Exam1 - Math 313 **- Fall** 2014

1. (a) Given a sequence $\{a_n\}$ of real numbers, give the definition of the statement: "$lim_{n \longrightarrow \infty}\, a_n$ exists."

   (b) Prove that $lim_{n \longrightarrow \infty}\, \frac{n^2}{n^2+1} = 1$.

2. (a) Give the definition of convergence for a series $S = \sum_{n=1}^{\infty} b_n$ where S is a real number and b_n is a real number for each natural number n.

   (b) Prove that the series $1 + 1/2 + 1/3 + 1/4 + 1/5 + ...$ diverges.

3. Define a function $f : N \longrightarrow Q$ from the natural numbers to the rational numbers by the formula $f(n) = 2^n$ if n is a prime number, and $f(n) = 1/2^n$ if n is not a prime number. Is the sequence $a_n = f(n)$ a bounded sequence or is it an unbounded sequence? Choose the correct answer and prove your claim.

4. (a) State the Completeness Axiom for the Real Numbers and use it to prove that the intersection of a countable collection of nested closed intervals in the real line is non-empty. That is, given intervals $I_n = [a_n, b_n]$ with $a_n < b_n$ and $I_{n+1} \subseteq I_n$ for all natural numbers n, show that $\cap_{n=1}^{\infty} I_n$ is not the empty set.

   (b) Show that the above intersection can be empty if we replace closed intervals by open intervals. Give a specific example of such an empty intersection of an infinite collection of nested open intervals where each individual interval is non-empty.

   (c) Use the result in part (a) of the problem to prove that the real numbers are uncountable.