

Math 215 -Fall 2010 - Assignment #7  
Due Wednesday, November 24, 2010.

1. Read Chapters 15,16,17 (we have done all this in class).  
Read Chapters 19,20,21 (this will be discussed in the week of  
November 15 - 19).

2. page 225. Problems 2,3,5,6

3. For the primes  $p = 2,3,5,7,11,13$   
make a chart for each prime showing the possible remainders on  
division by  $p$  of the square of a natural number. For example, for  
 $p=5$ , we have

$$0^2 = 0, \quad \text{remainder} = 0$$

$$1^2 = 1, \quad \text{remainder} = 1$$

$$2^2 = 4, \quad \text{remainder} = 4$$

$$3^2 = 9 = 5 + 4, \quad \text{remainder} = 4$$

$$4^2 = 16 = 15 + 1, \quad \text{remainder} = 1$$

$$5^2 = 25 = 25 + 0, \quad \text{remainder} = 0$$

The pattern repeats after this, and so we have only the remainders  
0,1,4. The numbers 2 and 3 do not appear as remainders on division  
of a square by 5. The problem asks you to determine this  
information for each prime in the set above (for division of squares  
by that prime number.)

4. This problem uses continued fractions which we define as follows.

$$[a] = a$$

$$[a,b] = a + 1/b$$

$$[a,b,c] = a + 1/(b + 1/c)$$

and recursively by

$$[a_1, a_2, a_3, \dots, a_n] = a_1 + 1/[a_2, a_3, \dots, a_n].$$

(a) Prove by induction that

$$[a_1, a_2, a_3, \dots, a_n, x] = (Rx + S)/(Px + Q)$$

where  $R,S,P$  and  $Q$  are functions only of  $a_1, a_2, \dots, a_n$  (and not  
involving  $x$ ).

(b) Show that  $x = [a_1, a_2, a_3, \dots, a_n, x]$  is always a quadratic  
equation. Solve explicitly  $x = [1,2,3,4,x]$ .

(c) Examine evidence (i.e. try some examples) for the

**Theorem.** If  $a_1, a_2, \dots, a_n$  are positive integers and  $[a_1, a_2, a_3, \dots, a_n] = P/Q$  where  $P$  and  $Q$  are positive integers with no common factor, then  $[a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1] = P/S$  where  $P$  and  $S$  are relatively prime positive integers and  $QS \equiv (-1)^{n+1} \pmod{P}$ .

For example  $[1,3] = 1 + 1/3 = 4/3$

$[3,1] = 3 + 1/1 = 4/1$

and  $1 \times 3 = 3 \equiv -1 \pmod{4}$ .

In your examples try some longer continued fractions like  $[1,2,3,4,5]$ .