Notational Correction!

$\boxed{A_{n\times n}}$

$$\text{CofactorMatrix}(A) = \text{Cof}(A)$$
$$= \left((-1)^{i+j} |M_{ij}(A)| \right).$$

(We erroneously called this the adjoint of A.)

$$\text{Adjoint}(A) = \text{Adj}(A) = \text{Cof}(A)^T.$$

Thus $|A| \neq 0 \Longrightarrow$

$$\boxed{A^{-1} = \frac{1}{|A|} \text{adj}(A)}$$

e.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\text{Cof}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$$\text{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$\Delta = ad - bc \neq 0$

$$\Longrightarrow A^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Vector Cross Product

A little more about $\boxed{a \times b}$.

$$a \times b = \begin{vmatrix} (i, j, k) \\ a \\ b \end{vmatrix} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$(a \times b) \cdot c = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} c \\ a \\ b \end{vmatrix}$$

$$= - \begin{vmatrix} a \\ c \\ b \end{vmatrix} = -(c \times b) \cdot a$$

$$= + \begin{vmatrix} a \\ b \\ c \end{vmatrix} = (b \times c) \cdot a$$

$= \text{etc.}$

Note: $X \cdot Y = Y \cdot X \Longrightarrow$

$$\boxed{(a \times b) \cdot c = a \cdot (b \times c)}.$$

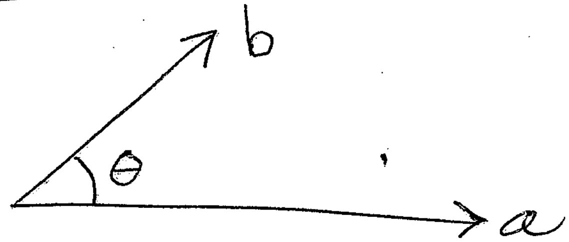
In $\mathbb{R}^n = \{(a_1,a_2,\dots,a_n)=a \mid a_i\in\mathbb{R}\}$

have <u>dot product</u> $a\cdot b = a_1b_1+\dots+a_nb_n$

and <u>length</u> $\|a\|=\sqrt{a\cdot a}$

$$\|a\|=\sqrt{a_1^2+a_2^2+\dots+a_n^2}.$$

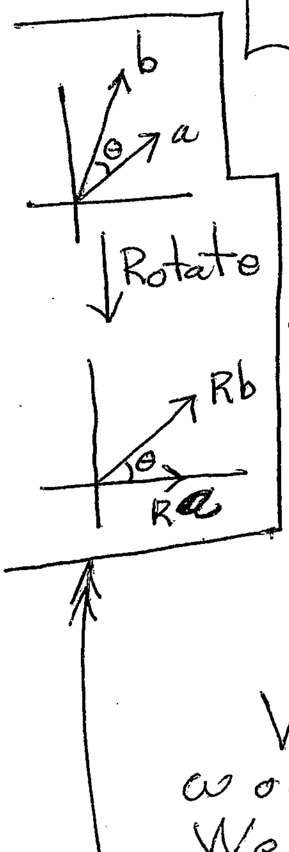
<u>Back to $\mathbb{R}^n$.</u>

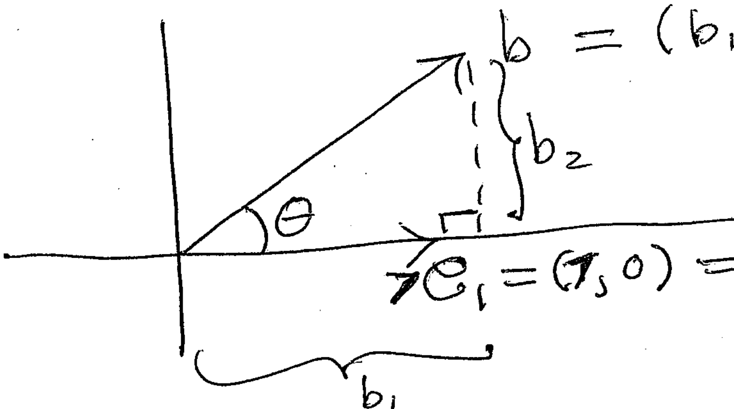


<u>Claim:</u> $a\cdot b=\|a\|\,\|b\|\cos(\theta)$

where θ = ∡ between a and b.

Lets verify in $\mathbb{R}^n$ $(n=2)$.





$b=(b_1,b_2)$ $\Big\}$ $\dfrac{b_1}{\|b\|}=\cos(\theta)$

$7e_1=(7,0)=a$, $\|a\|=7$ here

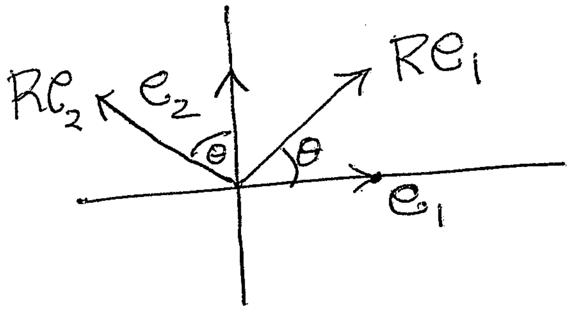
$$a\cdot b=(7,0)\cdot(b_1,b_2)=7b_1=7\|b\|\cos(\theta)$$

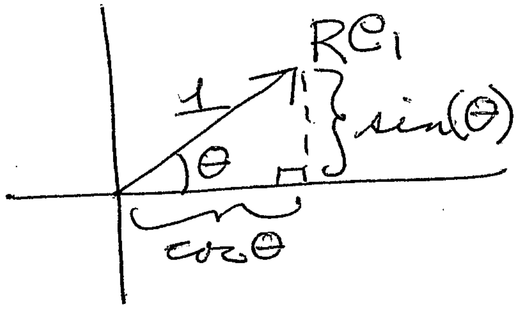
$$a\cdot b=\|a\|\,\|b\|\cos(\theta)$$

We see that it will continue to work if we multiply a by a constant.

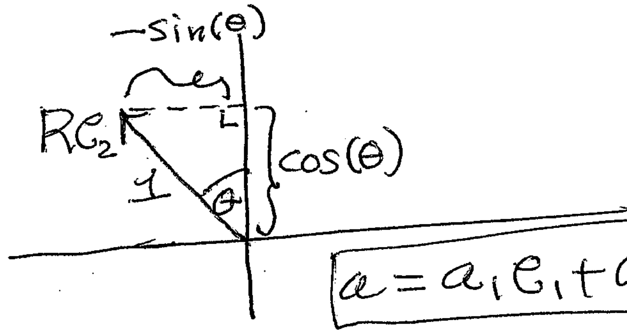
We finish the proof by showing first <u>that $a\cdot b$ does not change under a rotation.</u>

How to rotate by $\measuredangle\theta$ counterclockise?





$$Re_1 = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$



$$Re_2 = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

$$\boxed{a = a_1 e_1 + a_2 e_2}$$

$$\Rightarrow Ra = a_1 Re_1 + a_2 Re_2$$

$$= a_1 \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} + a_2 \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Rotations are linear transformations

$R(a+b) = R(a) + R(b)$,

$R(ka) = kR(a)$.

$$\Rightarrow R\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Exercise. $(Ra)\cdot(Rb) = a \cdot b$ for any $a, b \in \mathbb{R}^2$.

Example. $(Re_1)\cdot(Re_2) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$

$= -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)$

$= 0$

$= e_1\cdot e_2$ //

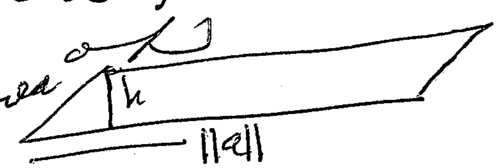
Claim: $\|a\times b\| = \|a\|\,\|b\|\sin(\theta)$

= area of parallelogram spanned by $a + b$.

$\dfrac{h}{\|b\|} = \sin(\theta)$

$\|a\|\,\|b\|\sin(\theta) = \|a\|\cdot h$

= area of



Note $(\|a\|\,\|b\|\sin(\theta))^2 = \|a\|^2\|b\|^2\sin^2\theta$

$= \|a\|^2\|b\|^2(1-\cos^2\theta)$

$= \|a\|^2\|b\|^2 - \|a\|^2\|b\|^2\cos^2(\theta)$

So should prove that $\|a\times b\|^2 = \|a\|^2\|b\|^2 - (a\cdot b)^2$

∴ Exercise: Prove that

$\|a\times b\|^2 + (a\cdot b)^2 = \|a\|^2\|b\|^2$

Vector Spaces

First the standard example.

$$\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \vec{a} \;\middle|\; a_i \text{ are real numbers} \right\}$$

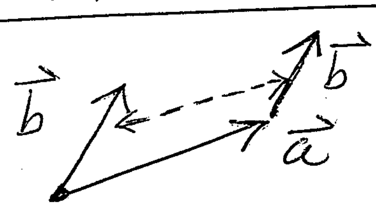
$$\left.\begin{aligned} (\vec{a}+\vec{b})_i &= a_i + b_i \\ (c\vec{a})_i &= c a_i \\ \vec{0} &= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}\right\}$$

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

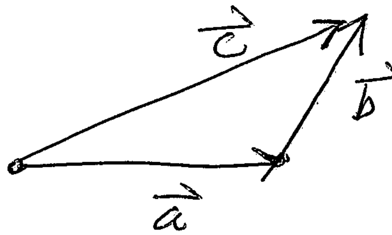
1 in the ith place

(Sometimes write $(a_1, a_2, \ldots, a_n)$)

Familiar Geometry

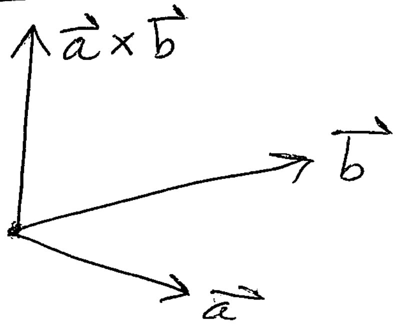


"A vector is a quantity with magnitude and direction."



$\vec{c} = \vec{a} + \vec{b}$

In $\mathbb{R}^3$:



$\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Fact: $\vec{a} \in \mathbb{R}^n \Longrightarrow$

$x \in S$ means "x is a member of S"

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$$

Proof. $a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$

$$= a_1\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\1\\0\\\vdots\\0\end{bmatrix} + \cdots + a_n\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}a_1\\0\\0\\\vdots\\0\end{bmatrix} + \begin{bmatrix}0\\a_2\\0\\\vdots\\0\end{bmatrix} + \begin{bmatrix}0\\0\\a_3\\0\\\vdots\\0\end{bmatrix} + \cdots + \begin{bmatrix}0\\0\\\vdots\\0\\a_n\end{bmatrix}$$

$$= \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} = \vec{a} \checkmark \quad //$$

Fact. $\sum_{K=1}^{n} a_i \vec{e}_i = \vec{0} \Rightarrow a_1 = a_2 = \cdots = 0.$

Proof. $\sum_{K=1}^{n} a_i\vec{e}_i = \vec{a}$. So $\sum_{K=1}^{n} a_i\vec{e}_i = \vec{0}$

$\Rightarrow \vec{a} = \vec{0} \Rightarrow a_i = 0$ for each

$i = 1, 2, \ldots, n.$ //

$\exists$: There exists
$\forall$: For all.
$a \in V$: a belongs to V.

Vector Space Axioms

V is a set with operations $+$, scalar multiplication.

Scalars $= \mathbb{R}$ = the real numbers.
(Later we will use other scalars.)

$x, y \in V \Longrightarrow x + y \in V$. [C2]

$\alpha \in \mathbb{R}, x \in V \Longrightarrow \alpha x \in V$. [C1]

A1. $x + y = y + x \quad \forall x, y \in V$

A2. $(x + y) + z = x + (y + z) \quad \forall x, y, z \in V$.

A3. $\exists\, \vec{0} \in V$ s.t. $x + \vec{0} = x \quad \forall x \in V$.

A4. $x \in V \Longrightarrow \exists\, -x \in V$ s.t. $x + (-x) = \vec{0}$.

A5. $\alpha(x + y) = \alpha x + \alpha y \quad \forall \alpha \in \mathbb{R}; x, y \in V$.

A6. $(\alpha + \beta)x = \alpha x + \beta x \quad \forall \alpha, \beta \in \mathbb{R}; x \in V$.

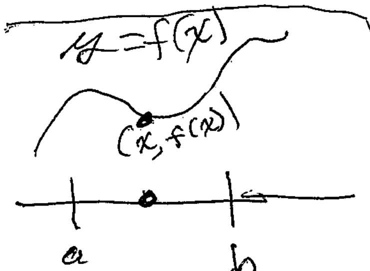
A7. $(\alpha\beta)x = \alpha(\beta x) \quad \forall \alpha, \beta \in \mathbb{R}; x \in V$.

A8. $1 \cdot x = x \quad \forall x \in V$.

Examples. (a) $\mathbb{R}^n \ni \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and usual operations.

(b) $\{n \times m$ matrices$\} = \mathcal{M}(n, m)$.
$+$ = addition of matrices as usual $(\alpha A)_{ij} = \alpha A_{ij}$.
scalar x matrix

(c) $C[a,b]$ = all real valued continuous functions $f : [a,b] \longrightarrow \mathbb{R}$.
$(f + g)(x) = f(x) + g(x)$.
$(\alpha f)(x) = \alpha f(x)$.



$f : [a,b] \to \mathbb{R}$
$g(x) = \sin(x)$
$f(x) = x^3 + 3$

Remarks

(a) $\mathbb{R}^n = V$

Vector space over $\mathbb{R}$.

$\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ with $a_i \in \mathbb{R}$.

$\vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$

$\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n$ <u>dot product</u>

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{\vec{a} \cdot \vec{a}}$ is the length of the vector $\vec{a}$.

We say $\vec{a} \perp \vec{b}$ ($\vec{a}$ is perpendicular to $\vec{b}$) when $\vec{a} \cdot \vec{b} = \vec{0}$.

(b) $\mathcal{M}(n, m) \longleftrightarrow \mathbb{R}^{n+m}$

e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow (a, b, c, d)$

This correspondence is an example of an <u>isomorphism</u> of <u>vector spaces</u>.

(c) $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous} \}$

Think of f as $\{ f(x) \mid x \in [a,b] \}$ but written "in order". Then you see that f is a generalization of a vector in $\mathbb{R}^n$.

(d) Let $V_n = \{ f : \{1, 2, \ldots, n\} \to \mathbb{R} \}$

$(f+g)(x) = f(x) + g(x) \quad x = 1, 2, \ldots, n$

$(kf)(x) = kf(x) \quad x = 1, 2, \ldots, n$.

Show V_n is a vector space isomorphic to $\mathbb{R}^n$.

Topics For Chapter 3

- Subspace
- Null Space
- Span of set of vectors
- Linear Independence
  - Calculations
  - Wronkskian
- Basis and Dimension
- Change of Basis
- Row Space and Column Space

Subspace

If $S \subseteq V$, V a vector space

and if

(i) $\alpha \vec{x} \in S$ $\forall \vec{x} \in S$ and $\alpha \in \mathbb{R}$

& (ii) $\vec{x} + \vec{y} \in S$ whenever $\vec{x} \in S$ and $\vec{y} \in S$

then S is itself a vector space, and we say that S is a subspace of V.

$$P_n = \{ p = c_1 + c_2 x + \cdots + c_{n-1} x^{n-1} \mid c_i \in \mathbb{R} \}.$$
$$= \{ \text{polynomials of degree} < n \}.$$

$C[\mathbb{R}]$ = all continuous functions $f: \mathbb{R} \to \mathbb{R}$.

P_n is a subspace of $C[\mathbb{R}]$.

$\mathbb{R}^n \supset \operatorname{Span}\{v_1, v_2, \dots, v_k\}$ Here $v_1, \dots, v_k \in \mathbb{R}^n$ is some set of vectors.

$W = \operatorname{Span}\{v_1, \dots, v_k\} = \{ c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_i \in \mathbb{R} \}$

Then W is a subspace of $\mathbb{R}^n$.

Theorem (3.1.1). V vector space, $\vec{x} \in V$ then (i) $0\vec{x} = \vec{0}$.

(ii) $\vec{x} + \vec{y} = \vec{0} \Rightarrow \vec{y} = -\vec{x}$.

(iii) $(-1)\vec{x} = -\vec{x}$.

Proof. (i) $\vec{x} = 1\vec{x} = (1+0)\vec{x} = 1\vec{x} + 0\vec{x} = \vec{x} + 0\vec{x}$

$\therefore -\vec{x} + \vec{x} = -\vec{x} + (\vec{x} + 0\vec{x})$

$\vec{0} = -\vec{x} + (\vec{x} + 0\vec{x})$

$\therefore \vec{0} = (-\vec{x} + \vec{x}) + 0\vec{x}$

$\therefore \vec{0} = \vec{0} + 0\vec{x} = 0\vec{x}$

$\therefore \vec{0}$

(ii) $\vec{x} + \vec{y} = \vec{0} \Rightarrow -\vec{x} + (\vec{x} + \vec{y}) = -\vec{x}$

$\Rightarrow (-\vec{x} + \vec{x}) + \vec{y} = -\vec{x}$

$\Rightarrow \vec{0} + \vec{y} = -\vec{x}$

$\Rightarrow \vec{y} = -\vec{x}$

(iii) $\vec{0} = 0\vec{x} = (1 + (-1))\vec{x} = 1\vec{x} + (-1)\vec{x} = \vec{x} + (-1)\vec{x}$

$\Rightarrow$ (ii) $(-1)\vec{x} = -\vec{x}$. //

Let V be a vector space and $\{v_1, v_2, \ldots, v_n\}$ a set of vectors in V. We say that $\{v_1, v_2, \ldots, v_n\}$ are <u>linearly independent</u> if <u>no one of them can be written as a linear combination of the others.</u> This is the same as saying

$$\boxed{\begin{array}{l} c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0} \\ \Longrightarrow c_1 = c_2 = \cdots = c_n = 0 \end{array}}$$

<u>Example.</u> $\{e_1, e_2, \ldots, e_n\}$ in $V = \mathbb{R}^n$ are linearly independent.

<u>Example.</u> $\{x, \sin(x), \cos(x)\}$ in $C[\mathbb{R}]$ are linearly independent.

<u>Solution:</u> $c_1 x + c_2 \sin(x) + c_3 \cos(x) = 0$ means this is true for all $x \in \mathbb{R}$.

Substitute values:

$x = 0$: $c_1 \cdot 0 + c_2 \sin(0) + c_3 \cos(0) = 0$

$\Longrightarrow c_3 = 0$ $(\cos(0) = 1)$.

So $c_1 x + c_2 \sin x = 0$. Let $x = \pi/2$.

$\Rightarrow c_1(\pi/2) + c_2 \sin(\pi/2) = 0 \Rightarrow \boxed{c_1 \frac{\pi}{2} + c_2 = 0}$

Let $x=\pi$. $c_1\pi + c_2\sin(\pi) = 0$

$$\Rightarrow c_1\pi + 0 = 0$$
$$\Rightarrow c_1\pi = 0$$
$$\Rightarrow \underline{c_1 = 0}.$$

Now $c_1\pi/2 + c_2 = 0 \Rightarrow c_2 = 0$.

So $c_1 = c_2 = c_3 = 0$ ✓

$\therefore$ $\{x, \sin(x), \cos(x)\}$ are lin ind.//

<u>Example</u>. Show that $\{1, x, x^2, \ldots, x^n\}$ are linearly independent in $\mathbb{C}[\mathbb{R}]$.

<u>Solution</u>. Suppose $c_1 + c_2x + c_3x^2 + \cdots + c_{n+1}x^n = 0$.

Let $x_1, x_2, \ldots, x_{n+1}$ be $(n+1)$ distinct real numbers. Then we have

$$\begin{cases} c_1 + c_2x_1 + c_3x_1^2 + \cdots + c_{n+1}x_1^n = 0 \\ \quad\cdots \\ c_1 + c_2x_{n+1} + c_3x_{n+1}^2 + \cdots + c_{n+1}x_{n+1}^n = 0 \end{cases}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ & & \cdots & & \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{bmatrix}}_{\text{This is the Vandermonde Matrix } V} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(P. 67, problem #28).

It follows from that V is non-singular.

So $V\vec{c} = \vec{0} \Rightarrow \vec{c} = \vec{0}$ ✓ $\therefore$

$\{1, x, x^2, \ldots, x^n\}$ is lin ind.//

(12.1)

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ & & \cdots & & \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{bmatrix}$$

$y = Vc = V\begin{bmatrix} c_1 \\ \vdots \\ c_{n+1} \end{bmatrix}$ Then

$$y_i = c_1 + c_2 x_i + \cdots + c_{n+1} x_i^n$$

$$y_i = p(x_i)$$

where $p(x) = c_1 + c_2 x + \cdots + c_{n+1} x^n$.

$\boxed{\begin{matrix} Vc = \vec{0} \\ \forall c \neq \vec{0} \end{matrix}} \Rightarrow p(x_i) = 0$ for $i = 1, \ldots, n+1$.

$\Rightarrow p(x)$ has $(n+1)$ distinct roots.

$\Rightarrow$ contradiction since $\deg(p) = n$.

$\therefore Vc = \vec{0} \Rightarrow c = \vec{0}$

$\Rightarrow \{1, x, x^2, \ldots, x^n\}$ are linearly independent.

See problem 28 page 67

Example. $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}$, $v_3 = \begin{bmatrix} 10 \\ 15 \\ 20 \end{bmatrix}$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\Leftrightarrow \begin{bmatrix} 1 & 7 & 10 \\ 2 & 9 & 15 \\ 3 & 11 & 20 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore row-reduce $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$.

$$M = \begin{bmatrix} 1 & 7 & 10 \\ 2 & 9 & 15 \\ 3 & 11 & 20 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & 10 \\ 0 & -5 & -5 \\ 0 & -10 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Since $R\vec{c} = \vec{0}$ has non-zero solutions, $\{v_1, v_2, v_3\}$ are dependent.

In fact $\boxed{v_3 = 3v_1 + v_2}$

and one can read off this dependency from the row-reduced matrix.

Row Space

$M = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$, M $n \times m$

Then $r_i \in \mathbb{R}^m$ are rows of M.

$$RS(M) \underset{\text{def}}{=} \text{Row Space}(M) = \text{Span}\{r_1, r_2, \ldots, r_n\} \underset{\text{subspace}}{\subseteq} \mathbb{R}^m$$

__Theorem.__ If M' is obtained from M by elementary row operations then $RS(M') = RS(M)$.

__Proof.__ Clear for mult of r_i by $\alpha \neq 0$, and for interchange of rows. Note the following:

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \xrightarrow[r_2 \mapsto r_2 + \alpha r_1]{} \begin{bmatrix} r_1 \\ r_2 + \alpha r_2 \end{bmatrix}$$

$$\text{Span}\{r_1, r_2 + \alpha r_1\} = \{c_1 r_1 + c_2(r_2 + \alpha r_1)\}$$
$$= \{(c_1 + c_2\alpha) r_1 + c_2 r_2\}$$
$$= \text{Span}\{r_1, r_2\}$$ (since can solve $d_1 = c_1 + c_2\alpha$, $d_2 = c_2$ for any d_1 & d_2). //

Ex: $M = \begin{bmatrix} 1 & 7 & 10 \\ 2 & 9 & 15 \\ 3 & 11 & 20 \end{bmatrix} \xrightarrow[\text{prev. page}]{} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow$

$RS(M) = \text{Span}\{(1,0,3), (0,1,1)\}$.

these are linearly ind.

The Wronskian

Show $\sin(x)$ & $\cos(x)$ are linearly independent.

Solution.

Suppose $a\sin(x) + b\cos(x) = 0$.

Differentiate: $a\cos(x) - b\sin(x) = 0$

So
$$\begin{bmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\downarrow \text{Det}$$

$$-\sin^2(x) - \cos^2(x) = -1$$

So matrix non-singular.

$\Longrightarrow \sin(x), \cos(x)$ ind.

Give $f_1, \dots, f_n \in C^{n-1}[a,b]$ define the Wronskian $W[f_1, \dots, f_n]$:

$$W[f_1, \dots, f_n](x) = \text{Det}\begin{bmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ & \vdots & \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{bmatrix}$$

Thm. $W[f_1, \dots, f_n] \neq 0 \Longrightarrow$ $\{f_1, f_2, \dots, f_n\}$ linearly independent

Use Wronskian to prove that $\{1, x, x^2\}$ is linearly independent.

Solution:

$$W[1,x,x^2] = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

Use Wronskian to show $\{1, x, x^2, \ldots, x^n\}$ is linearly ind.

Solution:

$$W[1,x,\ldots,x^n] = \begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1 & 2x & 3x^2 & \cdots & nx^{n-1} \\ 0 & 0 & 2 & 6x & \cdots & n(n-1)x^{n-2} \\ & & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & & (n!) \end{vmatrix}$$

upper triangular, all diag entries $\neq 0$

$$\Rightarrow W[1,x,\ldots,x^n] \neq 0.$$

Subspace Examples

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \ x_2 = 2x_1 \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$S = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \ x_2 = 2x_1 + 1 \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 2\alpha + 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

But W is <u>not</u> a subspace.

Among other things $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ must belong to a subspace. But

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = 0$$

$\Rightarrow$ contradiction.

$$S = \left\{ \begin{bmatrix} \alpha \\ \alpha^2 \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\}$$

<u>not</u> a subspace.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ but } 5 \neq 3^2.$$

So we do not have closure under addition.

$S = \{ p \in P_n \mid p(0) = 0 \}$

$p(0) = 0$, $q(0) = 0$

$\Rightarrow (p+q)(0) = p(0) + q(0) = 0 + 0 = 0$ ✓

$(\alpha p)(0) = \alpha(p(0)) = \alpha \cdot 0 = 0$ ✓

So S is a <u>subspace of P_n</u>.

$C^n[a,b]$ = all functions f with continuous n-th derivative on $[a,b]$.

$C^\infty[a,b]$ = all functions f with ∞ly many derivatives on $[a,b]$ (e.g. $f(x) = e^{7x}$)

$C^\infty[\mathbb{R}]$ = all functions f defined on $\mathbb{R} \longrightarrow \mathbb{R}$ with ∞ly many derivs.

$$S = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \middle| \, x \in \mathbb{R} \right\}$$

$$\begin{bmatrix} \cancel{x_1} \\ 1 \end{bmatrix} + \begin{bmatrix} \cancel{x_2}x' \\ 1 \end{bmatrix} = \begin{bmatrix} x+x' \\ 2 \end{bmatrix} \notin S$$

So S not a subspace.

$$S = \left\{ A \in \underbrace{\mathbb{R}^{2\times 2}}_{\text{2x2 matrices}} \middle| \, a_{12} = -a_{21} \right\}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ -a_{12} & a_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}, \quad B = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$$

$$A + B = \begin{bmatrix} a+d & b+e \\ -(b+e) & c+f \end{bmatrix} \checkmark$$

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ \alpha(-b) & \alpha c \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix} \checkmark$$

Thus S is a subspace of $\mathbb{R}^{2\times 2}$.

$$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} \subset \mathbb{R}^{2\times 2}$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathcal{B} = \Big\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{= I}, \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{= J} \Big\}$$

basis (spans & lind) for S.

$$\boxed{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = aI + bJ}$$

$$J^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$(aI + bJ)(cI + dJ)$$
$$= acI + adIJ + bcJI + bdJ^2$$
$$= (ac - bd)I + (ad + bc)J$$

$$(a+bi)(c+di) = (ac - bd) + (ad + bc)i$$

S has same multiplicative structure as the complex numbers.

Null Space of a Matrix

A an $m\times n$ matrix.

$$N(A)=\{x\in\mathbb{R}^n \mid Ax=0\}$$

$N(A)\subset\mathbb{R}^n$ subspace

$$\begin{cases} Av=0,\ Aw=0 \implies A(v+w) = Av+Aw=0+0=0 \ \checkmark \\ A(\alpha v)=\alpha Av=\alpha\cdot 0=0 \ \checkmark \end{cases}$$

Example: $A=\begin{bmatrix}1&1&1&0\\2&1&0&1\end{bmatrix}$

pivot columns

$$\left[\begin{array}{cccc|c}1&1&1&0&0\\2&1&0&1&0\end{array}\right]\longrightarrow\left[\begin{array}{cccc|c}1&1&1&0&0\\0&-1&-2&1&0\end{array}\right]$$

$$N(A)=\mathrm{Span}(v_1,v_2)\longrightarrow \begin{array}{ccccc} x_1 & x_2 & \alpha & \beta & \\ \end{array}\left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&2&-1&0\end{array}\right]$$

$$v_1=\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} \qquad v_2=\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}\alpha-\beta\\-2\alpha+\beta\\\alpha\\\beta\end{pmatrix}=\alpha\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

$$N(A)=\{\quad\}$$

$$S = \{ f \in C^2[a,b] \mid f''(x) + f(x) = 0 \}$$

$$\underbrace{f'(x) = \frac{df}{dx} = Df(x)}_{\text{notations for derivative.}}$$

Note that $\sin(x) \in S$ and $\cos(x) \in S$.

$$\boxed{\begin{array}{l} D(\alpha f) = \alpha D(f),\ \alpha \in \mathbb{R}. \\ D(f+g) = Df + Dg \end{array}} \quad D = d/dx.$$

$$S = \{ f \mid (D^2 + 1) f = 0 \}$$

$$(D^2+1)(f+g) = (D^2 f + f) + (D^2 g + g)$$

$\Rightarrow$ $f \in S, g \in S$ then $f+g \in S$.

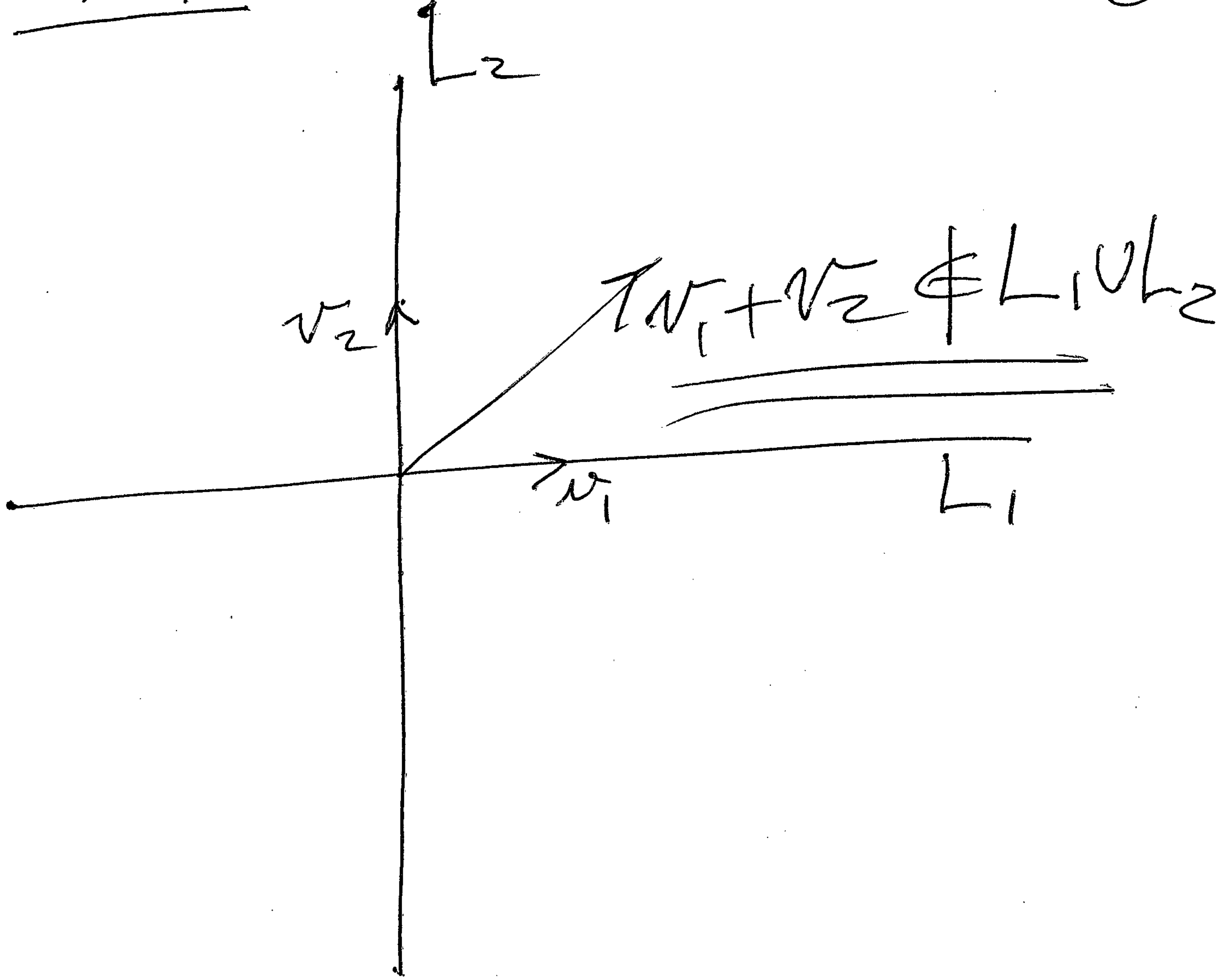
Similarly $\alpha f \in S$ when $\alpha \in \mathbb{R}$.

In fact, the general solution of the <u>differential equation</u>, $f'' + f = 0$ is $a \sin(x) + b \cos(x) = f(x)$.

Note that a & b are determined by the values $f(0)$ and $f'(0)$.

$$\boxed{\begin{array}{l} b = f(0) \\ a = f'(0) \end{array}} . \quad f'(x) = a\cos(x) - b\sin(x).$$

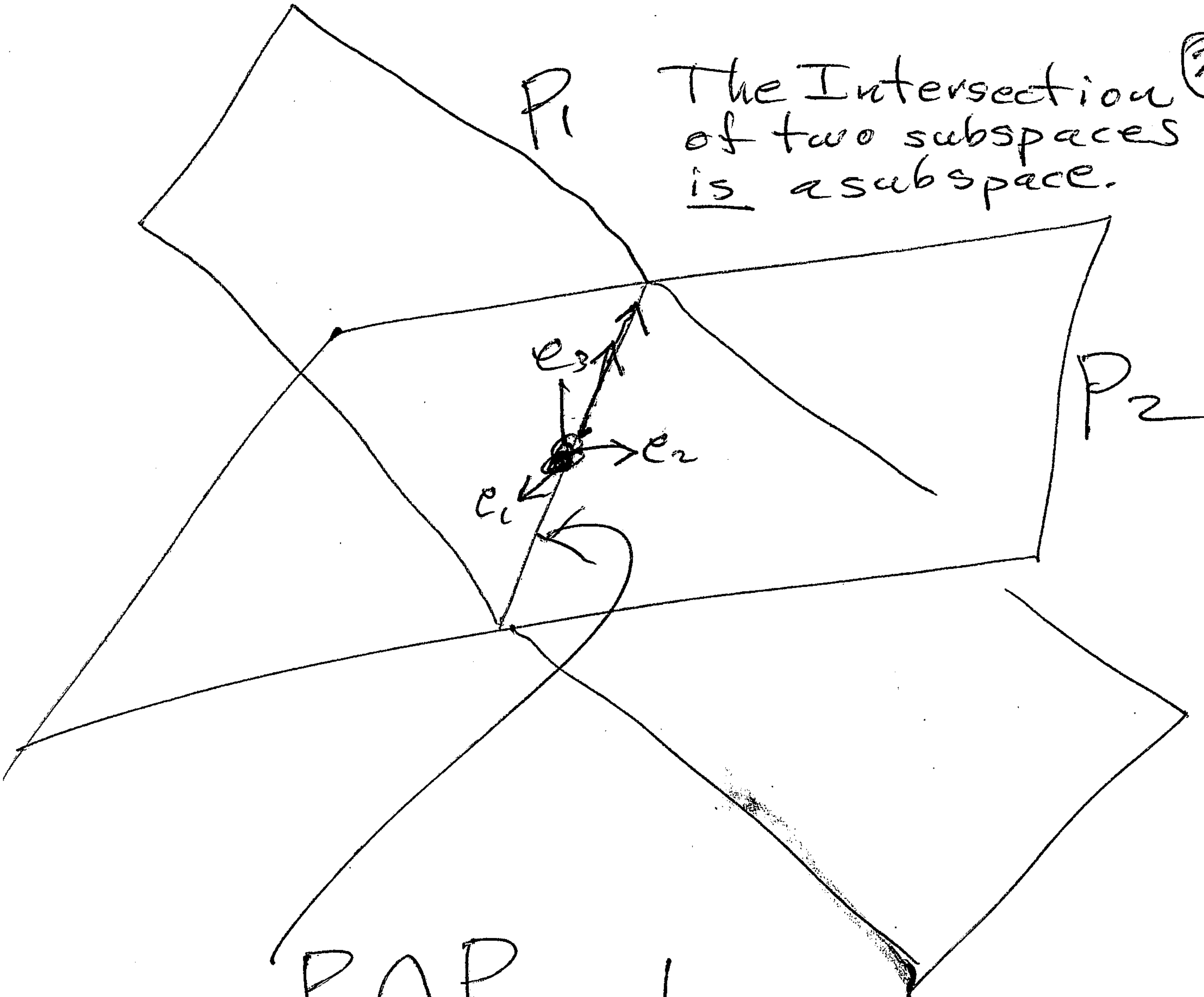
Example



$L_1 \cup L_2$ = Just the crossed lines.

The union of two subspaces is usually not a subspace.

The Intersection of two subspaces <u>is</u> a subspace.

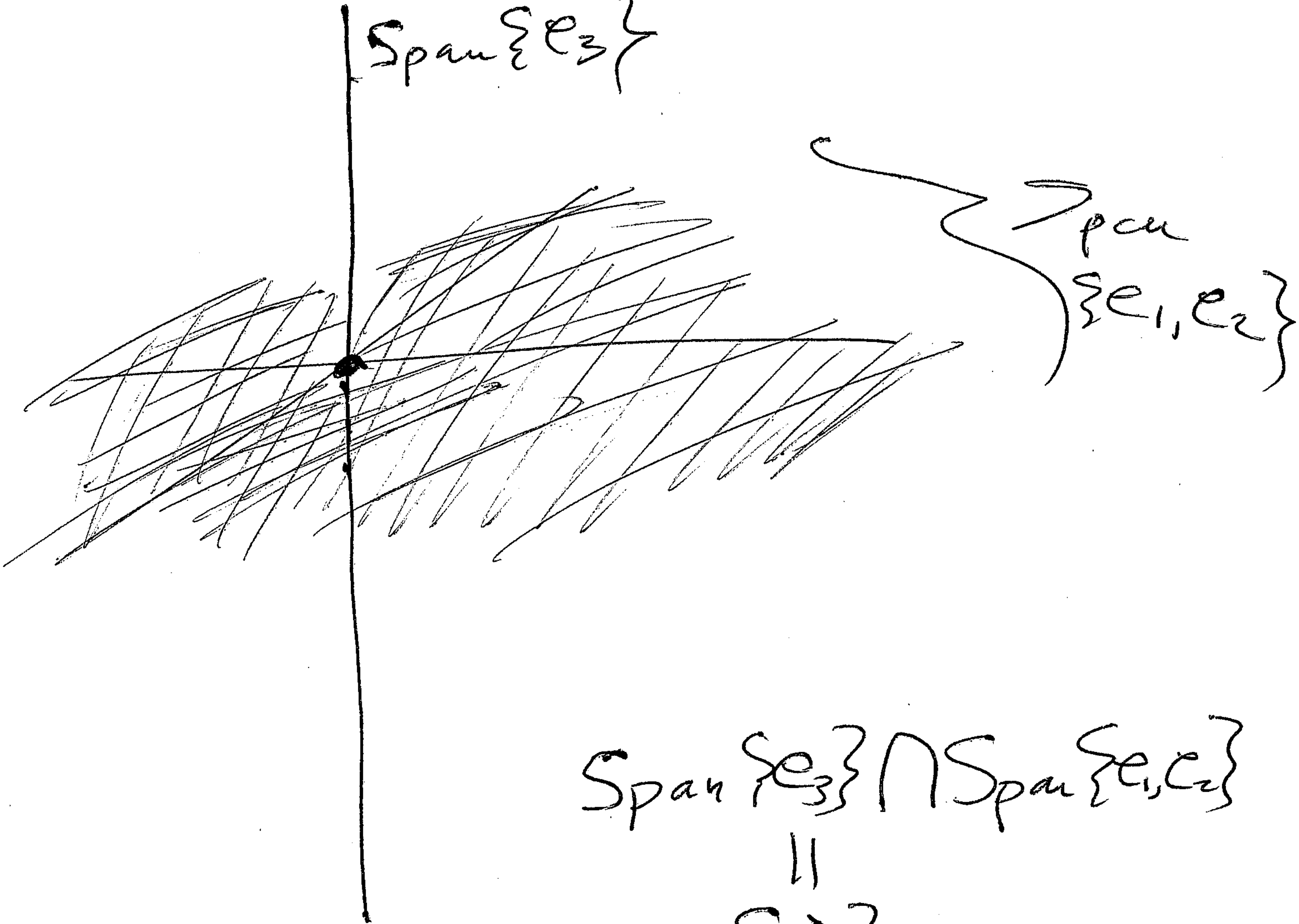


$P_1 \cap P_2 = L_1$

$W, W' \subset V$ are subspaces then $W \cap W'$ is a subspace.

<u>Pf.</u> $v, w \in W \cap W'$ then $v \in W$ and $w \in W$
$\Rightarrow v + w \in W$
Simil $v + w \in W'$
and so $v + w \in W \cap W'$
etc. //

$\{\vec{0}\}$ is a subspace



$$\text{Span}\{e_3\} \cap \text{Span}\{e_1, e_2\}$$
$$\|$$
$$\{\vec{0}\}$$

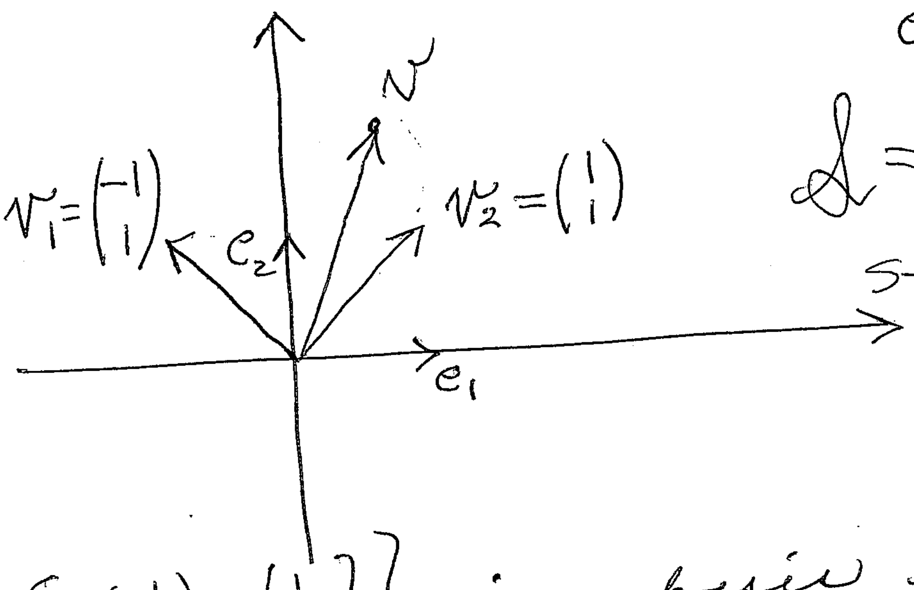
subspace!

Transition Matrix

$e_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$\mathcal{S} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ standard basis



$\beta = \left\{ \underset{v_1}{\begin{pmatrix}-1\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\1\end{pmatrix}} \right\}$ is a basis for $\mathbb{R}^2$.
(i.e. linearly ind & spans $\mathbb{R}^2$)

Problem. Given $v = \begin{pmatrix}1\\7\end{pmatrix}$, find $a, b \in \mathbb{R}$ s.t. $v = a v_1 + b v_2$.

$v = e_1 + 7e_2$

$\begin{pmatrix}1\\7\end{pmatrix} = [v]_{\mathcal{S}}$

$\begin{pmatrix}a\\b\end{pmatrix} = [v]_{\beta}$

Solution. $a v_1 + b v_2 = v$ is same as matrix equation

$M = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, $M\begin{bmatrix}a\\b\end{bmatrix} = v$

(v_1: 1st col, v_2: 2nd col)

So $\begin{bmatrix}a\\b\end{bmatrix} = M^{-1} v$.

Here

$$M = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det(M)} \operatorname{adj}(M) = \frac{1}{(-2)} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Check: $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \text{[redacted]} & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$$\begin{pmatrix} a \\ b \end{pmatrix} = M^{-1} v = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 6/2 \\ 8/2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Check $3v_1 + 4v_2 = \begin{pmatrix} -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ ✓

$M = [v_1 \; v_2]$ is called the transition matrix.

Note that $\boxed{\begin{aligned} Me_1 &= v_1 \\ Me_2 &= v_2 \end{aligned}}$.

We say that M is the transition matrix from $\beta = \{v_1, v_2\}$ (new basis) to $\mathcal{E} = \{e_1, e_2\}$ (standard basis).

Transition Matrix

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$U = [u_1\, u_2\, u_3] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

$$[v]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boxed{U e_i = u_i}$$

Suppose $v = a u_1 + b u_2 + c u_3$

Then $v = U \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

In this sense U translates from the basis $\beta = \{u_1, u_2, u_3\}$ to the standard basis $\{e_1, e_2, e_3\}$.

We call U the transition matrix from β to $\mathcal{S}$, and we write $\boxed{\beta \xrightarrow{U} \mathcal{S}}$.

$$v = U \begin{bmatrix} a \\ b \\ c \end{bmatrix} \longleftrightarrow \boxed{[v]_\mathcal{S} = U [v]_\beta}$$

Suppose $\{v_1, \ldots, v_n\}$ is a basis for V. Then every $v \in V$ has a unique expression as a linear combination of $v_1, \ldots, v_n$.

<u>Proof.</u> Suppose $v = c_1 v_1 + \cdots + c_n v_n$
& $v = d_1 v_1 + \cdots + d_n v_n$.

Then $c_1 v_1 + \cdots + c_n v_n = d_1 v_1 + \cdots + d_n v_n$

$\Rightarrow (c_1 - d_1) v_1 + \cdots + (c_n - d_n) v_n = 0$.

$\Rightarrow c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots, c_n - d_n = 0$
(since $v_1, \ldots, v_n$ lin ind).
$\therefore c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$. //

Next we show that any two bases for a vector space V must have the same number of elements.

Accordingly, we define

$$\boxed{\dim(V) = \#\text{ of basis elements in a basis for } V.}$$

To see the issue involved, suppose that $\{v_1, v_2\}$ and $\{w_1, w_2, w_3\}$ are bases for V. Then

$$\left.\begin{aligned} w_1 &= a v_1 + b v_2 \\ w_2 &= c v_1 + d v_2 \\ w_3 &= e v_1 + f v_2 \end{aligned}\right\} \text{for some real numbers } a, \ldots, f.$$

Consider solutions x, y, z to

$$x w_1 + y w_2 + z w_3 = 0.$$

$$\Longleftrightarrow \quad (xa + yc + ze) v_1 + (xb + yd + zf) v_2 = 0$$

$$\Longleftrightarrow \quad \left.\begin{aligned} xa + yc + ze &= 0 \\ \text{and} \quad xb + yd + zf &= 0 \end{aligned}\right\} \left(\begin{array}{l}\text{because } v_1 \\ \text{\& } v_2 \text{ are} \\ \text{lin ind}\end{array}\right)$$

$$\Longleftrightarrow \quad \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But we know that the row echelon form of a 2×3 matrix yields non-zero solutions. This means that $\{w_1, w_2, w_3\}$ is not lin ind & so not a basis. We get a contradiction if we assume bases with diff numbers of elements. //

We have indicated the proof of

Theorem 3.4.1. $\{v_1, \dots, v_n\}$ span V then any collection of m vectors in V with $m > n$ is linearly dependent.

Corollary 3.4.2. If $\beta = \{v_1, \dots, v_n\}$ & $\beta' = \{u_1, \dots, u_m\}$ are bases for $V \Longrightarrow \boxed{n = m}$.

$n = m = \dim(V)$

Theorem 3.4.3. V a vector space of dim $n > 0$, then

(I) *Any* set of n linearly independent vectors span V.

(II) Any n vectors that span V are linearly independent.

Pf. (I) Suppose $\{v_1, \dots, v_n\}$ lin ind. $v \in V$. Then 3.4.1 $\Rightarrow \{v_1, \dots, v_n, v\}$ dep. So $\exists$ $c_1, \dots, c_n, c$ s.t. not all zero &
$$c_1 v_1 + \dots + c_n v_n + cv = 0.$$
But if $c = 0 \Rightarrow \{v_1, \dots, v_n\}$ dep contrad.

$\therefore c \neq 0 \therefore v = (-c_1/c)v_1 + \dots + (-c_n/c)v_n$.

(II) Otherwise would have contrad to 3.4.2. //

Theorem 3.4.4. V vector space of dim $n > 0$.
Then (i) no fewer than n vectors can span V.

(ii) any subset of $< n$ lin ind vectors can be extended to form a basis for V.

(iii) any spanning set containing $> n$ vectors can be pared down to a basis for V.

Back to transition matrices.

Suppose $\{v_1, v_2, \ldots, v_n\} = \beta'$ is a basis for $\mathbb{R}^n$. $\{e_1, \ldots, e_n\} = \beta$ standard basis.

Assume that each v_i is a column vector in standard basis.

Form $U = [v_1 \, v_2 \ldots v_n]$ with cols v_k.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$$

Then $U e_i = v_i$.

& U is invertible. So

$$e_i = U^{-1} v_i.$$

This lets us solve the problem.

$$\boxed{\text{If } v \in V, \text{ find } c_1, \ldots, c_n \text{ such that } v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.}$$

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$\Rightarrow U^{-1} v = c_1 U^{-1} v_1 + c_2 U^{-1} v_2 + \cdots + c_n U^{-1} v_n$$

$$U^{-1} v = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$$

$$\Rightarrow U^{-1} v = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Example. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, v_n = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$

$$U = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 \cdots & 1 \\ & \cdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & -1 \\ & & \cdots & & & \\ 0 & 0 & 0 & & \cdots & 1 \end{bmatrix}$$

e.g. $v = \begin{bmatrix} n \\ n-1 \\ n-2 \\ \vdots \\ 1 \end{bmatrix}, \quad U^{-1} v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow$

$$v_1 + v_2 + \cdots + v_n = v.$$

$$\mathcal{S} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\beta = \left\{ u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\beta' = \left\{ v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}.$$

$$U = [u_1 u_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{tikzcd} & \mathcal{S} & \\ \beta \arrow[ur, "U"] \arrow[rr, bend right, "V^{-1}U"'] & & \beta' \arrow[ul, "V"'] \arrow[ll, "U^{-1}V"'] \end{tikzcd}$$

Note: 1) $v = a u_1 + b u_2$

$$\Longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = [v]_\beta$$

2) $v = x v_1 + y v_2$

$$\Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = [v]_{\beta'}$$

$$\left.\begin{array}{l} v = a u_1 + b u_2 = U \begin{bmatrix} a \\ b \end{bmatrix} \\ v = x v_1 + y v_2 = V \begin{bmatrix} x \\ y \end{bmatrix} \end{array}\right\} \quad U \begin{bmatrix} a \\ b \end{bmatrix} = V \begin{bmatrix} x \\ y \end{bmatrix}$$

So $\begin{bmatrix} a \\ b \end{bmatrix} = U^{-1} V \begin{bmatrix} x \\ y \end{bmatrix}$

& $\begin{bmatrix} x \\ y \end{bmatrix} = V^{-1} U \begin{bmatrix} a \\ b \end{bmatrix}$

---

example:

$v = u_1 + u_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}_{\mathcal{S}} \leftarrow$ standard basis.

$$[v]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[v]_{\beta'} = V^{-1} U [v]_\beta = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$4 v_1 - 2 v_2 = \begin{pmatrix} 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \checkmark$$

I want to find x, y
s.t. $v = x v_1 + y v_2$.

So $x v_1 + y v_2 = a u_1 + b u_2$

$$\Rightarrow x V e_1 + y V e_2 = a U e_1 + b U e_2$$

$$V(x e_1 + y e_2) = U(a e_1 + b e_2)$$

$$V\begin{bmatrix} x \\ y \end{bmatrix} = U\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = V^{-1} U \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} v \end{bmatrix}_{\beta'} = V^{-1} U [v]_{\beta}$$

The matrix $V^{-1}U$ transforms β-coordinates into β'-coordinates and so we call $V^{-1}U$ <u>the</u> <u>transition matrix from β to β'!</u>

One more point.

Basis $\{u_1, u_2\} = \beta$

& Basis $\{v_1, v_2\} = \beta'$

Suppose $v_1 = au_1 + cu_2$

$v_2 = bu_1 + du_2$

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = Transition matrix from β' to β.

If $v = xv_1 + yv_2$, $[v]_{\beta'} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow v = x(au_1 + cu_2) + y(bu_1 + du_2)$

$= (xa + yb)u_1 + (xc + yd)u_2$

$[v]_\beta = \begin{bmatrix} xa + yb \\ xc + yd \end{bmatrix}$

So $[v]_\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} [v]_{\beta'}$

Generalizing

$\{u_1, \ldots, u_n\} = \beta$, $\{v_1, \ldots, v_n\} = \beta'$ Bases for Vector Space W

Suppose $\boxed{v_i = a_{1i}u_1 + a_{2i}u_2 + \cdots + a_{ni}u_n}$

Then $A = (a_{ij})$ = transition matrix from $\beta' \longrightarrow \beta$.

$$[v]_\beta = A[v]_{\beta'}$$

e.g. $[v_1]_{\beta'} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$A\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = [v_1]_\beta$$

If you know transition matrices to standard basis S then you can determine transition matrices between bases.

$$\begin{tikzcd} & S & \\ \beta \arrow[ur, "U"] \arrow[rr, "V^{-1}U"] & & \beta' \arrow[ul, "V"'] \arrow[ll, bend left, "U^{-1}V"] \end{tikzcd}$$

~~Rank~~-Nullity Theorem

Consider a matrix in row echelon form.

e.g. $A = \begin{bmatrix} \boxed{1} & 3 & \boxed{0} & 7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad A_{m \times n}$

$\nu(A)$ = Nullity of A = $\dim N(A)$ = # of non-leading 1 cols.

$r(A)$ = rank(A) = dim(RowSpace(A)) = dim(RS(A))

n = total number of colums of A.

dim(RS(A)) = # of cols with leading 1's.
= # of rows with leading 1's.

Thus dim(RS(A)) = dim(CS(A)).

$\boxed{\text{dim(RowSpace}(A)) = \text{dim(ColSpace}(A))}$

nullity(A) + rank(A) = # of non-leading 1 cols.
+ # of leading 1 cols.

$\boxed{\text{nullity}(A) + \text{rank}(A) = \#\text{ cols}(A)}$

These formulas are true for any matrix A.

Example

Linear Algebra and Linear Diff Equations

Example. $x'(t) = dx/dt$.

Consider the system of diff eqns:

$$\boxed{\begin{aligned} x'(t) &= 4x(t) - 2y(t) \\ y'(t) &= x(t) + y(t) \end{aligned}}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$.

Then for $x, y \in C^\infty[\mathbb{R}]$ we have $\begin{pmatrix} x \\ y \end{pmatrix} \in C^\infty[\mathbb{R}]^2 = \{ f : \mathbb{R} \to \mathbb{R}^2 \mid f = (f_1, f_2)$ and each f_i is infinitely diff $\}$

$\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad \mathcal{H}$

We want to find the subspace $\mathcal{S}$ of $\mathcal{H}$ consisting of the solutions to $Df = Af$ where $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

Note: D is a linear transformation of $\mathcal{H}$ and so is M. So we want the null space of $(D - A) = T$.

We let $\mathcal{S} \subset \mathcal{H}$ be this nullspace.

$$\boxed{\begin{aligned} x'(t) &= 4x(t) - 2y(t) \\ y'(t) &= x(t) + y(t) \end{aligned}}$$

Try $x(t) = ae^{\lambda t}$
$y(t) = be^{\lambda t}$.

$$\lambda a e^{\lambda t} = 4ae^{\lambda t} - 2be^{\lambda t}$$
$$\lambda b e^{\lambda t} = ae^{\lambda t} + be^{\lambda t}$$

Divide by $e^{\lambda t}$:

$$\lambda a = 4a - 2b$$
$$\lambda b = a + b$$

$$(4-\lambda)a - 2b = 0$$
$$a + (1-\lambda)b = 0$$

$$\boxed{\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

This will have $\neq 0$ solutions $\begin{bmatrix} a \\ b \end{bmatrix}$ if $\mathrm{Det}\begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix} = 0$.

So $\lambda^2 - 5\lambda + 6 = 0$

So $(\lambda-2)(\lambda-3) = 0$

$\underline{\lambda = 2}$ or $\underline{\lambda = 3}$.

(33)

$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

$M = (A - \lambda I) = \begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}$

$\det(M) = 0 \iff \lambda = 2$ or 3

$\lambda = 2$: $M = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

So $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satis $Mv = \vec{0}$

$\lambda = 3$: $M = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

So $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ satis $Mv = \vec{0}$.

$\Rightarrow$ $v_1 e^{2t} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$ & $v_2 e^{3t} = \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions to the diff eqn.

The general solution is

$$c_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}.$$

So the null space of $D - A$ is Span$\left\{ \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} \right\}$.