Linear Algebra and Linear Diff Equations

Example. $x'(t) = dx/dt$.

Consider the system of diff eqns:

Exam 2
Nov. 4 (Mon.)

$$\boxed{\begin{aligned} x'(t) &= 4x(t) - 2y(t) \\ y'(t) &= x(t) + y(t) \end{aligned}}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$.

Then for $x, y \in C^\infty[\mathbb{R}]$ we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \in C^\infty[\mathbb{R}]^2 = \{ f : \mathbb{R} \to \mathbb{R}^2 \mid f = (f_1, f_2) \text{ and each } f_i \text{ is infinitely diff} \}$$

$$\| \quad \mathcal{H}$$

We want to find the subspace $\mathcal{S}$ of $\mathcal{H}$ consisting of the solutions to $Df = Af$ where $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

Note: D is a linear transformation of $\mathcal{H}$ and so is M. So we want the null space of $(D - A) = T$.

We let $\mathcal{S} \subset \mathcal{H}$ be this nullspace.

$$\boxed{\begin{aligned} x'(t) &= 4x(t) - 2y(t) \\ y'(t) &= x(t) + y(t) \end{aligned}}$$

Try $x(t) = ae^{\lambda t}$
$y(t) = be^{\lambda t}$.

$$\lambda a e^{\lambda t} = 4ae^{\lambda t} - 2be^{\lambda t}$$
$$\lambda b e^{\lambda t} = ae^{\lambda t} + be^{\lambda t}$$

Divide by $e^{\lambda t}$:

$$\lambda a = 4a - 2b$$
$$\lambda b = a + b$$

$$(4-\lambda)a - 2b = 0$$
$$a + (1-\lambda)b = 0$$

$$\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This will have $\neq 0$ solutions $\begin{bmatrix} a \\ b \end{bmatrix}$

if $\mathrm{Det}\begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix} = 0$.

$(4-\lambda)(1-\lambda)+2$
$\shortparallel$
$\lambda^2 - 5\lambda + 4 + 2$

So $\lambda^2 - 5\lambda + 6 = 0$

So $(\lambda-2)(\lambda-3) = 0$

$\underline{\lambda = 2}$ or $\underline{\lambda = 3}$.

$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

$M = (A - \lambda I) = \begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}$

$\det(M) = 0 \iff \lambda = 2$ or 3

$\lambda = 2$: $M = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

So $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satis $Mv = \vec{0}$

$\lambda = 3$: $M = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

So $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ satis $Mv = \vec{0}$.

$\Rightarrow$ $v_1 e^{2t} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$ & $v_2 e^{3t} = \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions to the diff eqn.

The general solution is

$$c_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}.$$

So the null space of $D - A$ is Span$\left\{ \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} \right\}$.

$\boxed{\text{Solns to Diff Eqns are determined by initial conds.}}$

Change of Basis

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \quad \mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \{e_1, e_2\}$$

$$\mathcal{B} = \{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} =$$ Transition matrix from $\mathcal{B} \longrightarrow \mathcal{S}$.

$$\boxed{\begin{aligned} Av_1 &= 2v_1 \\ Av_2 &= 3v_2 \end{aligned}}$$ $(\lambda_1 = 2, \lambda_2 = 3)$

$$Pe_i = v_i$$

$$Av_i = \lambda_i v_i$$

$$APe_i = \lambda_i Pe_i = P(\lambda_i e_i)$$

$$\boxed{P^{-1}APe_i = \lambda_i e_i} \Longrightarrow P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

So here $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

<u>Check</u>: $P^{-1} = \frac{1}{(-1)} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \checkmark$$

More generally, let A be an $n \times n$ matrix. Let $\{v_1, v_2, \ldots, v_n\} = \beta$ be a basis for $\mathbb{R}^n$. Thus

$$P = [v_1 \cdots v_n] = \text{transition matrix from } \beta \longrightarrow \mathcal{S}.$$

So $P^{-1}v_i = e_i$ & $Pe_i = v_i$.

Suppose $Av_i = B_{1i}v_1 + \cdots + B_{ni}v_n$

so that $B = [A]_\beta$.

Then $APe_i = B_{1i}Pe_1 + \cdots + B_{ni}Pe_n$

$$\Rightarrow P^{-1}APe_i = B_{1i}e_1 + \cdots + B_{ni}e_n$$

$$P^{-1}APe_i = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{ni} \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = B.$$

Thus $\boxed{P^{-1}[A]_{\mathcal{S}}P = [A]_\beta.}$

Theorem 4.3.1 $E = \{v_1, \dots, v_n\}$
$F = \{w_1, \dots, w_n\}$

bases for V, vector space.

$L : V \longrightarrow V$ a linear operator on V.

S = a transition matrix from F to E.

$(S[v]_F = [v]_E)$

Let $A = [L]_E$, $B = [L]_F$.

Then $B = S^{-1}AS$.

Proof. $x \in \mathbb{R}^n$ any vector.

Let $v = x_1 w_1 + \dots + x_n w_n$. $(x = [v]_F)$

$y = Sx$, $t = Ay$, $z = Bx$

$\Rightarrow y = [v]_E \Rightarrow v = y_1 v_1 + \dots + y_n v_n$.

$$y = [v]_E = Sx$$

$t = [Lv]_E$, $z = [Lv]_F$

$$\begin{tikzcd} y \arrow[r, "A"] & t \arrow[d, "S^{-1}"] \\ x \arrow[u, "S"] \arrow[r, "B"'] & z \end{tikzcd}$$

$S[\cdot]_F = [\cdot]_E$

$\therefore$ $t = Sz$

& $S^{-1}t = z$

so $S^{-1}Ay = Bx$

$S^{-1}ASx = Bx$

for all $x \in \mathbb{R}^n$.

$\Longrightarrow S^{-1}AS = B$. //

$$\begin{tikzcd} E \arrow[d, "{[L]_E}"'] & F \arrow[l, "S"'] \arrow[d, "{[L]_F}"] \\ E \arrow[r, "S^{-1}"'] & F \end{tikzcd}$$

$$[L]_F = S^{-1}[L]_E S$$

$$\beta = \left\{ \overset{w_1}{\overset{\shortparallel}{\begin{pmatrix}1\\1\end{pmatrix}}}, \overset{w_2}{\overset{\shortparallel}{\begin{pmatrix}1\\-1\end{pmatrix}}} \right\} \text{ basis for } \mathbb{R}^2.$$

$$\beta' = \left\{ \underset{v_1}{\underset{\shortparallel}{\begin{pmatrix}2\\1\end{pmatrix}}}, \underset{v_2}{\underset{\shortparallel}{\begin{pmatrix}1\\2\end{pmatrix}}} \right\} \text{ basis for } \mathbb{R}^2.$$

Suppose $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is specified by
$$\boxed{\begin{aligned} Lw_1 &= w_1 + w_2 \\ Lw_2 &= 2w_1 + 3w_2 \end{aligned}}.\quad [L]_\beta = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

a) Find $[L]_\mathcal{S}$ where $\mathcal{S}$ is the standard basis $\mathcal{S} = \{e_1, e_2\} = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

b) Find $[L]_\beta$.

c) Find $[L]_{\beta'}$

Solution. (a) Let $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so that

a) & b) $S: \beta \longrightarrow \mathcal{S}$ is the transition matrix from β to $\mathcal{S}$. We have

$w_1 = Se_1,\ w_2 = Se_2.$

So $[L]_\mathcal{S} Se_1 = Se_1 + Se_2$
$[L]_\mathcal{S} Se_2 = 2Se_1 + 3Se_2$

$$\Rightarrow (S^{-1}[L]_\mathcal{S} S)e_1 = e_1 + e_2 = \begin{bmatrix}1\\1\end{bmatrix}$$
$$(S^{-1}[L]_\mathcal{S} S)e_2 = 2e_1 + 3e_2 = \begin{bmatrix}2\\3\end{bmatrix}$$
$$\Longrightarrow S^{-1}[L]_\mathcal{S} S = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = [L]_\beta$$

Thus $[L]_{\mathcal{S}} = S[L]_{\mathcal{B}}S^{-1}$

Note:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "S"] \arrow[d, "{[L]_{\mathcal{B}}}"'] & \mathcal{S} \arrow[d, "{[L]_{\mathcal{S}}}"] \\
\mathcal{B} \arrow[r, "S"'] & \mathcal{S}
\end{tikzcd}$$

$$= S\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}S^{-1}, \quad S = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$S^{-1} = \begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}\left(\frac{1}{-2}\right) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{2}$$

$$= \begin{bmatrix}2 & 5\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{2} = \begin{bmatrix}7/2 & -3/2\\ -1/2 & 1/2\end{bmatrix} = [L]_{\mathcal{S}}.$$

<u>Check</u>

$$\frac{1}{2}\begin{bmatrix}7 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4\\ 0\end{bmatrix} = \begin{bmatrix}2\\ 0\end{bmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix} + \begin{pmatrix}1\\ -1\end{pmatrix} \checkmark$$

$$\frac{1}{2}\begin{bmatrix}7 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}10\\ -2\end{bmatrix} = \begin{bmatrix}5\\ -1\end{bmatrix}$$

$$2\begin{pmatrix}1\\ 1\end{pmatrix} + 3\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}5\\ -1\end{pmatrix} \checkmark$$

(c) Find $[L]_{\beta'}$.

$$\beta' \xrightarrow[S']{} \mathcal{S}$$

$$S' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad S'^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{tikzcd}
\beta' \arrow[r, "S'"] \arrow[d, "{[L]_{\beta'}}"'] \arrow[d, bend left=80] & \mathcal{S} \arrow[d, "{[L]_{\mathcal{S}}}"] \\
\beta' \arrow[r, "S'"] & \mathcal{S}
\end{tikzcd}$$

$$\Rightarrow [L]_{\beta'} = S'^{-1} [L]_{\mathcal{S}} S'$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \frac{1}{2}\begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 15 & -7 \\ -9 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[L]_{\beta'} = \frac{1}{6}\begin{bmatrix} 23 & 1 \\ -13 & 1 \end{bmatrix}$$

Inner Product on $\mathbb{R}^n$

$\vec{a}, \vec{b} \in \mathbb{R}^n$ $\quad \|\vec{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$

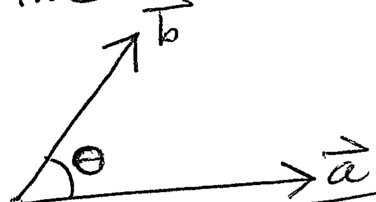
$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$

Fact (will prove later):

$-1 \leq \vec{a} \cdot \vec{b} / \|\vec{a}\| \|\vec{b}\| \leq 1$ for $\vec{a} \neq \vec{0}$ & $\vec{b} \neq \vec{0}$.

Note that $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$.

We define $\cos(\theta) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$



The cosine of the angle between $\vec{a}$ & $\vec{b}$.

$$\boxed{\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)}$$

Let $V \subset \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$.

Then $V^{\perp} = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \text{ for all } w \in V \}$

is the subspace orthogonal to V. (Note that $V^{\perp}$ is indeed a subspace of V.)

Example. $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^3$.

Then $V^{\perp} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Fact. $V \subseteq \mathbb{R}^n$ as above, then

$V \cap V^{\perp} = \{0\}$ and $\dim(V) + \dim(V^{\perp}) = n$.

$$V = \text{Span}\{\underset{\begin{pmatrix}1\\1\\1\end{pmatrix}}{\underset{\parallel}{v_1}}, \underset{\begin{pmatrix}1\\-1\\1\end{pmatrix}}{\underset{\parallel}{v_2}}\} \subseteq \mathbb{R}^3$$

Find basis for $V^\perp$.

$$V^\perp = \text{NullSpace}\begin{bmatrix}1&1&1\\1&-1&1\end{bmatrix}.$$

$$\begin{bmatrix}1&1&1\\1&-1&1\end{bmatrix} \to \begin{bmatrix}1&1&1\\0&-2&0\end{bmatrix} \to \begin{array}{c}\begin{array}{ccc}x&y&\alpha\end{array}\\ \begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}\end{array}$$

pivot cols.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-\alpha\\0\\\alpha\end{pmatrix} = \alpha\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

$$\Longrightarrow V^\perp = \text{Span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}.$$

So $V^\perp$ has basis

and $\dim(V^\perp) = 1$.

Problem.

Solve the diff equ

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Exam 2, Nov. 4

Next Chapter is Chapter 6. (We'll return to Chapter 5 later.)

Try $\boxed{\begin{aligned} x_1 &= a e^{\lambda t} \\ x_2 &= b e^{\lambda t} \end{aligned}}$

$$x_1' = \lambda a e^{\lambda t}$$
$$x_2' = \lambda b e^{\lambda t}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} e^{\lambda t}$$

$$\lambda \begin{pmatrix} a \\ b \end{pmatrix} e^{\lambda t}$$

$$= \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda t}$$

$$\lambda \begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Linear Alg Problem

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\frac{d}{d\theta}(e^{i\theta}) = i e^{i\theta}$$

$$i(\cos\theta + i\sin\theta)$$
$$= -\sin(\theta) + i\cos(\theta)$$

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta)$$
$$= -\sin(\theta) + i\cos\theta$$

$$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$e^{i(\theta+\varphi)}$$
$$= \cos(\theta+\varphi) + i\sin(\theta+\varphi)$$

$$e^{i\theta} e^{i\varphi}$$
$$= (c(\theta) + i s(\theta))(c(\varphi) + i s(\varphi))$$
$$= [c(\theta)c(\varphi) - s(\theta)s(\varphi)]$$
$$+ i[s(\theta)c(\varphi) + c(\theta)s(\varphi)]$$

$$\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

~~want~~ solns $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \lambda I \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{bmatrix}$$

non zero $\begin{pmatrix} a \\ b \end{pmatrix}$ solns $\iff$ matrix singular.

Need $\underbrace{\text{Det}\begin{pmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{pmatrix}}_{M} = 0$.

$$\lambda^2 - 12\lambda + 35 = 0$$
$$\parallel$$
$$(\lambda - 7)(\lambda - 5)$$

$$\boxed{A = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}}$$

$\lambda = 5$ or $\lambda = 7 \Rightarrow M$ sing.

$\lambda = 5$: $M = \begin{pmatrix} 6-5 & 1 \\ 1 & 6-5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

so. $NS(M) = \text{Span}\left\{ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

$\lambda = 7$: $M' = \begin{pmatrix} 6-7 & 1 \\ 1 & 6-7 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$NS(M') = \text{Span}\left\{ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

Genl Soln to DE:

$$\boxed{c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}}$$

$$\boxed{\begin{aligned} A v_1 &= 5 v_1 \\ A v_2 &= 5 v_2 \end{aligned}}$$

Problem. $$\boxed{\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \end{aligned}}$$

A

$c_A(\lambda) = |A - \lambda I|$

characteristic poly of A

- roots are the eigenvalues of A
- $\neq 0$ vectors v s.t. $Av = \lambda v$ some λ are called eigenvectors.

Try $$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\left.\begin{aligned} x(t) &= a e^{\lambda t} \\ y(t) &= b e^{\lambda t} \end{aligned}\right\} \rightsquigarrow \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\longrightarrow \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Singular when $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0.$

$$\iff \lambda^2 + 1 = 0$$

$$\lambda = i \text{ or } -i.$$

$$e^{\lambda t} = e^{it} \text{ or } e^{-it}.$$

We have to look for eigenvectors using complex nos.

$\lambda = i$: $$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow{r_1 \to r_1 i} \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \lambda = i$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = i \begin{pmatrix} i \\ 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} i e^{it} \\ e^{it} \end{pmatrix}$$

satisfies the Diff eqn.

<u>and</u> $\begin{pmatrix} i e^{it} \\ e^{it} \end{pmatrix} = \begin{pmatrix} i(\cos t + i \sin t) \\ (\cos t + i \sin t) \end{pmatrix}$

$$= \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

Each of $\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$ and $\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

are real solns of the equation.

and $\left\{ \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \right\}$

are <u>linearly independent</u> (<u>why?</u>)

and so form a basis for the space of solutions to

$$\boxed{\begin{aligned} x' &= -y \\ y' &= x \end{aligned}}.$$

$V = \mathbb{R}^n$, A an $n \times n$ matrix.

$\{v_1, \ldots, v_n\}$ a set (all $\neq 0$) of eigenvectors for A.

$Av_i = \lambda_i v_i$, $\lambda_i \neq \lambda_j$ for $i \neq j$.

<u>**Then**</u> $\{v_1, \ldots, v_n\}$ are linearly independent & hence a basis for $\mathbb{R}^n$.

<u>**Proof.**</u> $n=1$. $\{v_1 \neq 0\} \Rightarrow \{v_1\}$ basis for $\mathbb{R}^1$.

$n=2$. $\{v_1, v_2\}$. Suppose $c_1v_1 + c_2v_2 = 0$.

Then $c_1Av_1 + c_2Av_2 = 0$

$$\Rightarrow c_1\lambda_1v_1 + c_2\lambda_2v_2 = 0$$

<u>But also</u> $c_1\lambda_1v_1 + c_2\lambda_1v_2 = 0$

Subtract $\quad c_2(\lambda_2 - \lambda_1)v_2 = 0$.

$$\left.\begin{array}{l}\lambda_2 - \lambda_1 \neq 0 \\ v_2 \neq 0\end{array}\right\} \Rightarrow c_2 = 0.$$

& this implies by $n=1$ that $c_1 = 0$.

<u>**Inductive Step:**</u> Assume for <u>some</u> <u>n</u> that any n vectors ($\neq 0$) with distinct A eigenvalues are lin indep. Suppose $c_1v_1 + \cdots + c_nv_n + c_{n+1}v_{n+1} = 0$.
& $\{v_1, \ldots, v_{n+1}\}$ non-zero with distinct e.v. $\lambda_1, \ldots, \lambda_{n+1}$.

$\Rightarrow$ 1) $c_1\lambda_{n+1}v_1 + \cdots + c_n\lambda_{n+1}v_n + c_{n+1}\lambda_{n+1}v_{n+1} = 0$ (mult by λ_{n+1})

2) $c_1\lambda_1v_1 + \cdots + c_n\lambda_nv_n + c_{n+1}\lambda_{n+1}v_{n+1} = 0$ (apply A)

$\Rightarrow$ 3) $c_1(\lambda_{n+1} - \lambda_1)v_1 + \cdots + c_n(\lambda_{n+1} - \lambda_n)v_n = 0$.

$\Rightarrow c_1(\lambda_{n+1} - \lambda_1) = \cdots = c_n(\lambda_{n+1} - \lambda_n) = 0 \Rightarrow c_1 = \cdots = c_n = 0$

$\Rightarrow$ (1) $c_{n+1} = 0$ //

Characteristic Polynomial of a 3x3 Matrix.

$$\begin{vmatrix} (a-\lambda) & b & c \\ d & (e-\lambda) & f \\ g & h & k-\lambda \end{vmatrix}\begin{matrix} (a-\lambda) & b \\ d & (e-\lambda) \\ g & h \end{matrix} \qquad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

$$= (a-\lambda)(e-\lambda)(k-\lambda) + bfg + cdh - c(e-\lambda)g - (a-\lambda)fh - bd(k-\lambda)$$

$$\begin{aligned} = \; & aek - \lambda ek - a\lambda k - ae\lambda + a\lambda^2 + e\lambda^2 + k\lambda^2 - \lambda^3 \\ & + bfg \quad + \lambda cg + \lambda fh + \lambda bd \\ & + cdh \\ & - ceg \\ & - afh \\ & - bdk \end{aligned}$$

$$= |A| + \lambda\left[-\begin{vmatrix} a & c \\ g & k \end{vmatrix} - \begin{vmatrix} e & f \\ h & k \end{vmatrix} - \begin{vmatrix} a & b \\ d & e \end{vmatrix}\right] + (a+e+k)\lambda^2 - \lambda^3$$

$$|A - \lambda I| = -\left(\lambda^3 - tr(A)\lambda^2 + \widehat{tr}(A)\lambda - det(A)\right)$$

$$\widehat{tr}(A) = \begin{vmatrix} a & c \\ g & k \end{vmatrix} + \begin{vmatrix} e & f \\ h & k \end{vmatrix} + \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

<u>Ex</u>: Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$.

Compute and factor $C_A(\lambda) = |A - \lambda I|$. Find eigenvalues, eigenvectors and an invertible matrix P s.t. $P^{-1}AP$ is a diagonal matrix.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$tr(A) = 2$$

$$\widehat{tr}(A) = \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix}$$

$$= 1$$

$$\det(A) = 0$$

$$C_A(\lambda) = -(\lambda^3 - 2\lambda^2 + \lambda)$$

$$= -\lambda(\lambda^2 - 2\lambda + 1)$$

$$= -\lambda(\lambda - 1)^2$$

$$\lambda = 0 : A - 0I = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} : v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 1 : A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{pmatrix} 3\alpha - \beta \\ \alpha \\ \beta \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{\text{dim 2}} \ \underline{\text{eigenspace}}$$

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$C_A(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix} (a_{11}-\lambda) & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & & \vdots \\ a_{31} & & \ddots & \\ \vdots & & & a_{n-1\,n} \\ a_{n1} & a_{n-2} & \cdots & (a_{nn}-\lambda) \end{vmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\Longrightarrow C_A(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} tr(A) \lambda^{n-1} + \cdots + \det(A)$$

← the constant term

$$\boxed{tr(A) = a_{11} + a_{22} + \cdots + a_{nn}} \Rightarrow \boxed{tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n}$$

$$\| $$

$$(-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$$\Longrightarrow C_A(0) = \lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$$

Example: $A = \begin{bmatrix} 5 & -18 \\ 1 & -1 \end{bmatrix}$, $tr(A) = 4$

$\det(A) = -5 + 18 = 13$

$$C_A(\lambda) = \begin{vmatrix} 5-\lambda & -18 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13$$

roots: $\lambda = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

$\lambda_1 = 2 + 3i$

$\lambda_2 = 2 - 3i$

$\lambda_1 + \lambda_2 = 4$

$\lambda_1 \lambda_2 = 4 + 9 = 13$ ✓

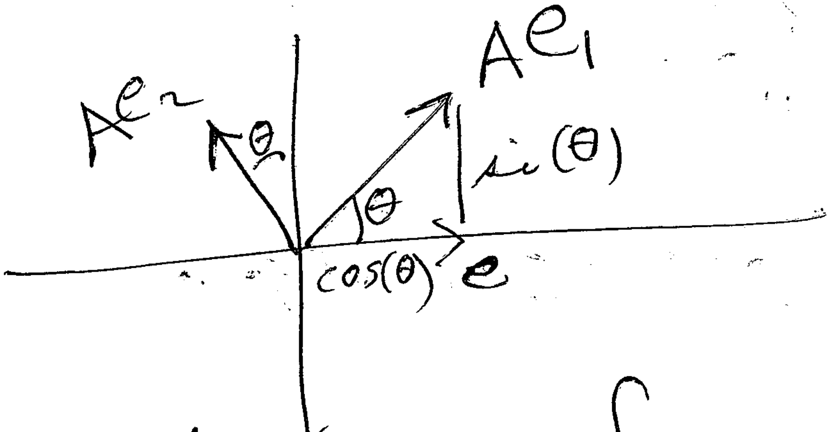
p294

(47.1)

B.. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R_\theta$

$\parallel$

rotation by $\measuredangle\theta$ ↺



We know from geometry that $R_\varphi R_\theta v = R_{\theta+\varphi} v$

$$\boxed{R_\varphi \circ R_\theta = R_{\theta+\varphi}}$$

prod of matrices

$$\Rightarrow \boxed{R_\varphi R_\theta = R_{\theta+\varphi}}$$

$$R_\theta R_\varphi = R_{\theta+\varphi}$$ (geometry of rotations)

$$\Downarrow$$

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{bmatrix} = \begin{bmatrix} c(\theta+\varphi) & -s(\theta+\varphi) \\ s(\theta+\varphi) & c(\theta+\varphi) \end{bmatrix}$$

$$\parallel$$

$$\begin{bmatrix} c(\theta)c(\varphi) - s(\theta)s(\varphi) & -s(\theta)c(\varphi) - c(\theta)s(\varphi) \\ s(\theta)c(\varphi) + c(\theta)s(\varphi) & -s(\theta)s(\varphi) + c(\theta)c(\varphi) \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} c(\theta+\varphi) &= c(\theta)c(\varphi) - s(\theta)s(\varphi) \\ s(\theta+\varphi) &= s(\theta)c(\varphi) + c(\theta)s(\varphi) \end{aligned}}$$

$B = S^{-1}AS$, $S_{n\times n}$ invertible

$A_{n\times n}$ $\quad (S^{-1}IS = I)$

$$\Rightarrow C_B(\lambda) = \mathrm{Det}(S^{-1}AS - \lambda I)$$
$$= \mathrm{Det}(S^{-1}(A-\lambda I)S)$$
$$= \mathrm{Det}(S^{-1})\,\mathrm{Det}(A-\lambda I)\,\mathrm{Det}(S)$$
$$= \mathrm{Det}(A-\lambda I)\,\mathrm{Det}(S^{-1})\,\mathrm{Det}(S)$$
$$= \mathrm{Det}(A-\lambda I)\,\mathrm{Det}(S)^{-1}\,\mathrm{Det}(S)$$

$$C_B(\lambda) = \mathrm{Det}(A-\lambda I)$$

$$\therefore \boxed{C_B(\lambda) = C_A(\lambda)}$$

---

ex: $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

$$B = S^{-1}AS = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -7 \\ -6 & 12 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ +6 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -2 \\ +6 & 6 \end{bmatrix}$$

$$\left.\begin{array}{l} \det(A) = \det(B) = 6 \\ \mathrm{tr}(A) = \mathrm{tr}(B) = 5 \end{array}\right\} \Rightarrow \begin{array}{l} C_A(\lambda) = C_B(\lambda) \\ = \lambda^2 - 5\lambda + 6 \\ = (\lambda-2)(\lambda-3). \end{array}$$

Example 1

$$A = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$C_A(\lambda) = \begin{vmatrix} (4-\lambda) & -5 & 1 \\ 1 & (-\lambda) & -1 \\ 0 & 1 & (-1-\lambda) \end{vmatrix} = (4-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - \begin{vmatrix} -5 & 1 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(\lambda+\lambda^2+1) - (5+5\lambda-1)$$

$$= \left\{ \begin{matrix} 4\lambda + 4\lambda^2 + 4 \\ -\lambda \quad -\lambda^2 \quad -\lambda^3 \\ -5\lambda \qquad -4 \end{matrix} \right\} = -2\lambda + 3\lambda^2 - \lambda^3$$

$$= -\lambda(\lambda^2 - 3\lambda + 2)$$

$$= -\lambda(\lambda-1)(\lambda-2)$$

$$C_A(\lambda) = -\lambda(\lambda-1)(\lambda-2)$$

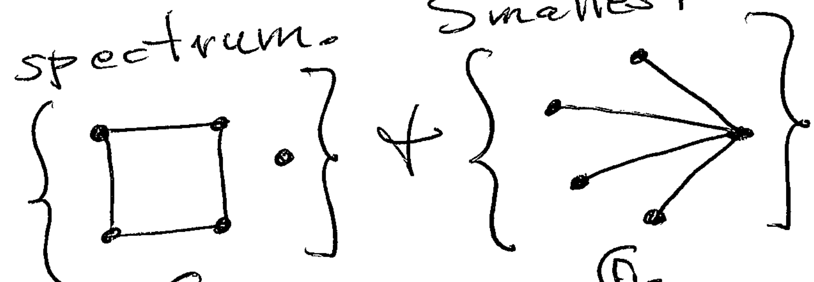
<u>example</u>: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ 

Γ

A is the <u>adjacency matrix</u> for Γ.
Then $C_A(\lambda)$ is called the <u>characteristic polynomial for the graph Γ</u>.

The roots of $C_A(\lambda)$ (eigenvalues of A) are called the <u>spectrum</u> of the graph Γ.

(See <http://en.wikipedia.org/wiki/Spectral_graph_theory>

- isomorphic graphs have same char poly hence same spectrum.
- $\exists$ pairs of distinct graphs with same spectrum. Smallest example is: 

Γ_1 Γ_2

<u>Exercise</u>. Work out the spectra of Γ_1 and Γ_2.

Example. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $\mathcal{E}_\lambda(A) = \lambda$-eigenspace of A

$c_A(\lambda) = (3-\lambda)^3$.

$\lambda = 3 : (A - 3I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\begin{aligned} (A-3I)e_1 &= 0 \\ (A-3I)e_2 &= e_1 \\ (A-3I)e_3 &= e_2 \end{aligned}}$

$\Rightarrow \dim \mathcal{E}_3(A) = 1$

$\mathcal{E}_3(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Consider $D = d/dt$ and

$\boxed{\begin{aligned} (D-3)f(t) \\ = f'(t) - 3f(t) \end{aligned}}$

$(D-3)^3 x(t) = 0$

$(D-3)e^{3t} = 3e^{3t} - 3e^{3t} = 0$ ✓

But note $(D-3)(te^{3t})$

$= D(te^{3t}) - 3te^{3t}$

$= e^{3t} + 3te^{3t} - 3te^{3t}$

$= e^{3t}$

and $(D-3)(t^2e^{3t}) = 2te^{3t} + 3t^2e^{3t} - 3t^2e^{3t}$

$= 2te^{3t}$

So $(D-3)^3(t^2e^{3t}) = (D-3)^2(2te^{3t})$

$= (D-3)(2e^{3t}) = 0$.

$$W = \text{Span}\{\underset{\overset{\|}{v_1}}{e^{3t}},\ \underset{\overset{\|}{v_2}}{te^{3t}},\ \underset{\overset{\|}{v_3}}{\tfrac{t^2}{2}e^{3t}}\}$$

$$\left.\begin{aligned}(D-3)v_3 &= v_2\\ (D-3)v_2 &= v_1\\ (D-3)v_1 &= 0\end{aligned}\right\} \quad [D-3]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[D]_\beta = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

This is a clue to finding good bases. If the eigenspace of A for eigenvalue λ is too small, take each eigenvector v such that $(A-\lambda I)v = 0$ and see if there is a v' s.t. $(A-\lambda I)v' = v$ and ad v'' s.t. $(A-\lambda I)v'' = v'$ etc.

We will see that these vectors $v, v', v'', \dots$ (to the extent that you can find them) are linearly independent.

Example: $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$.

$A=\begin{bmatrix}7&4\\-9&-5\end{bmatrix}$, $C_A(\lambda)=(\lambda^2-2\lambda+1)$
$=(\lambda-1)^2$.

$\lambda=1$: $A-\lambda I=\begin{bmatrix}6&4\\-9&-6\end{bmatrix}\to\begin{bmatrix}3&2\\-3&-2\end{bmatrix}\to\begin{bmatrix}3&2\\0&0\end{bmatrix}$

$\Rightarrow \dim E_1(A)=1$

spanned by $v=\begin{pmatrix}-2\\3\end{pmatrix}$.

Now look for v' $(A-I)v'=\begin{pmatrix}-2\\3\end{pmatrix}$.

$\left[\begin{array}{cc|c}6&4&-2\\-9&-6&3\end{array}\right]\to\left[\begin{array}{cc|c}3&2&-1\\-3&-2&1\end{array}\right]\to\left[\begin{array}{cc|c}3&2&-1\\0&0&0\end{array}\right]$

$3x+2y=-1$
take $x=-1$, $y=+1$
$\begin{pmatrix}-1\\+1\end{pmatrix}=v'$.

Check:

$Av=\begin{bmatrix}7&4\\-9&-5\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}$ ✓

$Av'=\begin{bmatrix}7&4\\-9&-5\end{bmatrix}\begin{bmatrix}-1\\+1\end{bmatrix}=\begin{bmatrix}-3\\+4\end{bmatrix}=v+v'$.

$Av=v$
$Av'=v+v'$
$\beta=\{v,v'\}$ $[A]_\beta=\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

Suppose $\beta=\{v,v'\}$
$Av=3v$
$Av'=v+3v'$
$[A]_\beta=\begin{bmatrix}3&1\\0&3\end{bmatrix}$

One More Example

$A=\begin{bmatrix}48&25\\-81&-42\end{bmatrix}$

$\begin{pmatrix}x_1'(t)\\x_2'(t)\end{pmatrix}=\begin{bmatrix}48&25\\-81&-42\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$

Suppose we have A 2×2 and vectors $\{v_1, v_2\}$ lin ind s.t.

$$\left.\begin{aligned} Av_1 &= \lambda v_1 \\ Av_2 &= \lambda v_2 + v_1 \end{aligned}\right\} \quad [A]_{\{v_1,v_2\}} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Then we can solve ($D = d/dt$)

$$Dx = Ax$$

as follows: $x_1 = e^{\lambda t} v_1$

$$x_2 = e^{\lambda t}[t v_1 + v_2]$$

Check: $Dx_1 = \lambda e^{\lambda t} v_1 = \lambda x_1$

$$Dx_2 = \lambda x_2 + \underbrace{e^{\lambda t} v_1}_{x_1} = \lambda x_2 + x_1 \quad \checkmark$$

$$Ax_1 = e^{\lambda t} A v_1 = \lambda e^{\lambda t} v_1 = \lambda x_1$$

$$\begin{aligned} Ax_2 &= e^{\lambda t}[tAv_1 + Av_2] \\ &= e^{\lambda t}[\lambda t v_1 + \lambda v_2 + v_1] \\ &= \lambda e^{\lambda t}[t v_1 + v_2] + e^{\lambda t} v_1 \\ &= \lambda x_2 + x_1 \quad \checkmark \end{aligned}$$

> Exercise: Show that x_1 & x_2 above are linearly indep.

Exercise: Solve the differential equation $Dx = Ax$ when $A = \begin{bmatrix} 7 & 4 \\ -9 & 5 \end{bmatrix}$, and when $A = \begin{bmatrix} 48 & 25 \\ -81 & -42 \end{bmatrix}$.

Suppose
$$\begin{cases} Av_1 = \lambda v_1 \\ Av_2 = \lambda v_2 + v_1 \\ Av_3 = \lambda v_3 + v_2 \\ \dots \\ Av_n = \lambda v_n + v_{n-1} \end{cases}$$
$$\beta = \{v_1, v_2, \dots, v_n\}$$
then β is linearly independent if $v_1 \neq 0$.

<u>Exercise</u>. Prove this statement for $n=2$ and $n=3$.

$$\Rightarrow [A]_\beta = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}.$$

Let
$$\begin{cases} x_1 = e^{\lambda t} v_1 \\ x_2 = e^{\lambda t}\left[\frac{t}{1!} v_1 + v_2\right] \\ x_3 = e^{\lambda t}\left[\frac{t^2}{2!} v_1 + \frac{t}{1!} v_2 + v_3\right] \\ x_4 = e^{\lambda t}\left[\frac{t^3}{3!} v_1 + \frac{t^2}{2!} v_2 + \frac{t}{1!} v_3 + v_4\right] \\ \dots \\ x_n = e^{\lambda t}\left[\frac{t^{n-1}}{(n-1)!} v_1 + \frac{t^{n-2}}{(n-2)!} v_2 + \dots + \frac{t}{1!} v_{n-1} + v_n\right] \end{cases}$$

<u>Exercise</u>: Show that the $x_1, \dots, x_n$ are linearly independent solutions to
$$Dx = Ax.$$

Example: $A=\begin{pmatrix}2&1&0\\-1&3&1\\-1&0&4\end{pmatrix}$ $P=\begin{pmatrix} & ? & \end{pmatrix}$

$$C_\lambda(A) = -\left(\lambda^3 - tr(A)\lambda^2 + \hat{tr}(A)\lambda - det(A)\right)$$

$tr(A) = 9$

$$\hat{tr}(A) = \begin{vmatrix}3&1\\0&4\end{vmatrix} + \begin{vmatrix}2&0\\-1&4\end{vmatrix} + \begin{vmatrix}2&1\\-1&3\end{vmatrix}$$
$$= 12 + 8 + (6+1) = 27.$$

$$det(A) = 2\begin{vmatrix}3&1\\0&4\end{vmatrix} - \begin{vmatrix}-1&1\\-1&4\end{vmatrix} = 2\cdot 12 - (-4+1)$$
$$= 24 + 3 = 27$$

$$C_\lambda(A) = -(\lambda^3 - 9\lambda^2 + 27\lambda - 27)$$
$$= -(\lambda-3)^3$$

$$\left[\begin{array}{l}(\lambda-3)^3 = \lambda^3 + 3(-3)\lambda^2 + 3(-3)^2\lambda + (-3)^3 \\ \quad = \lambda^3 - 9\lambda^2 + 27\lambda - 27 \quad \checkmark\end{array}\right]$$

Thus $\lambda = 3$ is the only eigenvalue.

$$A - 3I = \begin{bmatrix}-1&1&0\\-1&0&1\\-1&0&1\end{bmatrix} \to \begin{bmatrix}1&-1&0\\1&0&-1\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&-1&0\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\to \begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix} : v = \begin{pmatrix}1\\1\\1\end{pmatrix} \text{ spans } \mathcal{E}_1(A)$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & a \\ -1 & 0 & 1 & b \\ -1 & 0 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -a \\ 0 & -1 & 1 & b-a \\ 0 & -1 & 1 & c-a \end{array}\right]$$

$$(A-3I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \qquad \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -b \\ 0 & 1 & -1 & a-b \\ 0 & 0 & 0 & c-b \end{array}\right]$$

So need $b = c$.

<u>Case 1.</u>

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha-1 \\ \alpha \\ \alpha \end{pmatrix}$$

Let $v' = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

<u>Check:</u> $$Av' = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now use $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = v' = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha-1 \\ \alpha \end{pmatrix}$$

Can use $v'' = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$$(A-3I)v'' = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = v'.$$

So we have

$$\left.\begin{aligned}(A-3I)v&=0\\(A-3I)v'&=v\\(A-3I)v''&=v'\end{aligned}\right\}\quad\begin{aligned}Av&=3v\\Av'&=3v'+v\\Av''&=3v''+v'\end{aligned}$$

$$\beta=\{v, v', v''\}$$

$$P=[v\,v'\,v'']=\begin{bmatrix}1&-1&0\\1&0&-1\\1&0&0\end{bmatrix}\qquad [A]_\beta=\begin{bmatrix}3&1&0\\0&3&1\\0&0&3\end{bmatrix}$$

Find P^{-1}:

$$\left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\1&0&-1&0&1&0\\1&0&0&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\0&1&-1&-1&1&0\\0&1&0&-1&0&1\end{array}\right]$$

$$\to\left[\begin{array}{ccc|ccc}1&0&0&0&0&+1\\0&1&0&-1&0&1\\0&0&-1&0&1&-1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&0&0&+1\\0&1&0&-1&0&1\\0&0&1&0&-1&1\end{array}\right]$$

Check $\begin{bmatrix}1&-1&0\\1&0&-1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&+1\\-1&0&1\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ ✓

$$P^{-1}AP=\begin{bmatrix}0&0&1\\-1&0&1\\0&-1&1\end{bmatrix}\begin{bmatrix}2&1&0\\-1&3&1\\-1&0&4\end{bmatrix}\begin{bmatrix}1&-1&0\\1&0&-1\\1&0&0\end{bmatrix}=\begin{bmatrix}-1&0&4\\-3&-1&4\\0&-3&3\end{bmatrix}\begin{bmatrix}1&-1&0\\1&0&-1\\1&0&0\end{bmatrix}$$

$$=\begin{bmatrix}3&1&0\\0&3&1\\0&0&3\end{bmatrix}$$ ✓

$\boxed{\text{Exercise: Solve the diff equ } DX=AX.}$

If $A = PDP^{-1}$

$$\Rightarrow A^2 = PDP^{-1}PDP^{-1}$$

$$= PDIDP^{-1}$$

$$A^2 = PD^2P^{-1}$$

& similarly $\boxed{A^n = PD^nP^{-1}}$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ adjacency matrix for 

$$C_A(\lambda) = \lambda^2 - \lambda - 1$$

$$\Rightarrow \lambda = \underbrace{\frac{1+\sqrt{5}}{2}}_{\varphi}, \underbrace{\frac{1-\sqrt{5}}{2}}_{\overline{\varphi}}$$

$$\boxed{\begin{aligned} \varphi^2 &= \varphi + 1 \\ \overline{\varphi}^2 &= \overline{\varphi} + 1 \end{aligned}}$$

<u>Note:</u> $\varphi\overline{\varphi} = -1$, $\varphi + \overline{\varphi} = 1$, $\varphi - \overline{\varphi} = \sqrt{5}$

$$\lambda = \varphi : A - \varphi I = \begin{bmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{bmatrix} \longrightarrow \begin{bmatrix} \varphi - \varphi^2 & \varphi \\ 1 & -\varphi \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -1 & \varphi \\ 1 & -\varphi \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\varphi \\ 0 & 0 \end{bmatrix} : v_1 = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}.$$

$$\lambda = \overline{\varphi} : A - \overline{\varphi} I = \begin{bmatrix} 1-\overline{\varphi} & 1 \\ 1 & -\overline{\varphi} \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\varphi} - \overline{\varphi}^2 & \overline{\varphi} \\ 1 & -\overline{\varphi} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -\overline{\varphi} \\ 0 & 0 \end{bmatrix} : v_2 = \begin{pmatrix} \overline{\varphi} \\ 1 \end{pmatrix}.$$

58

$$P=\begin{pmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{pmatrix}, \quad P^{-1}AP=\begin{pmatrix} \varphi & 0 \\ 0 & \bar{\varphi} \end{pmatrix}$$

$$P^{-1}=\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\bar{\varphi} \\ -1 & \varphi \end{pmatrix} \qquad A=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D=\begin{pmatrix} \varphi & 0 \\ 0 & \bar{\varphi} \end{pmatrix}$$

$$\Rightarrow A^n=PD^nP^{-1}=\begin{pmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^n & 0 \\ 0 & \bar{\varphi}^n \end{pmatrix}\begin{pmatrix} 1 & -\bar{\varphi} \\ -1 & \varphi \end{pmatrix}\frac{1}{\sqrt{5}}$$

$$=\begin{pmatrix} \varphi^{n+1} & \bar{\varphi}^{n+1} \\ \varphi^n & \bar{\varphi}^n \end{pmatrix}\begin{pmatrix} 1 & -\bar{\varphi} \\ -1 & \varphi \end{pmatrix}$$

$$A^n=\begin{pmatrix} \frac{\varphi^{n+1}-\bar{\varphi}^{n+1}}{\sqrt{5}} & \frac{\varphi^n-\bar{\varphi}^n}{\sqrt{5}} \\ \frac{\varphi^n-\bar{\varphi}^n}{\sqrt{5}} & \frac{\varphi^{n-1}-\bar{\varphi}^{n-1}}{\sqrt{5}} \end{pmatrix}=\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

$$\left.\begin{array}{l} f_0=0 \\ f_1=1 \\ f_2=1 \end{array}\right\} f_{n+1}=f_n+f_{n-1}$$

$$\begin{array}{ccccccccccccc} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12} \\ 0, & 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & 89, & 144, \dots \end{array}$$

<u>Note:</u>

$$\varphi^2=\varphi+1$$
$$\varphi^3=\varphi^2+\varphi=\varphi+1+\varphi=2\varphi+1$$
$$\varphi^4=2\varphi^2+\varphi=2(\varphi+1)+\varphi=3\varphi+2$$
$$\varphi^5=5\varphi+3 \qquad \boxed{\varphi^n=f_n\varphi+f_{n-1}}$$
$$\varphi^6=8\varphi+5$$

Let $\mathcal{M}_n = \{A \mid A$ is an $n\times n$ matrix with coeffs in $\mathbb{R}$ = real nos.$\}$

Then $\mathcal{M}_n$ is a vector space over $\mathbb{R}$ via $0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, $A+B$ = usual matrix addition.

Exercise: $\boxed{\dim(\mathcal{M}_n) = n^2}$ with basis

$$E_{ij} = \begin{pmatrix} \text{all zeroes} \\ \text{except a 1} \\ \text{in the } ij \text{ position} \end{pmatrix}$$

e.g. $E_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
$(n=4)$

Show that there are n^2 matrices E_{ij} and that they are all linearly independent in $\mathcal{M}_n$.

$\{I, A, A^2, \ldots, A^{n^2}\}$ for any matrix $A \in \mathcal{M}_n$ is <u>dependent</u> since $n^2+1 > \dim(\mathcal{M}_n)$ and there are (n^2+1) vectors in this list. This implies that there are constants (<u>not all 0</u>) <u>c_i s.t.</u>

$$c_0 \underline{I} + c_1 A + c_2 A^2 + \cdots + c_{n^2} A^{n^2} = 0.$$

We can do better than this.

Cayley Hamilton Theorem.

A $n \times n$ matrix in M_n.

$C_A(\lambda) = (-1)^n(c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n)$

Then $c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1}A^{n-1} + A^n = 0$.

Every matrix satisfies its own char polynomial. "$C_A(A) = 0$."

Proof Discussion. Suppose A is 2×2.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad C_A(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc).$$

$\{e_1, e_2\} = \mathcal{S}$ standard basis.

$$\left.\begin{aligned} Ae_1 &= ae_1 + ce_2 \\ Ae_2 &= be_1 + de_2 \end{aligned}\right\} \Rightarrow \begin{aligned} (a-A)e_1 + ce_2 &= 0 \\ be_1 + (d-A)e_2 &= 0 \end{aligned}$$

$$\Rightarrow \underbrace{\begin{bmatrix} aI - A & cI \\ bI & dI - A \end{bmatrix}}_{M} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} DET(M) &= (aI - A)(dI - A) - bcI \\ &= A^2 - (a+d)A + (ad-bc)I \end{aligned}$$

$$DET(M) = C_A(A).$$

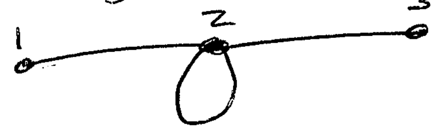
But $$adj(M)\,M = \begin{bmatrix} DET(M) & 0 \\ 0 & DET(M) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} DET(M) & 0 \\ 0 & DET(M) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} &DET(M)e_1 = 0 \\ &\quad \& \\ &DET(M)e_2 = 0 \end{aligned}\right\} \Rightarrow \underline{\underset{\displaystyle C_A(A)}{\overset{\|}{DET(M)}} = 0}$$

QED

Exercise. Let $A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = A$

be adjacency matrix for the graph



G.

Find $C_A(\lambda)$ and use Cayley-Hamilton Theorem to determine A^{10}.

How many walks of length 10 are there on G from 1 to 3?

$$C_A(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & (1-\lambda) & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2-\lambda-1) - \begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix}$$

$-\lambda(\lambda-2)(\lambda+1)$
$\parallel$
$-\lambda(\lambda^2-\lambda-2)$
$\parallel$

$$= -\lambda(\lambda^2-\lambda-1)+\lambda$$

$$= -\lambda^3+\lambda^2+2\lambda = \lambda(-\lambda^2+\lambda+2)$$

So CHT says $A^3 = A^2+2A$

<u>Check:</u> $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 5 & 3 \\ 1 & 3 & 1 \end{bmatrix}$

$A^2+2A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 5 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A^3$ ✓

$A^4 = A^3+2A^2 = A^2+2A+2A^2 = 3A^2+2A$

$A^5 = 3A^3+2A^2 = 3(A^2+2A)+2A^2 = 5A^2+6A$

$A^6 = 5A^3+6A^2 = 5(A^2+2A)+6A^2 = 11A^2+10A$

$A^7 = 11A^3+10A^2 = 11(A^2+2A)+10A^2 = 21A^2+22A$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^3 = A^2 + 2A$$

$$A^4 = 3A^2 + 2A$$
$$A^5 = 5A^2 + 6A$$
$$A^6 = 11A^2 + 10A$$
$$A^7 = 21A^2 + 22A$$
$$A^8 = 21(A^2+2A) + 22A^2 = 43A^2 + 42A$$
$$A^9 = 43(A^2+2A) + 42A^2 = 85A^2 + 86A$$
$$A^{10} = 85(A^2+2A) + 86A^2 = 171A^2 + 170A$$

The pattern is now clear. We have

$$A^n = a_n A^2 + (a_n + (-1)^{n-1})A$$

& $$A^{n+1} = a_n(A^2+2A) + (a_n + (-1)^{n+1})A^2$$
$$= [2a_n + (-1)^{n+1}]A^2 + 2a_n A$$

Thus $$\boxed{\begin{aligned} a_{n+1} &= 2a_n + (-1)^{n+1} \\ a_3 &= 1 \end{aligned}}$$

$$a_4 = 2a_3 + (-1)^4 = 2+1 = 3$$
$$a_5 = 2\cdot 3 + (-1)^5 = 5$$
$$a_6 = 2\cdot 5 + (-1)^6 = 11$$
$$\cdots$$

Can we get a closed formula for a_n? (next page)

Try $a_n = x2^n + y(-1)^n + z$ ← not needed

$n=3$: $8x - y + z = 1$
$n=4$: $16x + y + z = 3$
$n=5$: $32x - y + z = 5$

$$\left[\begin{array}{ccc|c} 8 & -1 & 1 & 1 \\ 16 & 1 & 1 & 3 \\ 32 & -1 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 8 & -1 & 1 & 1 \\ 0 & 3 & -1 & 1 \\ 0 & 3 & -3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 24 & -3 & 3 & 3 \\ 0 & 3 & -1 & 1 \\ 0 & 3 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 24 & 0 & 0 & 4 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 24 & 0 & 0 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \Longrightarrow \begin{array}{l} x = \frac{4}{24} = 1/6 \\ y = 1/3 \\ z = 0 \end{array}$$

$$\boxed{a_n = \frac{2^n + 2(-1)^n}{6}}$$

$$A^n = \left[\frac{2^n + 2(-1)^n}{6}\right] A^2 + \left[\left(\frac{2^n + 2(-1)^n}{6}\right) + (-1)^{n-1}\right] A$$

Exercise: Write an explicit formula for A^n by putting in the matrices for A^2 and A.

(63.1)

Another way to do this problem is to diagonalize the matrix A.

We have $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{aligned} C_A(\lambda) &= -\lambda^3 + \lambda^2 + 2\lambda \\ &= -\lambda(\lambda^2 - \lambda - 2) \\ &= -\lambda(\lambda+1)(\lambda-2) \end{aligned}$$

Eigenvalues: $\lambda = 0, -1, 2.$

$\lambda=0$: $A - 0I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda=-1$: $A-(-I) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} : v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda=2$: $A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} : v_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

<u>Check</u>: $Av_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

$Av_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ✓

$Av_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ✓

~~Let~~ $P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix}$

Then $P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$

or $\boxed{A = PDP^{-1}}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 2 & 1 \\ 0 & 1 & -2 & 0 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 6 & 2 & 2 & 0 \\ 0 & 3 & -6 & 0 & -3 & 0 \\ 0 & 0 & 6 & 1 & 2 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 & -1 & 1 \\ 0 & 0 & 6 & 1 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1/6 & 1/3 & 1/6 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 1/3 & -1/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{pmatrix}$$

Check $\begin{pmatrix} 1/2 & 0 & -1/2 \\ 1/3 & -1/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

$$A^K = PD^KP^{-1} = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^K & 0 \\ 0 & 0 & 2^K \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^K & 0 \\ 0 & 0 & 2^K \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 0 & (-1)^K & 2^K \\ 0 & (-1)^{K+1} & 2^{K+1} \\ 0 & (-1)^K & 2^K \end{pmatrix} \begin{pmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \frac{1}{6} = \frac{1}{6} \begin{pmatrix} 2(-1)^K + 2^K & 2(-1)^{K+1} + 2^K & 2(-1)^K + 2^K \\ 2(-1)^{K+1} + 2^{K+1} & 2(-1)^K + 2^{K+1} & 2(-1)^{K+1} + 2^{K+1} \\ 2(-1)^K + 2^K & 2(-1)^{K+1} + 2^K & 2(-1)^K + 2^K \end{pmatrix}.$$

Markov Chains

Markov Process

I. Set of possible outcomes or states is finite.

II. Probability of next outcome depends only on previous outcome.

III. Probabilities remain constant over time.

Initial vector X_0 gives a distribution of values at the start. Matrix A with a_{ij} = probability to switch from <u>state j to state i</u>

e.g.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} a_{11}S_1 + a_{12}S_2 \\ a_{21}S_1 + a_{22}S_2 \end{bmatrix} = \begin{bmatrix} S_1' \\ S_2' \end{bmatrix}$$

$$\left.\begin{tikzcd}[row sep=small] & S_1' \\ S_1 \arrow[ur, "a_{11}"] \arrow[dr, "a_{21}"'] & \\ & S_2' \end{tikzcd}\right\} \text{need } \left.\begin{matrix} a_{11} \\ +a_{21} \end{matrix}\right\} = 1$$

$$\left.\begin{tikzcd}[row sep=small] & S_2' \\ S_2 \arrow[ur, "a_{22}"] \arrow[dr, "a_{12}"'] & \\ & S_1' \end{tikzcd}\right\} \underline{\text{need}} \left.\begin{matrix} a_{12} \\ + \\ a_{22} \end{matrix}\right\} = 1$$

Each column is a probability vector.

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \qquad \begin{array}{ll} p+q=1 & 0 \le p,q \le 1 \\ r+s=1 & 0 \le r,s \le 1 \end{array}$$

$$C_A(\lambda) = \lambda^2 - (p+s)\lambda + ps - qr$$

$$\begin{pmatrix} s = 1-r \\ q = 1-p \end{pmatrix}$$

$$= \lambda^2 - (p-r+1)\lambda + \underbrace{\underbrace{p(1-r) - (1-p)r}_{p - pr - r + pr}}_{(p-r)}$$

$$= \lambda^2 - ((p-r)+1)\lambda + (p-r)$$

$$C_A(\lambda) = (\lambda - 1)(\lambda - (p-r))$$

$$\lambda = 1: \; A - I = \begin{bmatrix} p-1 & q \\ r & s-1 \end{bmatrix} = \begin{bmatrix} (1-q)-1 & q \\ r & (1-r)-1 \end{bmatrix}$$

$$= \begin{bmatrix} -q & q \\ r & -r \end{bmatrix} \longrightarrow \begin{bmatrix} r & -r \\ 0 & 0 \end{bmatrix}$$

$$v = \begin{pmatrix} r \\ q \end{pmatrix} \Longrightarrow Av = v$$

<u>Check</u> $A\begin{pmatrix} r \\ q \end{pmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{pmatrix} r \\ q \end{pmatrix} = \begin{bmatrix} pr + qr \\ qr + sq \end{bmatrix}$

$$= \begin{bmatrix} (p+q)r \\ (r+s)q \end{bmatrix} = \begin{bmatrix} 1 \cdot r \\ 1 \cdot q \end{bmatrix} = \begin{bmatrix} r \\ q \end{bmatrix}.$$

This suggests that the proportion $v = \binom{r}{q}$ corres to steady state behaviour where $A^n v_0 \approx A^{n+1} v_0$. For then $A(A^n v_0) \approx A^n v_0$ & $A^n v_0$ would be an eigenvector for eigenvalue 1. //

<u>Theorem</u>. Let M be a ~~Markov~~ Markov matrix. Then $\lambda = 1$ is an eigenvalue for M.

<u>Proof</u>. Each column of M sums to 1.

$\therefore$ Each row of M^T sums to 1.

$$\therefore \quad M^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

$\therefore$ $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 1$ for $\underline{M^T}$. This implies ~~that~~ $C_{M^T}(\lambda)$ has a root $\lambda = 1$.

But $C_{M^T}(\lambda) = C_M(\lambda)$ & so $C_M(\lambda)$ has a root $\lambda = 1$.

This means $\lambda = 1$ is an eigenvalue for M. //

<u>Example</u>: $M = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{2}{3} \end{pmatrix}$

$$M - I = \begin{pmatrix} -2/3 & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & -1/3 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} -8 & 3 & 2 \\ 4 & -6 & 2 \\ 4 & 3 & -4 \end{pmatrix}$$

$$\to \frac{1}{12}\begin{pmatrix} 4 & 3 & -4 \\ 0 & 9 & -6 \\ 0 & -9 & 6 \end{pmatrix} \to \frac{1}{12}\begin{pmatrix} 4 & 3 & -4 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \text{next page}$$

$$\to \frac{1}{12}\begin{pmatrix} 4 & 0 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} : v_1 = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$$Mv_1 = \frac{1}{12}\begin{pmatrix} 4 & 3 & 2 \\ 4 & 6 & 2 \\ 4 & 3 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 3\cdot 12 \\ 4\cdot 12 \\ 6\cdot 12 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} \checkmark$$

Now suppose we have $\overset{\text{any}}{M}$ and a basis of eigenvectors $\{v_1, v_2, \ldots, v_n\}$ with $Mv_i = \lambda_i v_i$.

Then if v is any vector, then

$$v = c_1 v_1 + \cdots + c_n v_n$$

$$\Rightarrow M^k v = c_1 \lambda_1^k v_1 + \cdots + \lambda_n^k v_n.$$

Suppose $\lambda_1 = 1$ & that 1 <u>dominates</u> the other eigenvalues in the sense that $\lambda_1 = 1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n > 0$.

Then $\lambda_r^k \to 0$ as $k \to \infty$ for $r > 1$.

$$\Rightarrow \lim_{k\to\infty} M^k v = c_1 v_1.$$

& note $M(c_1 v_1) = c_1 v_1$.

Thus the Markov process will always converge to <u>an $\lambda = 1$ eigenvector.</u>

e^A – Exponentiating Matrices

For a number $x \in \mathbb{R}$,

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$e^x = \sum_{n=0}^{\infty} x^n/n!$$

Euler generalized e^x to allow complex numbers in the exponent.

$$e^{i\theta} = 1+\frac{i\theta}{1!}+\frac{i^2\theta^2}{2!}+\frac{i^3\theta^3}{3!}+\frac{i^4\theta^4}{4!}+\cdots$$

$$= 1+\frac{i\theta}{1!}-\frac{\theta^2}{2!}-\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{i\theta^5}{5!}-\frac{\theta^6}{6!}\pm\cdots$$

$$= \underbrace{\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}\pm\cdots\right)}_{\cos(\theta)} + i\underbrace{\left(\frac{\theta}{1!}-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}\pm\cdots\right)}_{\sin(\theta)}$$

$$e^{i\theta} = \cos(\theta)+i\sin(\theta)$$

We generalize further to (A an $n\times n$ matrix)

$$e^A = I+\frac{A}{1!}+\frac{A^2}{2!}+\frac{A^3}{3!}+\cdots$$

Some observations:

1° $$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I+\frac{At}{1!}+\frac{A^2t^2}{2!}+\cdots\right)$$

$$= \frac{A}{1!}+\frac{2A^2t}{2!}+\frac{3A^3t^2}{3!}+\cdots$$

$$= A\left(I+\frac{At}{1!}+\frac{A^2t^2}{2!}+\cdots\right) = Ae^{At}$$

Then $\frac{d}{dt}(e^{At}) = Ae^{At}$

$\Rightarrow$ If $y(t)$ is a column of e^{At}, then

$$y'(t) = Ay.$$

So e^{At} solves diffl eqns!

2° Example. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Then $A^2 = -I$ and so A behaves like $i = \sqrt{-1}$.

$$\Rightarrow e^{At} = \cos(t)I + \sin(t)A$$
$$= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

3° Since we have the Cayley Hamilton Theorem, we can find out a lot about e^{At} by using $C_A(A) = 0$.

What can you say about $e^{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]t}$?