. Then if M is an $n \times n$ matrix, then

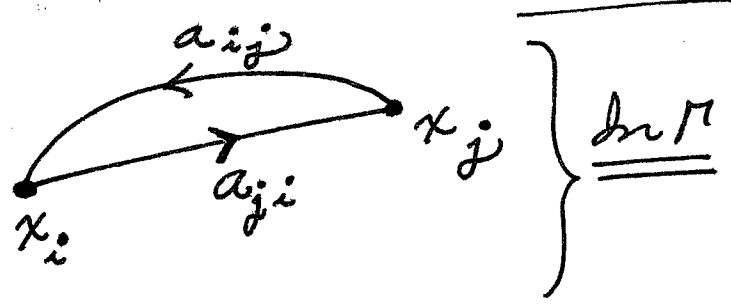
$$|M| = \sum_{\sigma} \epsilon(\sigma) M_{1\sigma_1} M_{2\sigma_2} \dots M_{n\sigma_n}$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of $(1, 2, \dots, n)$.



$$M_{ii} = 1$$

$$M_{ij} = -a_{ij} \quad (i \neq j)$$



$$|M(G)| = \sum_{\sigma} \epsilon(\sigma) M_{1\sigma_1} M_{2\sigma_2} \dots M_{n\sigma_n}$$

A permutation is a product of cycles. These correspond to the loops in the graph!

example : $n=5, \sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$
 $= (3, 2, 1, 4, 5)$

- 1 → 3 → 1
- 2 → 2
- 4 → 4
- 5 → 5

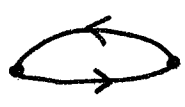
$$\epsilon(\sigma) M_{1\sigma_1} M_{2\sigma_2} M_{3\sigma_3} M_{4\sigma_4} M_{5\sigma_5}$$

$$\parallel$$

$$\epsilon(\sigma) M_{13} M_{31} M_{22} M_{44} M_{55}$$

$$\parallel$$

$$\epsilon(\sigma) (-a_{31})(-a_{13}) (1)(1)(1)$$



$$\parallel$$

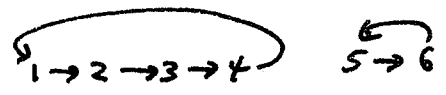
$$\epsilon(\sigma) a_{31} a_{13}$$

Here $\epsilon(\sigma) = -1$

example: n = 6

$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$$

$$= (2, 3, 4, \underline{1}, 6, 5)$$



Note that the sign of a cycle of length k is $(-1)^{k+1}$, and in the expansion, we will get k (-1) 's from the individual terms. Thus each cycle contributes (-1) to the eventual sign of a term.

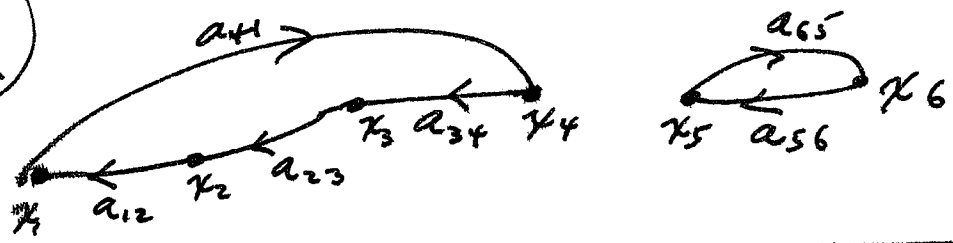
Here $\epsilon(\sigma) = (-1)^{4+1} (-1)^2 = (-1)(-1) = +1$

$$\epsilon(\sigma) M_{1\sigma_1} \dots M_{6\sigma_6}$$

$$(+1) (-a_{12}) (-a_{23}) (-a_{34}) (-a_{41}) (-a_{56}) (-a_{65})$$

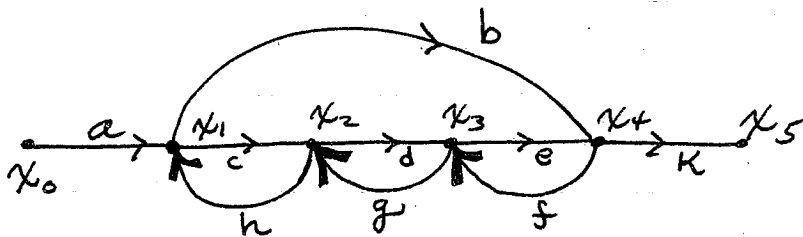
$$= (a_{12} a_{23} a_{34} a_{41}) (a_{56} a_{65})$$

These products are called cycle gains



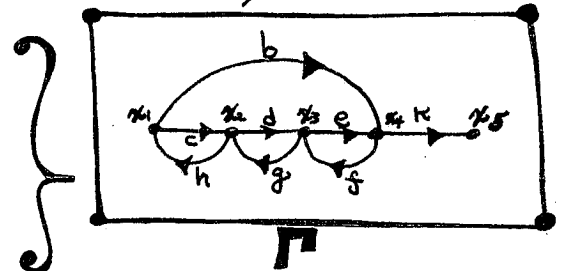
Conclusion: $|M(\Gamma)| = 1 - L_1 + L_2 - L_3 \pm \dots + (-1)^n L_n$
 where $n = \#$ of vertex-disjoint circuits in Γ .
 $L_1 =$ products of labels on single circuits.
 $L_2 =$ products of labels on vertex disjoint pairs.
 \vdots
 $L_n =$ product of labels on vertex disjoint n -tuples.

Now let's continue the analysis a bit further. Take the example:



So x_0 is input, and x_5 is output.

$$\begin{cases} x_1 = a x_0 + h x_2 \\ x_2 = c x_1 + g x_3 \\ x_3 = d x_2 + f x_4 \\ x_4 = b x_1 + e x_3 \\ x_5 = k x_4 \end{cases}$$



$$\begin{bmatrix} a x_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -h & 0 & 0 & 0 \\ -c & 1 & -g & 0 & 0 \\ 0 & -d & 1 & -f & 0 \\ -b & 0 & -e & 1 & 0 \\ 0 & 0 & 0 & -k & 1 \end{bmatrix}}_{M(\Gamma)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Now by Cramer's rule,

$$x_5 = \frac{\begin{vmatrix} 1 & -h & 0 & 0 & a x_0 \\ -c & 1 & -g & 0 & 0 \\ 0 & -d & 1 & -f & 0 \\ -b & 0 & -e & 1 & 0 \\ 0 & 0 & 0 & -k & 0 \end{vmatrix}}{\underline{\underline{|M(\Gamma)|}}}$$

The determinant of the numerator

$$\text{is } a x_0 \begin{vmatrix} -c & 1 & -g & 0 \\ 0 & -d & 1 & -f \\ -b & 0 & -e & 1 \\ 0 & 0 & 0 & -k \end{vmatrix} = (a x_0) (k) \begin{vmatrix} -c & 1 & -g \\ 0 & -d & 1 \\ -b & 0 & -e \end{vmatrix}$$

$$= (a x_0) (k) \left(-c \begin{vmatrix} -d & 1 \\ 0 & -e \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -b & -e \end{vmatrix} - g \begin{vmatrix} 0 & -d \\ -b & 0 \end{vmatrix} \right)$$

$$= (a x_0) (k) (-cde - b + gbd)$$

$$= x_0 (+acdek + abk - agbdk)$$

$$= x_0 (+acdek + abk [1 - gbd])$$

If you now look back at the graph you will see that there is a pattern to this.

- $\left\{ \begin{array}{l} P_1: acdek \leftrightarrow \text{path from } x_0 \text{ to } x_5. \\ P_2: abk \leftrightarrow \text{second path from } x_0 \text{ to } x_5. \end{array} \right.$

There are no loops in the graph that do not touch P_1 .

There is the loop gd that does not touch P_2 .

This is a general pattern!

The numerator determinant for the "output" will be a sum of the form $\sum_k T_k \Delta_k$ where T_k is

the product of edge labels (hence the transfer function for) for a directed path from input node to output node. And Δ_K is the signed loop-sum for the complement of this path.

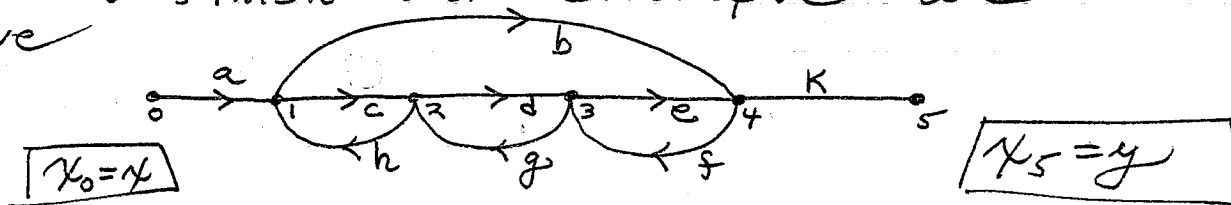
Thus we can say in general that if we have source node x and output node y for graph G then

$$y/x = \frac{\sum_K T_K \Delta_K}{\Delta}$$

where Δ is the loop-sum for G (-input) and $\sum_K T_K \Delta_K$ is as defined above.

This is called MASON'S FORMULA.

To finish our example we have



$$y/x = \frac{acdeK + abK(1 - dg)}{(1 - (ch) - (dg) - (ef) + (ch)(ef) - (bfg h))}$$

On the next page we have explicitly calculated the loop-sum $\Delta = |M(M)|$ for comparison with the denominator in this formula for Mason's rule.

$$\Delta = |M(p)| = \begin{vmatrix} 1 & -h & 0 & 0 & 0 \\ -c & 1 & -g & 0 & 0 \\ 0 & -d & 1 & -f & 0 \\ -b & 0 & -e & 1 & 0 \\ 0 & 0 & 0 & -k & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -g & 0 & 0 \\ -d & 1 & -f & 0 \\ 0 & -e & 1 & 0 \\ 0 & 0 & -k & 1 \end{vmatrix} + h \begin{vmatrix} -c & -g & 0 & 0 \\ 0 & 1 & -f & 0 \\ -b & -e & 1 & 0 \\ 0 & 0 & -k & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -f & 0 \\ -e & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} + g \begin{vmatrix} -d & -f & 0 \\ 0 & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} + h \left[\begin{vmatrix} -c & 1 & -f & 0 \\ -e & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} + g \begin{vmatrix} 0 & -f & 0 \\ -b & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} \right]$$

$$= 1 + f \begin{vmatrix} -e & 0 \\ 0 & 1 \end{vmatrix} - g d \begin{vmatrix} 1 & 0 \\ -k & 1 \end{vmatrix}$$

$$+ h \left[-c \begin{vmatrix} 1 + f & -e & 0 \\ 0 & 1 \end{vmatrix} + g f \begin{vmatrix} -b & 0 \\ 0 & 1 \end{vmatrix} \right]$$

$$= 1 - ef - gd - ch[(1 - ef)] + ghf(-b)$$

$$= 1 - ef - gd - ch - gbhf + chf$$

$$\therefore \Delta = 1 - (ef) - (gd) - (ch) - (bghf) + (ch)(ef)$$

This checks with the application of the signed loop-sums.

Thus we have seen that by "graphing" a system of linear equations it can be seen as a problem of input/output with feedback. The graph provides a simple pattern for the combination of the solution.

MASON'S FORMULA $y/x = \sum T_k \Delta_k / \Delta$

summarizes this version of Cramer's Rule.

ZK Jan 1, 1986