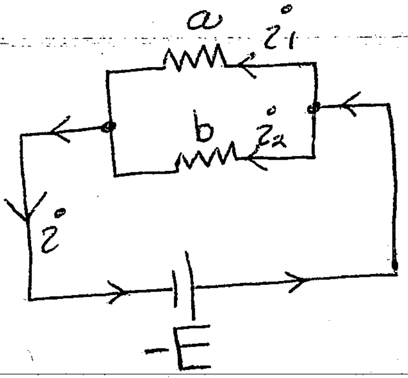
Electrical Circuits + Matrices ①

```
MM := {
   {-1, 1, 1, 0},
   {0, a, 0, E},
   {0, 0, b, E}
  }
MatrixForm[MM]
MatrixForm[RowReduce[MM]]
```



$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & a & 0 & e \\ 0 & 0 & b & e \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{a e+b e}{a b} \\ 0 & 1 & 0 & \frac{e}{a} \\ 0 & 0 & 1 & \frac{e}{b} \end{pmatrix}$$

$$R = \text{Simplify}\left[E/\left(\frac{a e+b e}{a b}\right)\right]$$

$$\frac{a b}{a+b}$$

loop equations

$$a i_1 = E$$
$$b i_2 = E$$

nodal equations

$$i = i_1 + i_2$$

$$\begin{array}{ccc|c} i & i_1 & i_2 & \\ \hline -1 & 1 & 1 & 0 \\ 0 & a & 0 & E \\ 0 & 0 & b & E \end{array}$$

$a \neq 0$
and
$b \neq 0$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & a & 0 & E \\ 0 & 0 & b & E \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & a & 0 & E \\ 0 & 0 & b & E \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & E/a \\ 0 & 0 & 1 & E/b \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & +E/a \\ 0 & 1 & 0 & E/a \\ 0 & 0 & 1 & E/b \end{array}\right]$$

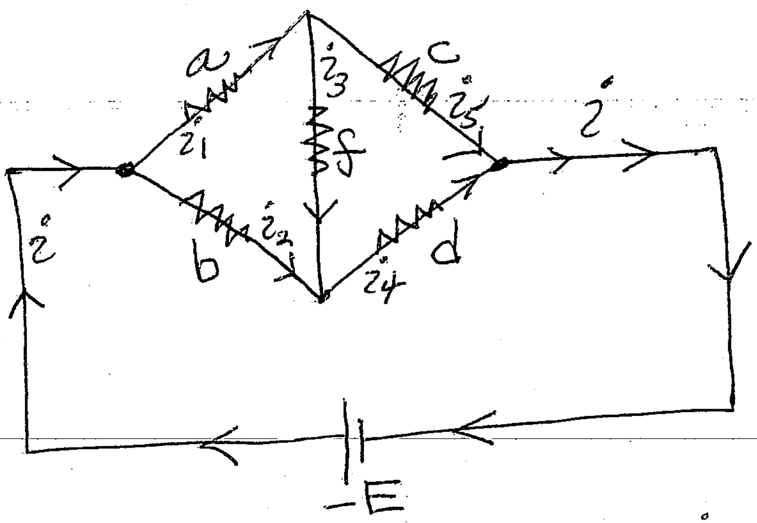
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & +E/a+E/b \\ 0 & 1 & 0 & E/a \\ 0 & 0 & 1 & E/b \end{array}\right]$$

and

$$\frac{E}{a}+\frac{E}{b}=E\left(\frac{a+b}{ab}\right)$$

$$i = E\left(\frac{a+b}{ab}\right) \Rightarrow i\left(\frac{ab}{a+b}\right) = E \Rightarrow \boxed{\begin{array}{c} R = \frac{ab}{a+b} \\ \text{effective resistance} \end{array}}$$

```
M := {
   {-1, 1, 1, 0, 0, 0, 0},
   {0, -1, 0, 1, 0, 1, 0},
   {0, 0, 1, 1, -1, 0, 0},
   {-1, 0, 0, 0, 1, 1, 0},
   {0, a, 0, 0, 0, c, E},
   {0, 0, b, 0, d, 0, E},
   {0, a, 0, f, d, 0, E},
   {0, 0, b, -f, 0, c, E}
  }
MatrixForm[M]
MatrixForm[RowReduce[M]]
```



$$\begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & a & 0 & 0 & 0 & c & e \\ 0 & 0 & b & 0 & d & 0 & e \\ 0 & a & 0 & f & d & 0 & e \\ 0 & 0 & b & -f & 0 & c & e \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{ace+bce+ade+bde+aef+bef+cef+def}{abc+abd+acd+bcd+abf+bcf+adf+cdf} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{bce+bde+bef+def}{abc+abd+acd+bcd+abf+bcf+adf+cdf} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{ace+ade+aef+cef}{abc+abd+acd+bcd+abf+bcf+adf+cdf} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{bce-ade}{abc+abd+acd+bcd+abf+bcf+adf+cdf} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{ace+bce+aef+cef}{abc+abd+acd+bcd+abf+bcf+adf+cdf} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{ade+bde+bef+def}{abc+abd+acd+bcd+abf+bcf+adf+cdf} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$i = i_1 + i_2$

$i_1 = i_3 + i_5$

$i_2 + i_3 = i_4$

$i_4 + i_5 = i$

nodes

$a i_1 + c i_5 = E$

$b i_2 + d i_4 = E$

$a i_1 + f i_3 + d i_4 = E$

$b i_2 - f i_3 + c i_5 = E$

loops

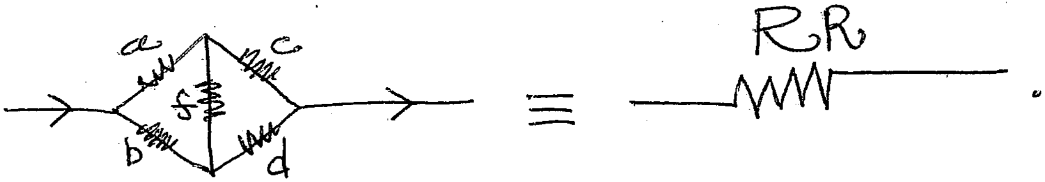
In[22]:=

```
RR = Simplify[E/((a c e + b c e + a d e + b d e + a e f + b e f + c e f + d e f)/(a b c + a b d + a c d + b c d + a b f + b c f + a d f + c d f))]
```

Out[22]=

$$\frac{c\,(d\,f + b\,(d + f)) + a\,(d\,(c + f) + b\,(c + d + f))}{(c + d)\,f + a\,(c + d + f) + b\,(c + d + f)}$$

This is the effective resistance of



```
In[59]:= S = a+b + f
         f ' = ab /S
         b ' = af /S
         a ' = bf /S
         RRR = Simplify[Expand[Simplify[f ' + (b ' + c ) (a ' + d) / (a ' + b ' + d + c)]]]
         Simplify[Expand[RRR / RR]]
```

Out[59]= $a+b+f$

Out[60]= $\frac{ab}{a+b+f}$

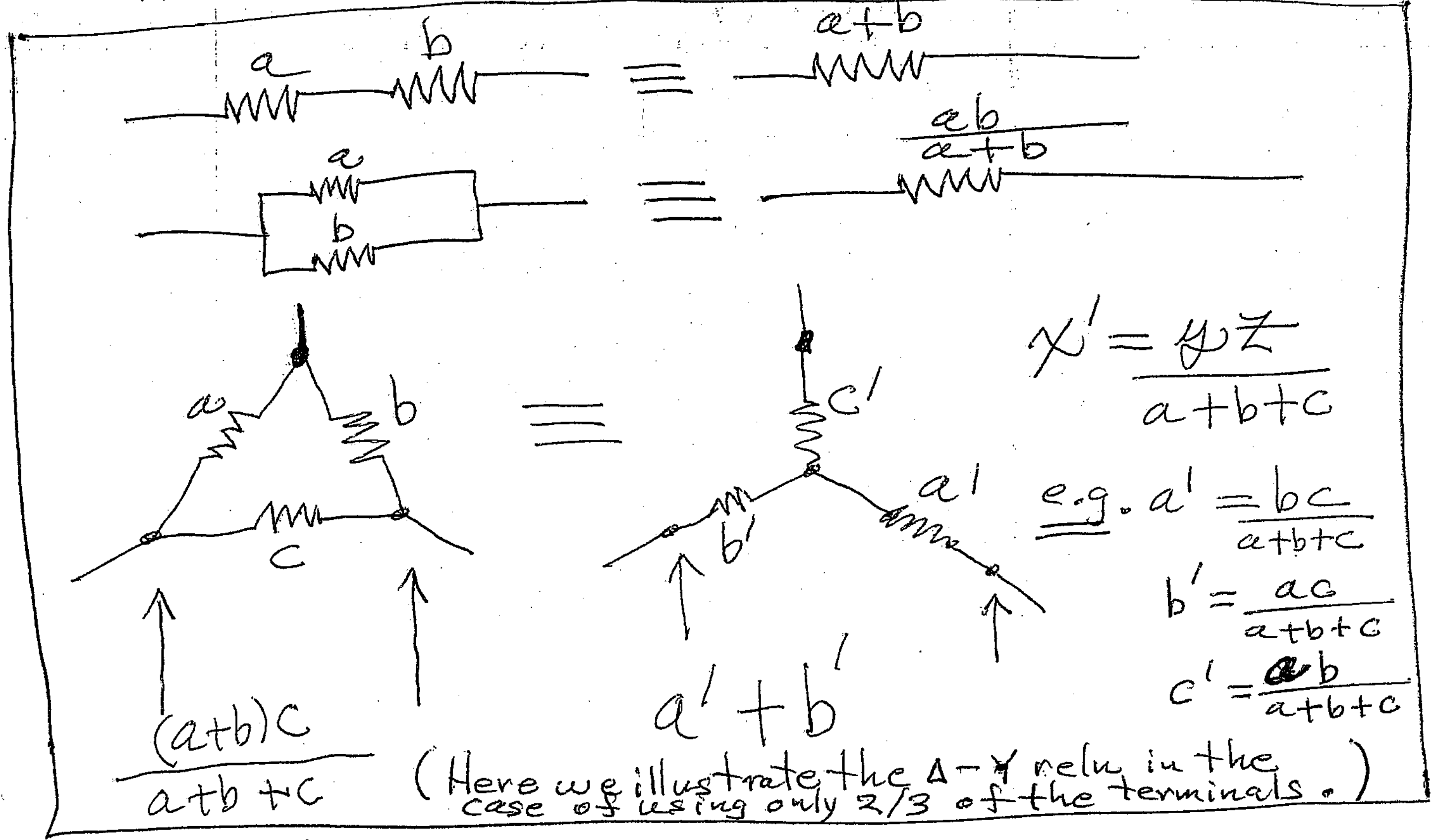
Out[61]= $\frac{af}{a+b+f}$

Out[62]= $\frac{bf}{a+b+f}$

Out[63]= $\frac{c\,(d\,f+b\,(d+f))+a\,(d\,(c+f)+b\,(c+d+f))}{(c+d)\,f+a\,(c+d+f)+b\,(c+d+f)}$

Out[64]= 1

This page illustrates how the effective resistance computed on page ② can be obtained just by using Series, Parallel and Star-Triangle relations.



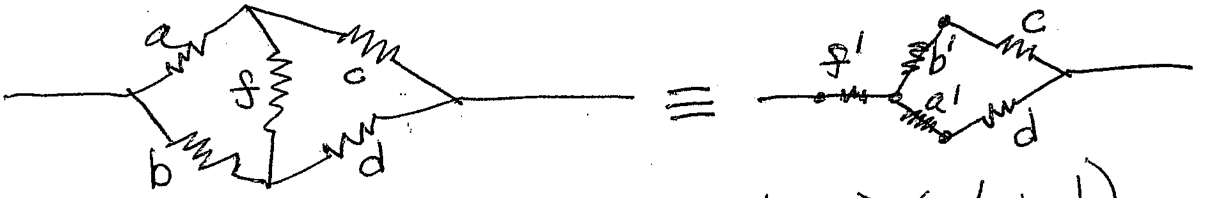
$$x' = \frac{yz}{a+b+c}$$

e.g. $a' = \frac{bc}{a+b+c}$

$$b' = \frac{ac}{a+b+c}$$

$$c' = \frac{ab}{a+b+c}$$

(Here we illustrate the Δ-Y reln in the case of using only 2/3 of the terminals.)



$$\Rightarrow \quad R = f' + \frac{(b'+c)(a'+d)}{a'+b'+c+d}$$

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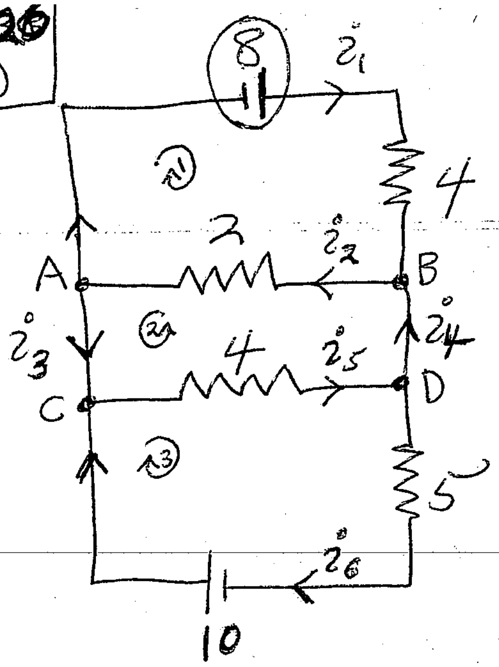
22.(c)



A: $i_2 = i_1 + i_3$

B: $i_2 = i_1 + i_4$

C: $i_5 = i_3 + i_6$

D: $i_5 = i_4 + i_6$

①: $-8 + 4i_1 + 2i_2 = 0$

②: $2i_2 + 4i_5 = 0$

③: $4i_5 + 5i_6 - 10 = 0$

(Homework Solution)

////	i_1	i_2	i_3	i_4	i_5	i_6	
A	1	−1	1	0	0	0	0
B	1	−1	0	1	0	0	0
C	0	0	1	0	−1	1	0
D	0	0	0	1	−1	1	0
①	4	2	0	0	0	0	+8
②	0	2	0	0	4	0	0
③	0	0	0	0	4	5	10

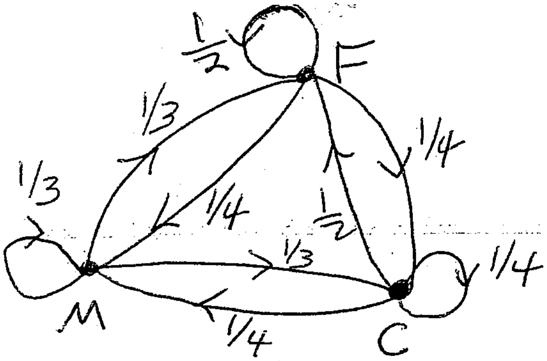
$\xrightarrow{}$ row reduces to

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$i_1 = 2$ $i_4 = -2$

$i_2 = 0$ $i_5 = 0$

$i_3 = -2$ $i_6 = 2$



Formulating the Economics Example Using Matrices

$$F' = \tfrac{1}{2}F + \tfrac{1}{3}M + \tfrac{1}{2}C$$

$$M' = \tfrac{1}{4}F + \tfrac{1}{3}M + \tfrac{1}{4}C$$

$$C' = \tfrac{1}{4}F + \tfrac{1}{3}M + \tfrac{1}{4}C$$

$$\left.\begin{pmatrix}F'\\M'\\C'\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{3} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}F\\M\\C\end{pmatrix}\right\}$$

$$\vec{v} = \begin{pmatrix}F\\M\\C\end{pmatrix}$$

$$A = \begin{pmatrix}\frac{1}{2} & \frac{1}{3} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4}\end{pmatrix}$$

$$I = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\boxed{\vec{v}\,' = A\vec{v}}$$

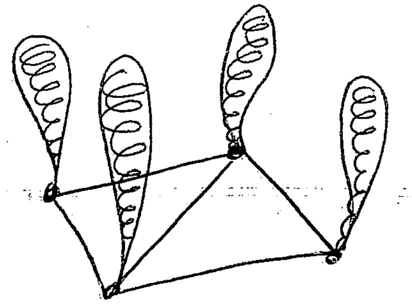
$$\text{Equilibrium} \Longleftrightarrow \vec{v}\,' = \vec{v}$$

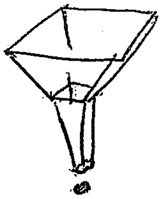
$$\Updownarrow$$

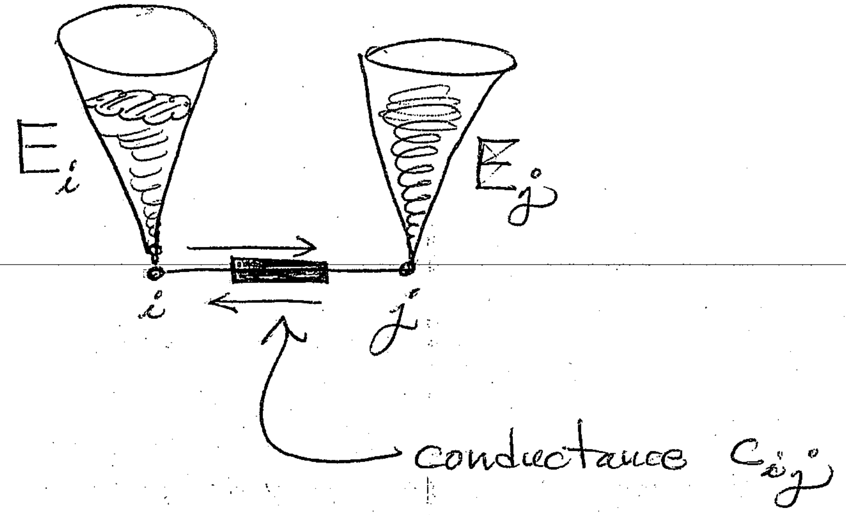
$$\vec{v} = A\vec{v} \Longleftrightarrow A\vec{v} = I\vec{v}$$

$$(A - I)\vec{v} = 0$$

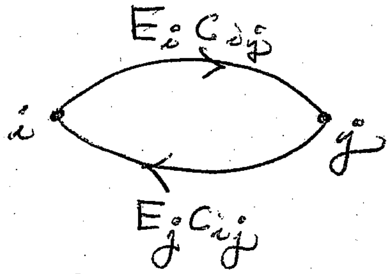
$$\boxed{A - I = \begin{pmatrix}-1/2 & 1/3 & 1/2\\ 1/4 & -2/3 & 1/4\\ 1/4 & 1/3 & -3/4\end{pmatrix}}$$







$$\boxed{\begin{array}{c} i = EC \\ \underline{\text{current}} \end{array}}$$



$$E_j'$$

$$E_j' = E_j + \sum_{\langle ij \rangle} c_{ij}[E_i - E_j]$$

Equilibrium

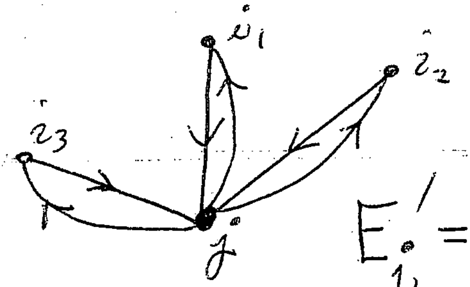
$$\boxed{\mathcal{K}E = 0}$$

$$E' = E + \mathcal{K}E$$

$$\boxed{\begin{array}{l} \mathcal{K}_{ii} = \sum_{\langle ij \rangle} c_{ij} \\ \mathcal{K}_{ij} = -c_{ij} \quad i \neq j \end{array}}$$

$$(\mathcal{K}E)_j = \sum_i \mathcal{K}_{ji} E_i = \mathcal{K}_{jj} E_j + \sum_{i \neq j} \mathcal{K}_{ji} E_i$$

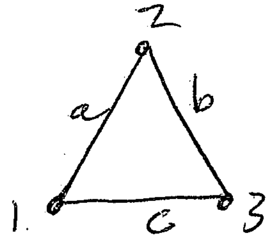
$$= \sum_{\langle ij \rangle} c_{ij} E_j + \sum_{i \neq j} -c_{ij} E_i$$



$$E_j' = E_j + c_{i_1 j}(E_{i_1} - E_j) + c_{i_2 j}(E_{i_2} - E_j) + \cdots + c_{i_n j}(E_{i_n} - E_j)$$

$$= E_j + \underbrace{\sum_{\langle ij\rangle} c_{ij} E_i \cancel{+} - \Big(\sum_{\langle ij\rangle} c_{ij}\Big) E_j}$$

$$\boxed{\begin{aligned} K_{ii} &= -\sum_{\langle ij\rangle} c_{ij} \\ K_{ij} &= c_{ij} \end{aligned}}$$



$$E_1' = E_1 + a(E_2 - E_1) + c(E_3 - E_1)$$
$$E_2' = E_2 + a(E_1 - E_2) + b(E_3 - E_2)$$
$$E_3' = E_3 + \underbrace{b(E_2 - E_3) + C(E_1 - E_3)}$$

$$\Delta E_1 = (-a-c)E_1 + aE_2 + cE_3$$
$$\Delta E_2 = aE_1 + (-a-b)E_2 + bE_3$$
$$\Delta E_3 = cE_1 + bE_2 + (-b-c)E_3$$

$$\Delta E = \begin{pmatrix} -a-c & a & c \\ a & -a-b & b \\ c & b & -c-b \end{pmatrix} E$$

③✓

Row Reduction Example

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 2 & -2 & 1 & 2 & 8 \\ 3 & 1 & 2 & -1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -1 & 0 & 2 \\ 0 & -8 & -1 & -4 & -10 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5/8 & 1 & 3+\frac{3}{4} \\ 0 & -8 & -1 & 0 & 2 \\ 0 & 0 & 0 & -4 & -12 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5/8 & 1 & 15/4 \\ 0 & 1 & 1/8 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{ccccc} \textcircled{x} & \textcircled{y} & z & \textcircled{w} & \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5/8 & 0 & 3/4 \\ 0 & 1 & 1/8 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \leftarrow$$

$$\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \end{array}$$

$$\boxed{z = \alpha}$$

This is in row reduced echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

$$x = -\frac{5}{8}\alpha + 3/4$$
$$y = -1/8\,\alpha - 1/4$$
$$z = \alpha$$
$$w = 3$$

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A matrix is said to be in row echelon form if

(i) $1^{st} \neq 0$ entry of each $\neq 0$ row is $\underline{1}$.

(ii) If $\vec{r}_k \neq \vec{0}$ then the number of leading zero entries in $\vec{r}_{k+1}$ is greater than the number of leading zero entries in r_k.

(iii) All zero rows are below the non-zero rows.

---

e.g. $\left[\begin{array}{cccc|c} 1 & 0 & 7 & 0 & 9 \\ 0 & 1 & 8 & 0 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$ is in row-echelon form.

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

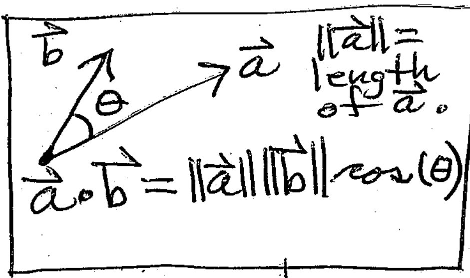
$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \underbrace{a_1b_1 + a_2b_2 + \cdots + a_nb_n}$$

same n

This is sometimes called the <u>dot product</u> of two vectors. If

$$\vec{a} = (a_1, a_2, \ldots, a_n)$$
$$\vec{b} = (b_1, b_2, \ldots, b_n)$$

then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$



$\|\vec{a}\|$ = length of $\vec{a}$.

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos(\theta)$

If $\vec{a}$ is a row vector and $\vec{b}$ is a column vector, then we write just $\vec{a}\vec{b}$ as above. e.g. $\vec{a} = (1, 2, 3)$

$$\vec{b} = \begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$$

then
$$\vec{a}\vec{b} = (1\ 2\ 3)\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix} = 1\cdot 3 + 2\cdot 7 + 3\cdot 9$$
$$= 3 + 14 + 27$$
$$= 44.$$

General Matrix Multiplication

$$\overset{\text{row}}{\downarrow}\quad a_{ij} \leftarrow \text{column}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$ an $n \times m$ matrix.

A has n rows and m columns.

We may write $A = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ r_n \end{pmatrix}$ or $A = (\vec{c}_1\ \vec{c}_2 \dots \vec{c}_m)$

e.g. $\vec{r}_2 = (a_{21}\ a_{22} \dots a_{2m})$

$$\vec{c}_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{n3} \end{pmatrix}.$$

Suppose $A = \begin{pmatrix} \vec{r}_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ is $n \times n$.

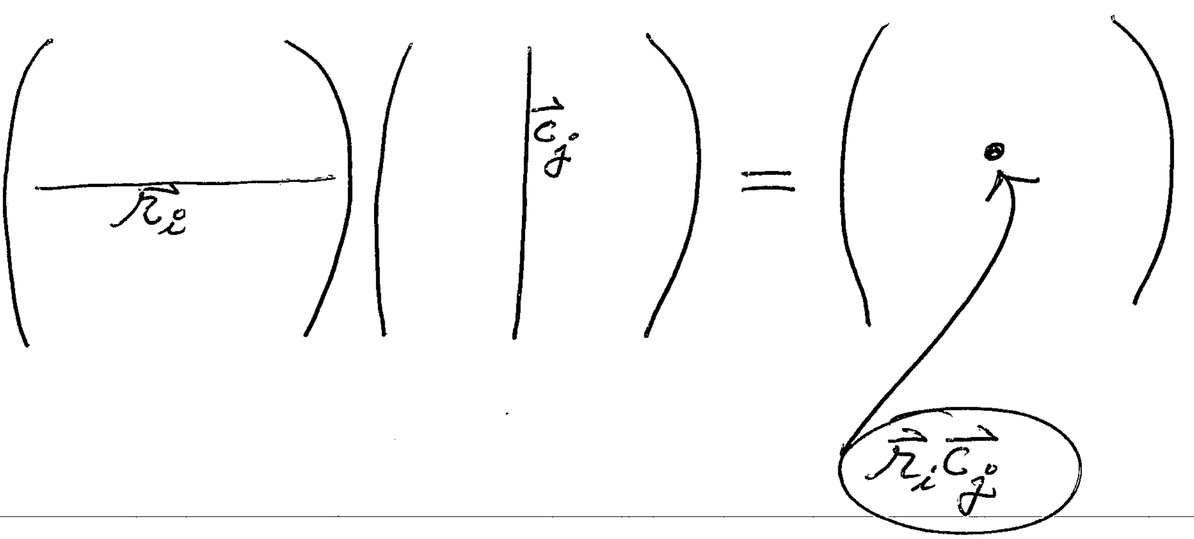
& $B = (\vec{c}_1\ \vec{c}_2 \dots \vec{c}_n)$ is $n \times n$.

We define a new matrix

$C = AB$ by $c_{ij} = \vec{r}_i \vec{c}_j$.

e.g. $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 1\cdot1+2\cdot1 & 1\cdot1+2\cdot7 \\ 3\cdot1+4\cdot1 & 3\cdot1+4\cdot7 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 15 \\ 7 & 31 \end{pmatrix}.$$



Matrices are generalized numbers.

We say A $n\times m$ and B $n'\times m'$ are the "same size" if $n=n'$ & $m=m'$.

Then $(A+B)_{ij} = A_{ij}+B_{ij}$ makes sense.

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 7 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 10 \\ 9 & 4 & 5 \end{pmatrix}.$$

If A in $n\times \textcircled{m}$ and B is $\textcircled{m}\times p$

then AB is $n\times p$. e.g.

$$\underset{\boxed{2\times 3}}{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}} \underset{\boxed{3\times 2}}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}} = \begin{pmatrix} 1\cdot 1+1\cdot 2+1\cdot 3 & 1\cdot 2+1\cdot 1+1\cdot 4 \\ 2\cdot 1+3\cdot 2+4\cdot 3 & 2\cdot 2+3\cdot 1+4\cdot 4 \end{pmatrix}$$

$$= \underset{\boxed{2\times 2}}{\begin{pmatrix} 6 & 7 \\ 20 & 23 \end{pmatrix}}.$$

Note: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

$I_n = \begin{pmatrix} 1 & & & \bigcirc \\ & 1 & & \\ & & 1 & \\ \bigcirc & & & \ddots \\ & & & & 1 \end{pmatrix}$ is called the $n \times n$ *identity matrix*.

Note: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

So it can happen that $AB \neq BA$.

Matrix multiplication is not commutative.

However. It is true that matrix multiplication is *associative*.

$A_{n\times m}$, $B_{m\times p}$, $C_{p\times q}$

$\Rightarrow (AB)C = A(BC).$

We will prove this later.

Note. If $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then

$$J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{J^2 = -I} \qquad \boxed{7\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \underset{ex}{=} \begin{pmatrix} 7 & 14 \\ 21 & 28 \end{pmatrix}}$$

The matrix J is a "square root of -1"

and so matrices can be used to model and make sense of complex numbers like $3+4i$ where $i = \sqrt{-1}$. We will come back to this. $a+bi \Longleftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = aI + bJ$.

Note. A system of linear equations

$$ax + by = e$$
$$cx + dy = f$$

can be expressed as a matrix equation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\boxed{M\vec{x} = \vec{k}}$$

$$\boxed{M\vec{x} = \vec{k}}$$

Can we find "M^{-1}" such that

$M^{-1}M = I$?

If so, then

$$M^{-1}(M\vec{x}) = M^{-1}\vec{k}$$
$$(M^{-1}M)\vec{x} = M^{-1}\vec{k}$$
$$I\vec{x} = M^{-1}\vec{k}$$
$$\boxed{\vec{x} = M^{-1}\vec{k}}$$

We could use matrix multiplication to solve the system of equations.

<u>Note</u>: $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$$= (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (ad-bc)\,I$$

Thus, if $\underline{ad-bc \neq 0}$, then

$$M^{-1} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>and</u>

$$\boxed{\begin{vmatrix}a&b\\c&d\end{vmatrix} \overset{\text{def}}{=} ad-bc}$$

If $ad-bc \neq 0$, then

given $\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}e\\f\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\begin{pmatrix}e\\f\end{pmatrix}$$

$$=\begin{pmatrix}(de-bf)/(ad-bc)\\(-ce+af)/(ad-bc)\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\begin{vmatrix}e&b\\f&d\end{vmatrix}\Big/\begin{vmatrix}a&b\\c&d\end{vmatrix}\\ \begin{vmatrix}a&e\\c&f\end{vmatrix}\Big/\begin{vmatrix}a&b\\c&d\end{vmatrix}\end{pmatrix}$$

This is called "Cramer's Rule".

For example,

$$\begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}7\\9\end{pmatrix}$$

$M=\begin{pmatrix}2&3\\1&2\end{pmatrix}$

$|M|=4-3=1$

$M^{-1}=\begin{pmatrix}2&-3\\-1&2\end{pmatrix}$

$$\Longrightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\begin{pmatrix}7\\9\end{pmatrix}=\begin{pmatrix}14-27\\-7+18\end{pmatrix}=\begin{pmatrix}-13\\11\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-13\\11\end{pmatrix}.$$

Finding inverses by row-reduction

If M is $n \times n$ and M row reduces to I_n then M has an inverse.
If you row-reduce $(M|I_n)$ you will get $(I_n|M^{-1})$.

Example:

$$\left(\underbrace{\begin{matrix}1&2\\3&4\end{matrix}}_{M}\middle|\begin{matrix}1&0\\0&1\end{matrix}\right) \longrightarrow \left(\begin{array}{cc|cc}1&2&1&0\\0&-2&-3&1\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc}1&0&-2&1\\0&-2&-3&1\end{array}\right) \longrightarrow \left(\begin{array}{cc|cc}1&0&-2&1\\0&1&3/2&-1/2\end{array}\right)$$

Check: $\begin{pmatrix}-2&1\\3/2&-1/2\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ ✓

Example: $M = \begin{pmatrix}1&1&2\\1&3&1\\2&1&1\end{pmatrix}$

$$\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\1&3&1&0&1&0\\2&1&1&0&0&1\end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&2&-1&-1&1&0\\0&-1&-3&-2&0&1\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&1&3&2&0&-1\\0&2&-1&-1&1&0\end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc}1&0&-1&-1&0&1\\0&1&3&2&0&-1\\0&0&-7&-5&1&2\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc}1&0&-1&-1&0&1\\0&1&3&2&0&-1\\0&0&1&5/7&\frac{-1}{7}&\frac{-2}{7}\end{array}\right) \longrightarrow \left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\middle|\underbrace{\begin{matrix}-2/7&-1/7&5/7\\-1/7&3/7&-1/7\\5/7&-1/7&-2/7\end{matrix}}_{M^{-1}}\right)$$

Continued from previous page.

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} -2/7 & -1/7 & 5/7 \\ -1/7 & 3/7 & -1/7 \\ 5/7 & -1/7 & -2/7 \end{pmatrix}$$

Problem. Solve

$$\begin{aligned} x+y+2z &= 1 \\ x+3y+z &= 1 \\ 2x+y+z &= 1 \end{aligned}$$

Solution: $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\Longrightarrow \quad M^{-1}M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/7 & -1/7 & 5/7 \\ -1/7 & 3/7 & -1/7 \\ 5/7 & -1/7 & -2/7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/7 \\ 1/7 \\ 2/7 \end{pmatrix}$$

Return to Matrix Multiplication

Notation: $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$

e.g. $\sum_{k=1}^{3} k = 1+2+3 = 6$

When we multiply AB where A is $n \times m$ and B is $m \times p$, we have

$$(AB)_{ij} = r_i(A)\,c_j(B)$$

$$= (a_{i1}\ a_{i2} \cdots a_{im})\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

$$\boxed{(AB)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}}$$

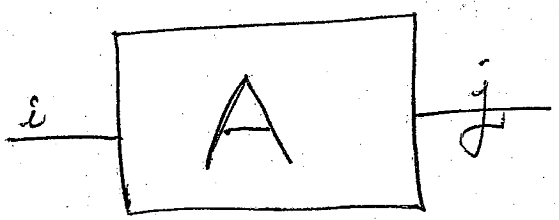
Note the corresponding indices.

We can also write

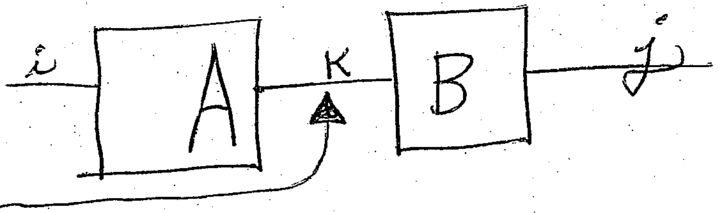
$$(AB)_{ij} = \sum_{K} A_{iK} B_{Kj}$$

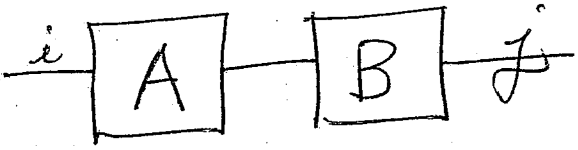
where we understand that K runs from 1 to m if A is n×m and B is m×p.

Another Notation



$\equiv A_{ij}$

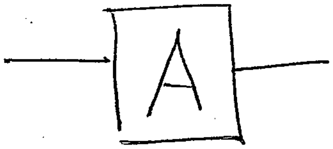
Then 

or just 

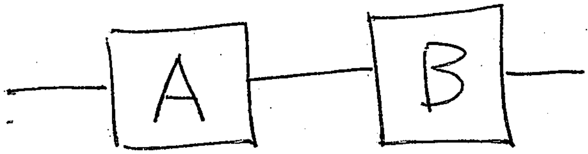
means $\sum_{K} A_{iK} B_{Kj}$

We sum over all indices on the connecting edge.

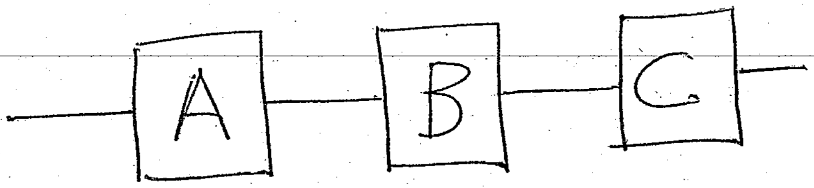
So



$\Longleftrightarrow A$



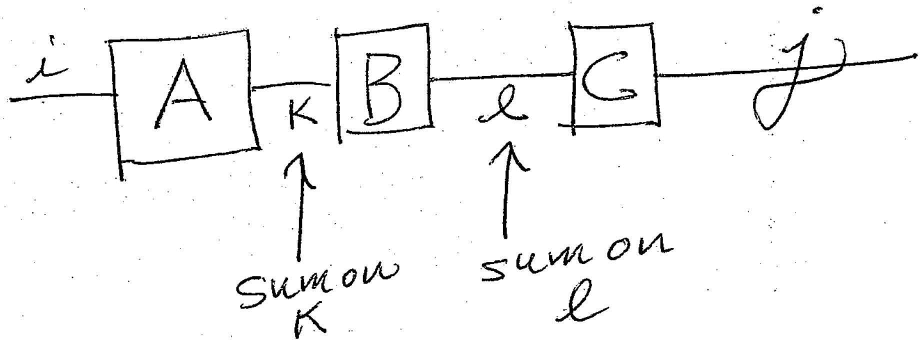
$\Longleftrightarrow AB$



$\Longleftrightarrow ABC$

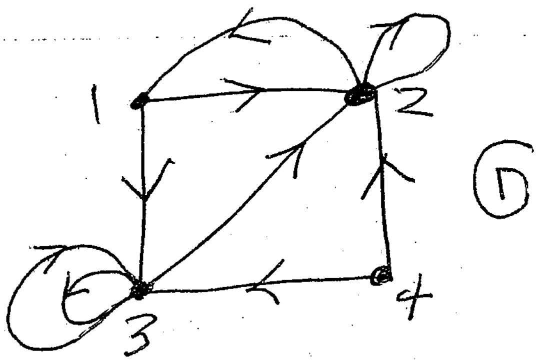
Say $A_{n\times m}$, $B_{m\times p}$, $C_{p\times r}$

Then $(ABC)_{ij} = \sum_{k,l} A_{ik} B_{kl} C_{lj}$



From this you can <u>see</u> why matrix multiplication is associative.

Adjacency Matrix of a Graph



Directed

Graph G with vertices (nodes) $\{1, 2, 3, 4\}$

$A = A(G)$ Adjacency Matrix of G

A_{ij} = # of directed edges in G from i to j.

$$A = \begin{array}{c|c|c|c|c|} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ \hline 2 & 1 & 1 & 0 & 0 \\ \hline 3 & 0 & 1 & 2 & 0 \\ \hline 4 & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

Theorem. If $A = A(G)$ the adjacency matrix of a directed graph G, then $(A^n)_{ij}$ is equal to the number of directed walks of length n on G from node i to node j.

Example 1. 

①

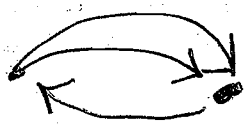
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...

Example 2. 

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

...

Example 3. 

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$$

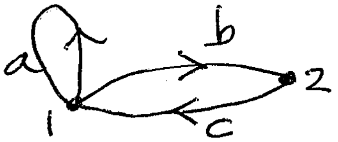
...

Note the appearance of Fibonacci Numbers

$$\underset{f_1}{\overset{1}{\parallel}}, \underset{f_2}{\overset{1}{\parallel}}, \underset{f_3}{\overset{2}{\parallel}}, \underset{f_4}{\overset{3}{\parallel}}, \underset{f_5}{\overset{5}{\parallel}}, \underset{f_6}{\overset{8}{\parallel}}, \underset{f_7}{\overset{13}{\parallel}}, \underset{f_8}{\overset{21}{\parallel}}, \underset{f_9}{\overset{34}{\parallel}}, \ldots$$

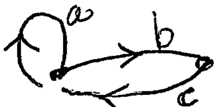
$$\boxed{\begin{gathered} f_1 = f_2 = 1 \\ f_{n+1} = f_n + f_{n-1} \end{gathered}}$$

Example 4. You can actually do much more. Label each edge of the graph with a letter.



①

Then a walk is an ordered sequence of edges of ①. For example

abc means


and corresponds to a walk
of length 3 from ① to ①.

Form a generalized adjacency matrix $B=\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ where

B_{ij} = sum of the labels for edges going from i to j.

Now you will find that if you form B^n and keep track of orders of multiplicatn of labels, then

$(B^n)_{ij}$ = "sum" of all walks of length n from i to j.

e.g. $B^2=\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}=\begin{pmatrix} a^2+bc & ab \\ ca & cb \end{pmatrix}$

$B^3=\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a^2+bc & ab \\ ca & cb \end{pmatrix}=\begin{pmatrix} a^3+abc+bca & a^2b+bcb \\ ca^2+cbc & cab \end{pmatrix}$

Note how each term is a specific walk.