Math 215 - Supplement on Finite and Infinite Sets
by LK

We take as given (for this discussion) the natural numbers

$$N = \{1,2,3,4,5,...\}$$

and we let $N_k = \{1,2,3,...,k\}$ be the set of the numbers from 1 to k. It is assumed that the natural numbers are pairwise distinct from one another so that the sets N_k are all distinct from one another for different values of k. We let N_0 denote the empty set. We say that a set X has (finite) *cardinality k* is there is a bijection between X and N_k for some k = 0,1,2,... .

We write |X| for the cardinality of a set X.

We say that a set X is *finite* if X has cardinality k for some k = 0,1,2,.... Thus X is finite if X is in bijective correspondence with N_k for some k.

Theorem 1. If X is a proper subset of N_n for some n > 0, then X has cardinality k for some k < n with k greater than or equal to 0.
Proof. We prove this theorem by induction on n.
For the base of the induction: If n = 1, then the only proper subset of $N_1 = \{1\}$ is the empty set. Since the empty set has cardinality 0 < 1, the theorem is true for n=1.
Now assume, as induction hypothesis, that the theorem is true for all n less than or equal to m. We wish to prove the theorem for n = m +1. Let X be a subset of N_{m+1}. We consider two cases.
Case 1: (m+1) is not a member of X. In this case X is in fact a subset of Nm, and by the induction hypothesis, X has cardinality k less than or equal to m (less if it is a proper subset and equal if X is equal to Nm). But since k is less than or equal to m, we have that k < m+1, and this proves the induction step for Case 1.
Case 2: (m+1) is a member of X. Now let X' denote the set obtained from X by removing the element (m+1). We have that the cardinality of X is 1 more than the cardinality of X' and X' is a *proper* subset of Nm. So by induction ,the cardinality of X' is less than m. Therefore the cardinality of X (being one greater) is less than m +1. This completes the induction step for Case 2.
This completes the proof of the Theorem. //

Theorem 2. If f: N_k -----> N_l is an injection, then k is less than or equal to l. If f is an injection, but not a surjection then k < l.

Proof. N_k is in bijection with a subset $f(N_k)$ of N_l. By Theorem 1, we know that the cardinality of $f(N_k)$ is less than or equal to l. Therefore the cardinality of N_k is less than or equal to l and less exactly when $f(N_k)$ is not equal to N_l.//

Theorem 3. No finite set X can be in bijective correspondence with a proper subset of itself.
Proof. This is a restatement of Theorem 1. If X is a proper subset of Y and Y is finite. Then there exists a bijection of F : Y -----> Nn for some n. This bijection takes X bijectively to a proper subset F(X) of Nn. By Theorem 1, F(X) has cardinality $k < n$. Hence X has cardinality $k < n$, while Y has cardinality n. Therefore X and Y have different cardinalities, and we have proved that no finite set can be put into bijective correspondence with a proper subset of itself. //

Theorem 4. The natural numbers N is an infinite set.
Proof. The mapping F : N ----> N defined by F(n) = 2n is a bijection between N and E where E is the set of even natural numbers. Thus N is in bijective correspondence with the proper subset E. By Theorem 3, this proves that N is not finite. //

Theorem 5. If X is an infinite set, then there exists an injection f:N -----> X.
Proof. We prove this result by constructing the mapping f by induction. To see, this first note that there is no problem mapping N_1 ---> X since certainly X is not empty. This is the beginning of the induction. Now suppose we have constructed an injective mapping $f_k : N_k$ -----> X. Then I claim that we can now construct an extension of f_k to $f_{k+1} : N_{k+1}$ -----> X that is also an injection. By extension I mean that $f_{k+1}(i) = f_k(i)$ for i=1,2,...,k. It is only $f_{k+1}(k+1)$ that is a new value. It is possible to choose a new value for $f_{k+1}(k+1)$, distinct from $f_k(1),\ldots,f_k(k)$. Otherwise f_k would be a surjective mapping from N_k to X, contradicting the assumption that X is infinite. We have proved that it is possible (by making a choice at each stage) to make an infinite collection of injective maps $f_k:N_k$ -----> X so that each f_l with $l > k$ is an extension of f_k. We now define f:N -----> X by $f(i) = f_i(i)$ for each natural number i. It follows by our construction that f is the desired injection from N to X. //

Remark. Note that the injection constructed in Theorem 5 is not necessarily surjective and in some cases (where X is not countable) it can never be surjective.

Theorem 6. If X is an infinite set, then there exists a bijection between X and a proper subset of X. Hence a set is infinite if and only if it is in bijective correspondence with a proper subset of itself.
Proof. We leave this proof as an exercise!
Hint: Use Theorem 5. //

Remark. Sometimes treatments of infinite sets take the conclusion of Theorem 6 as the definition of being infinite. That is, one can take the property of being in bijective correspondence with a proper subset as the definition of an infinite set.