1. Give a proof by mathematical induction of the following statement:

\[ 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \]

for all \( n = 1, 2, 3, \cdots \).

2. Suppose that there are \( n \) straight lines in the plane, positioned so that each line intersects each of the other lines once. Prove that the total number of intersection points among these \( n \) lines is equal to \( n(n - 1)/2 \) for \( n = 1, 2, 3, \cdots \). (Hint: You can proceed by induction on \( n \) and ask: If there are already \( n \) lines in the plane, how many new intersection points will occur when a new line is added to the set of \( n \) lines?)

3. Find integers \( r \) and \( s \) such that \( 30r + 43s = 1 \).

4. Recall that a natural number \( p \) is said to be prime if it has no divisors other than 1 and itself. By convention, the number 1 is not taken to be a prime, so the prime numbers begin with 2, 3, 5, 7, 11, 13, \cdots. Prove that there are infinitely many distinct prime numbers.

5. Prove that there exist irrational numbers \( a \) and \( b \) such that \( a^b \) is rational.

6. Prove that the following two statements are equivalent:

\[(A \Rightarrow B) \Rightarrow C\]

and

\[(A \lor C) \land (B \Rightarrow C).\]

In your proof, do not use truth tables. Use the facts that
\[A \Rightarrow B = (\sim A) \lor B\] and \[\sim (A \land B) = (\sim A) \lor (\sim B),\] and give a completely algebraic proof.

7. (a) Give the definitions of the terms injective and surjective for a function \( f : X \rightarrow Y \) from a set \( X \) to a set \( Y \).
(b) We define the composition of the function \( f : X \rightarrow Y \) and the function \( g : Y \rightarrow Z \) to be the function \( g \circ f : X \rightarrow Z \) with \( g \circ f(x) = g(f(x)) \) for all \( x \in X \). A map \( f : X \rightarrow Y \) between two sets is said to be bijective if it is both injective and surjective. Prove that if \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) are both bijective, then \( g \circ f : X \rightarrow Z \) is also bijective.

8. Let there be given an infinite list of sequences of 0’s and 1’s

\[ s^1, s^2, s^3, \ldots \]

That is, for each natural number \( n \) we have

\[ s^n = (s^n_1, s^n_2, s^n_3, \ldots) \]

where each entry \( s^n_k \) is equal either to 0 or to 1. Construct a sequence \( s, \)

\[ s = (s_1, s_2, s_3, \ldots) \]

of 0’s and 1’s such that \( s \neq s^n \) for any \( n = 1, 2, 3, \ldots \).

9. Let \( X \) be any set. Let \( P(X) \) denote the set of subsets of \( X \). Let

\[ F : X \rightarrow P(X) \]

be any well-defined mapping from \( X \) to its power set \( P(X) \). Show that \( F \) is not surjective.

10. Recall that we say that two integers \( n \) and \( m \) are congruent modulo \( p \)

\[ n \equiv m \ (\text{mod } p) \]

exactly when

\[ n - m = kp \]

for some integer \( k \).
(a) Prove that if \( a \equiv b \ (\text{mod } p) \) and \( b \equiv c \ (\text{mod } p) \), then \( a \equiv c \ (\text{mod } p) \).
(b) Prove that for any integer \( x \), \( (x - p)^2 \equiv x^2 \ (\text{mod } p) \).