SampleFinalExam - Math 313 **- Fall** 2014

Nota Bene. In each of the **True or False** problems: If true, give a proof of the statement. If false, give a counterexample or a proof that the statement is false.

1. **True or False**: Given any set X, there exists a surjective mapping from X to $P(X)$ where $P(X)$ denotes the set of subsets of X.

2. Prove that $44/333 = .132132132\cdots$ by using only the definition of limit. Prove that $1 = .999\cdots$ by using only the definition of limit. (You can use the fact that a monotone increasing sequence with an upper bound has a limit that is equal to the least upper bound of the sequence. If you use this result, you should explain exactly what the terms mean and how you are using it.) Note that you will have to explain what is the meaning of an expression like $.132132132\cdots$ as a limit of a certain sequence.

3. Let $\mathcal{R}$ denote the real numbers. A function $f : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be continuous at a point $a \in \mathcal{R}$ if $lim_{x\longrightarrow a} f(x) = f(a)$. In more detail this means that given an $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

   The function f is said to be continuous on a subset S of $\mathcal{R}$ if it is continuous at every point of S. Thus f is continuous on $\mathcal{R}$ if it is continuous at every point of $\mathcal{R}$.

   (a) Show that $f(x) = 3x$ is continuous at every point in $\mathcal{R}$.

   (b) Define $g : \mathcal{R} \longrightarrow \mathcal{R}$ by $g(x) = 1/x$ when $x \neq 0$ and $g(0) = 0$. Show that g is continuous at every point in $\mathcal{R}$ except $a = 0$. Describe how the definition of continuity fails for g at $a = 0$.

   (c) Give an example of a function f on the closed interval $[0, 1]$ such that $f(0) < 0$ and $f(1) > 0$ and $f(x) \neq 0$ for all $x \in [0, 1]$.

   (d) **True or False**: One can not find continuous functions f defined on the the closed interval $[0, 1]$ such that $f(0) < 0$ and $f(1) > 0$ and $f(x) \neq 0$ for all $x \in [0, 1]$.

   (e) Give the definition of the derivative$f'(x)$ at a point x of a function $f : \mathcal{R} \longrightarrow \mathcal{R}$. Give an example of a function that is continuous at 0 but the derivative of the function at 0 does not exist.

4. (a) Given a sequence $\{a_n\}$ of real numbers, give the definition of the statement: "$lim_{n\longrightarrow\infty}\, a_n$ exists."

   (b)**True or False**: $lim_{n\longrightarrow\infty}\, \frac{n!}{(n+1)!} = 1$.

   (c)**True or False**: $lim_{n\longrightarrow\infty}\, \frac{(1/7)^n}{(1/9)^n} = 0$.

5. (a) Give the definition of convergence for a series $S = \sum_{n=1}^{\infty} b_n$ where S is a real number and b_n is a real number for each natural number n.

   (b) Give an example of a series $\sum_{n=1}^{\infty} b_n$ that converges, but $\sum_{n=1}^{\infty} |b_n|$ diverges.

   (c)**True or False**: $\sum_{n=0}^{\infty}(-1)^n = 1/2$.

   (d)**True or False**: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges and its value is 4/3.

   (e)**True or False**: $\sum_{n=1}^{\infty} \frac{2}{3^n}$ converges and its value is 1.

6. Consider the following proof that $-1 = 1 + 2 + 2^2 + 2^3 + \cdots$: Let $s = 1 + 2 + 2^2 + 2^3 + \cdots$. Then $2s = 2 + 2^2 + 2^3 + \cdots$. Therefore $2s = s - 1$ and so $s = -1$. Explain as precisely as you can, what is wrong with this proof.

7. (a) **True or False**: The intersection of an arbitrary collection of open subsets of the real line is always an open set.

   (b) Give a specific example of an *empty* intersection of an infinite collection of nested open intervals where each individual interval is non-empty.

   (c) Explain the definition of the Cantor Set. Prove that the Cantor set is not countable.

8. Let $\mathcal{U}$ denote the collection of sets U in the real line such that each U is the complement of a finite number of points in the real line. Prove that any union of elements of $\mathcal{U}$ is an element of $\mathcal{U}$. Prove that any intersection of a finite number of elements of $\mathcal{U}$ is an element of $\mathcal{U}$.

9. (a) **True or False** Let X denote the union of the open intervals $(0, 1]$ and $(1, 2)$. Then X is a dis-connected subset of the real line.

   (b) **True of False** Path connected sets are connected. (Recall that a subset S of R^n is said to be path connected if , given points p and q in S, there is a function $f : [0, 1] \longrightarrow R^n$ such that $f(0) = p, f(1) = q$ and $f([0, 1])$ is contained in S. In other words, given any two points in S, there is a continous path from one to the other that is contained entirely in S.)

   (c) **True of False** Connected sets are path connected. Recall that we gave an example in class of the graph of the function $f(x) = 0$ for $x \leq 0$ and $f(x) = sin(1/x)$ for $x > 0$.

10. (a) Suppose that A is an open subset of the real line.

    True or False: For every point $x \in A$ there is an $\epsilon > 0$ so that the open interval $(x - \epsilon, x + \epsilon)$ is a subset of A.

    (b) **True or False**: There exist no subsets of the real numbers that are both open and closed.

A function is said be bounded if the set of its function values is a bounded subset of the real line.

(c) **True or False**: A continuous real-valued function, defined on an open interval is bounded on that interval.

(d) **True or False**: Every open cover of the the closed interval $[0, 1]$ has an infinite subcover.

(e) **True or False**: Given $\epsilon > 0$, if S is a countable subset of the real line, then S has an open cover by open intervals such that the sum of the lengths of these intervals can be made less than ϵ..

(f) **True or False**: Every open cover for the set of rational numbers in the real line has the property that the sum of the lengths of the intervals in the cover is greater than $1/N!$ where $N = 2^{2^{2^{137}}}$.