Elementary Matrices

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c+ra & d+rb \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a & 3b \\ c & d \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & K \end{pmatrix} = \begin{pmatrix} a & b & c \\ d+sa & e+sb & f+sc \\ g & h & K \end{pmatrix}$$

Let $E_{ij} = \begin{pmatrix} 1 & & & \\ \bullet & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ one entry s in the ij place, $i \neq j$

$$\Rightarrow E_{ij}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_i{}' = \vec{r}_i + s\vec{r}_j \\ \vdots \\ \vec{r}_n \end{pmatrix}$$

When $i = j$, we assume $s \neq 0$.

For example, $E_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (4×4)

$$E_{32}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vec{r}_4 \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 + s\vec{r}_2 \\ \vec{r}_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix} = \begin{pmatrix} \vec{r}_2 \\ \vec{r}_3 \\ \vec{r}_1 \end{pmatrix}$$

Permutation Matrices are also <u>elementary</u>.

All row operations can be accomplished by multiplation (on the left) by elementary matrices.

Example: $\begin{pmatrix}1&2\\2&5\end{pmatrix}\xrightarrow[r_2\mapsto r_2-2r_1]{}\begin{pmatrix}1&2\\0&1\end{pmatrix}$

$$\boxed{\underbrace{\begin{pmatrix}1&0\\-2&1\end{pmatrix}}_{=E_1}\begin{pmatrix}1&2\\2&5\end{pmatrix}=\begin{pmatrix}1&2\\0&1\end{pmatrix}}$$

$$\begin{pmatrix}1&2\\0&1\end{pmatrix}\xrightarrow[r_1\mapsto r_1-2r_2]{}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\boxed{\overbrace{\begin{pmatrix}1&-2\\0&1\end{pmatrix}}^{E_2=}\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.}$$

Note $\underbrace{\left[\begin{pmatrix}1&-2\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-2&1\end{pmatrix}\right]}_{E_2E_1=}\begin{pmatrix}1&2\\2&5\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$

$$\begin{pmatrix}5&-2\\-2&1\end{pmatrix}=\begin{pmatrix}1&2\\2&5\end{pmatrix}^{-1}$$

If a sequence of elementary matrices $E_1, E_2, \ldots, E_K$ transforms M into row echelon form R, then $(E_K E_{K-1}\cdots E_1)M = R$. If $R=I$, then $(E_K\cdots E_1)=M^{-1}$.

6'

$$E = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-7} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

21 entry

$$r_2 \longmapsto r_{②} - 7 r_{①}$$

$$EM = \begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 7 & 9 & 11 \\ 4 & 5 & 6 \end{pmatrix}$$

What is E^{-1}? $\underline{E^{-1}E = I}$

$$\underline{E^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

Example.

$$M=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\xrightarrow[r_2\to r_2-r_1]{}\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}$$

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}=E_1$$

$$\xrightarrow{r_3\to r_3-r_1}\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}\qquad \begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}=E_2$$

$$\xrightarrow{r_3\to r_3-r_2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}$$

$$\Rightarrow M^{-1}=E_3E_2E_1=\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\-1&1&0\\0&-1&1\end{pmatrix}.$$

Check: $\begin{pmatrix}1&0&0\\-1&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ ✓

Exercise: What row operation is encoded by

$$\begin{bmatrix}1&0&0&0\\0&1&0&-2\\0&0&1&0\\0&0&0&1\end{bmatrix}?$$

c)

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}$$

$$E_3 E_2 E_1 A = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix} = U$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad r_2 \to r_2 - \tfrac{1}{2} r_1$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad r_3 \to r_3 - 2 r_1$$

$$E_2 E_1 A = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{pmatrix} \xrightarrow[r_3 \to r_3 + 3 r_2]{} \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$E_3 E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

C''

$$E_3E_2E_1A = U$$

$$\Rightarrow \quad E_2E_1A = E_3^{-1}U$$

$$E_1A = E_2^{-1}E_3^{-1}U$$

$$\underline{A = E_1^{-1}E_2^{-1}E_3^{-1}U}$$

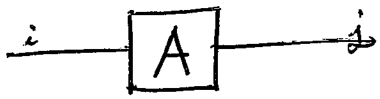
$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} = L$$

Thus
$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}}_{A}$$

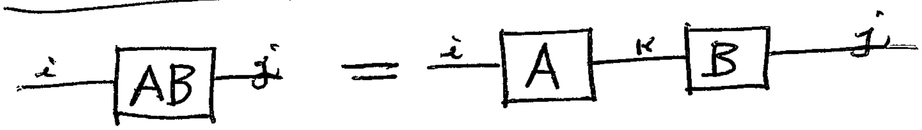
More Diagrammatics

A : 

A^T : 

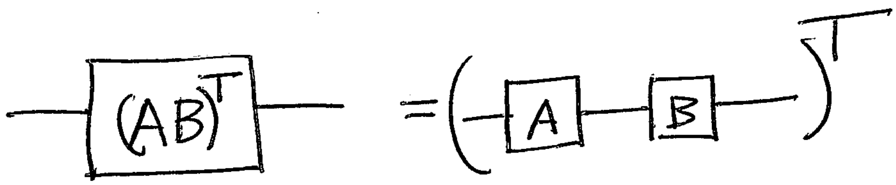
(rotate A by 180°)

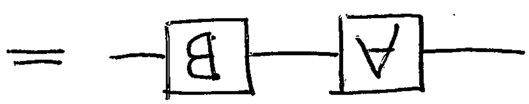
$$(A^T)_{ij} = A_{ji} \,,\ (A^T)_{ji} = A_{ij}$$



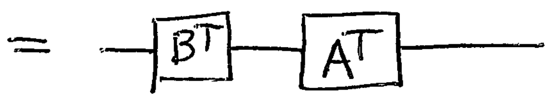
$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

Theorem. $(AB)^T = (B^T)(A^T)$.

Proof. 



(rotate by 180°)



$$\Rightarrow (AB)^T = B^T A^T \quad \text{QED}$$

Compare with

$$(AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$$

$$= \sum_k A^T_{kj} B^T_{ik} = \sum_k B^T_{ik} A^T_{kj} = (B^T A^T)_{ij}$$

$$\therefore (AB)^T = B^T A^T \quad //$$

e

$|AB| = |A||B|$

For 2×2 Matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$|AB| = (ae+bg)(cf+dh) - (af+bh)(ce+dg)$$

$$= aecf + aedh - afce - afdg$$
$$+ bgcf + bgdh - bhce - bhdg$$

$$= aedh + bgcf - afdg - bhce$$

$$|A||B| = (ad-bc)(eh-fg)$$
$$= adeh + bcfg - bceh - adfg$$

$$\therefore \quad |AB| = |A||B|$$

Determinants : $\boxed{2\times 2 \text{ dets defined by } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc}$

Calculational definition.

Suppose $(n-1)\times(n-1)$ dets already defined. If M is $n\times n$ then the *minors* $M(i,j)$ are the matrices obtained from M by *eliminating* *row i* + *col j*.

$$M=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 9 \\ 3 & 4 & 5 \end{pmatrix}: \quad M(1,1)=\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{2} & 7 & 9 \\ \cancel{3} & 4 & 5 \end{pmatrix}=\begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix}$$

$$M(2,2)=\begin{pmatrix} 1 & \cancel{2} & 3 \\ \cancel{2} & \cancel{7} & \cancel{9} \\ 3 & \cancel{4} & 5 \end{pmatrix}=\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$$

$$M(2,3)=\begin{pmatrix} 1 & 2 & \cancel{3} \\ \cancel{2} & \cancel{7} & \cancel{9} \\ 3 & 4 & \cancel{5} \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

etc.

Then $|M| = m_{11}|M(1,1)| - m_{12}|M(1,2)|$
$(M\ n\times n)$ $\quad + m_{13}|M(1,3)|$
$\quad - m_{14}|M(1,4)|$
$\quad \pm \cdots + (-1)^{n+1}|M(1,n)|$

Using
$$\begin{pmatrix} + & - & + & - & + & - & + & \cdots \\ - & + & - & + & - & + & - & \cdots \\ + & - & + & - & + & - & + & \cdots \\ & & & \cdots & & & & \end{pmatrix}$$

you can compute the det by expanding <u>any</u> row by minors.

<u>Example.</u>
$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 7 & 9 \\ 3 & 4 & 5 \end{vmatrix} = 1 \cdot \begin{vmatrix} 7 & 9 \\ 4 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & 9 \\ 3 & 5 \end{vmatrix} + 3 \begin{vmatrix} 2 & 7 \\ 3 & 4 \end{vmatrix}$$
$$= (35-36) - 2(10-27) + 3(8-21)$$
$$= -1 - 2(-17) + 3(-13)$$
$$= -1 + 34 - 39$$
$$= -6$$

<u>or</u>
$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 7 & 9 \\ 3 & 4 & 5 \end{vmatrix} \underset{(2^{nd}\ row)}{=\!=} -2 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} + 7 \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} - 9 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$
$$= -2(10-12) + 7(5-9) - 9(4-6)$$
$$= -2(-2) + 7(-4) - 9(-2)$$
$$= 4 - 28 + 18$$
$$= -6$$

There is a better deffinition of determinants. Think of the matrix $(n \times n)$ as n rows of vectors of size n.

$$M = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix}$$

then $|M| = \mathrm{Det}(M) = D\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix}$

is a function of the n-rows that <u>satisfies the following properties</u>:

1) $D\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = 1 \quad (\mathrm{Det}(I_n) = 1)$

2) If you <u>interchange</u> any two rows, the det changes sign.

e.g. $D\begin{pmatrix} \vec{r}_2 \\ \vec{r}_1 \\ \vec{r}_3 \\ \vdots \\ \vec{r}_n \end{pmatrix} = -D\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vdots \\ \vec{r}_n \end{pmatrix}$

3) D is a <u>linear function of each row</u>. (go to next page)

A function $F(\vec{v})$ defined on vectors is said to be <u>linear</u> if (a) $F(\vec{v}+\vec{w}) = F(\vec{v}) + F(\vec{w})$

(b) $F(k\vec{v}) = k\,F(\vec{v})$

where $\vec{v}, \vec{w}$ are vectors and k is a constant.

Thus we mean that e.g.

$$D\begin{pmatrix}\vec{r}_1\\ \vec{r}_2 + k\vec{r}_2{}'\\ \vec{r}_3\end{pmatrix} = D\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\\ \vec{r}_3\end{pmatrix} + k\,D\begin{pmatrix}\vec{r}_1\\ \vec{r}_2{}'\\ \vec{r}_3\end{pmatrix}$$

These 3 rules allow us to calculate any det.

e.g. $D\begin{pmatrix}a & b\\ c & d\end{pmatrix} = D\begin{pmatrix}a(1,0)+b(0,1)\\ (c,d)\end{pmatrix}$

$$= a\,D\begin{pmatrix}1 & 0\\ c & d\end{pmatrix} + b\begin{pmatrix}0 & 1\\ c & d\end{pmatrix}$$

$$= a\left[c\,D\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + d\,D\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right] + b\left[c\,D\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + d\,D\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right]$$

But Consider $\left[\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \xrightarrow[\text{Interchage}]{} \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right] \Rightarrow D\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = 0$

and $D\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = 0$ also.

So $D\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad\,D\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bc\,D\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

But $D\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

and $D\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -D\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(by interchange).

So $D\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$!

as expected

<u>Note</u>: Interchange (rule 2) $\Rightarrow$ If M has <u>two equal rows</u>, then $|M| = \phi$.

$$\boxed{\begin{array}{c}\text{Why?} \\ \text{Because } D = -D \\ \Rightarrow D = \phi.\end{array}}$$

Linearity (rule 3) $\Rightarrow$ If M has <u>a zero row</u> then $|M| = \phi$.

(Why?)

(because $F(\vec{0}) = F(\vec{0} + \vec{0}) = F(\vec{0}) + F(\vec{0})$
$\Rightarrow \phi = F(\vec{0})$ if
F is linear.)

Row Operations

$$D\begin{pmatrix} k\vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix} = k\,D\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix} \quad \left(\text{same for any row}\right)$$

$$D\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 + k\vec{r}_1 \\ \vec{r}_3 \\ \vdots \\ \vec{r}_n \end{pmatrix} \underset{\text{Linearity}}{=\!=\!=} D\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vdots \\ \vec{r}_n \end{pmatrix} + D\begin{pmatrix} \vec{r}_1 \\ k\vec{r}_1 \\ \vec{r}_3 \\ \vdots \\ r_n \end{pmatrix}$$

$$= D\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vdots \\ \vec{r}_n \end{pmatrix} + \cancel{k\,D\begin{vmatrix} \vec{r}_1 \\ \vec{r}_1 \\ \vec{r}_3 \\ \vdots \\ \vec{r}_n \end{vmatrix}}^{\;0} \quad \text{same}$$

$$= D\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ r_n \end{pmatrix}$$

The basic row operation of replacing a row by a multiple of another row does not change the determinant.

Using the Rules

6!

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 3 \end{vmatrix} = 2\cdot 3\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{vmatrix} = 6\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 = (6 - \cancel{2} - 2)$$

$$= \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}$$

e.g. $\begin{vmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ d & 0 & f \\ g & 0 & k \end{vmatrix}$

Then it is not hard to see that the above will calculate to $\begin{vmatrix} d & f \\ g & k \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 0 \\ \boxed{1} & 0 & \boxed{0} \\ \boxed{0} & 0 & \boxed{1} \end{vmatrix}$

$+ \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$

This is the source of the minor determinant <u>and</u> <u>the sign</u> in the row-expansion formula.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a \begin{vmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & k \end{vmatrix} + b \begin{vmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & k \end{vmatrix} + c \begin{vmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & k \end{vmatrix}$$

$$= a \begin{vmatrix} 1 & 0 & 0 \\ 0 & e & f \\ 0 & h & k \end{vmatrix} + b \begin{vmatrix} 0 & 1 & 0 \\ d & 0 & f \\ g & 0 & k \end{vmatrix} + c \begin{vmatrix} 0 & 0 & 1 \\ d & e & 0 \\ g & h & 0 \end{vmatrix} =$$ next page.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Adjoint Matrix

$$Ad(A) = B, \quad A_{n\times n}$$

$$B_{ij} = (-1)^{i+j}\left|M(A)(i,j)\right|$$

e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 9 \\ 3 & 4 & 5 \end{pmatrix}$, $B = Ad(A)$

$$B = \begin{pmatrix} +\begin{vmatrix} 7 & 9 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 9 \\ 3 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 7 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 7 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 17 & -13 \\ 2 & -4 & 2 \\ -3 & -3 & 3 \end{pmatrix}$$

$$AB^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 9 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} -1 & 2 & -3 \\ 17 & -4 & -3 \\ -13 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix} = (-6)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -6I_3$$

Theorem. $\boxed{B = Ad(A),\ A_{n\times n},\ \text{then } AB^T = \det(A)\, I_n.}$

$$\boxed{A\,Ad(A)^T = Det(A)\,I_n.}$$

Thus if $Det(A) \neq \emptyset$, then

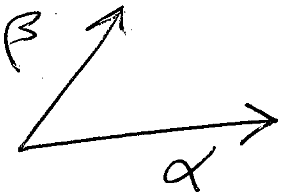
$$\boxed{A^{-1} = \frac{1}{Det(A)}\,ad(A)^T}.$$

Thus we have an explicit formula for the inverse of an $n \times n$ matrix.

$$\boxed{\text{So why is this correct } \text{~~that~~ } AB^T = Det(A)\,I\ ?}$$

Exercise. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Find A^{-1} by using $ad(A)$.

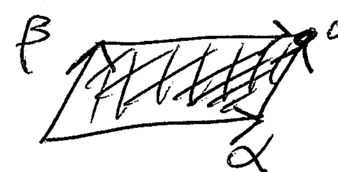
Hermann Grassmann (≈ 1880) 10

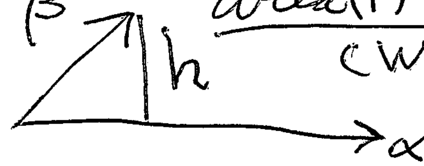


Invent a product for vectors

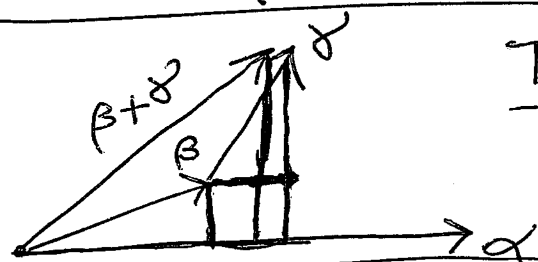
$$\alpha \wedge \beta$$

that will calculate the area of the parallelogram P spanned by them.





Area$(P) = h \times \|\alpha\|$
(why?)



(i) Triangle Areas Add.

So $\boxed{\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma}$

+ $\boxed{\alpha \wedge k\beta = k(\alpha \wedge \beta)}$

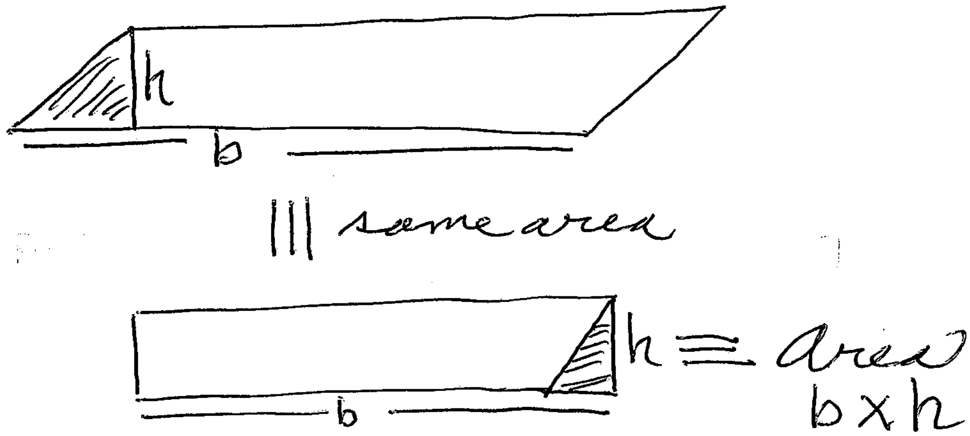
(ii)  $\alpha = \beta$ zero area $\boxed{\alpha \wedge \alpha = 0}$

But then:

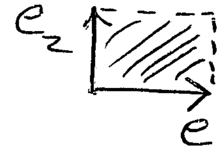
$$\begin{aligned} 0 &= (\alpha+\beta)\wedge(\alpha+\beta) \\ &= (\alpha+\beta)\wedge\alpha + (\alpha+\beta)\wedge\beta \\ &= \alpha\wedge\alpha + \beta\wedge\alpha + \alpha\wedge\beta + \beta\wedge\beta \\ &= 0 + \beta\wedge\alpha + \alpha\wedge\beta + 0 \end{aligned}$$

$$0 = \beta\wedge\alpha + \alpha\wedge\beta$$

So $\boxed{\alpha \wedge \beta = -\beta \wedge \alpha}$



Suppose $e_1 = (1,0)$
$e_2 = (0,1)$.



Then we want $e_1 \wedge e_2 = 1$.

<u>Now</u>. $(ae_1 + be_2) \wedge (ce_1 + de_2)$

$$= (ae_1 + be_2) \wedge (ce_1) + (ae_1 + be_2) \wedge (de_2)$$

$$= ac(e_1 \wedge e_1) + bc(e_2 \wedge e_1) + ad(e_1 \wedge e_2) + ad(e_2 \wedge e_2)$$

$$= 0 + bc(e_2 \wedge e_1) + ad(e_1 \wedge e_2) + 0$$

$$= bc(-e_1 \wedge e_2) + ad(e_1 \wedge e_2)$$

$$= (ad - bc)(e_1 \wedge e_2)$$

$$= ad - bc.$$

So Grassmann's method works! And it is "the same" as our theory of determinants.