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# ON MANIFOLDS HOMEOMORPHIC TO THE 7-SPHERE

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The object of this note will be to show that the 7-sphere possesses several distinct differentiable structures.

In §1 an invariant $\lambda$ is constructed for oriented, differentiable 7-manifolds $M^7$ satisfying the hypothesis (*) $H^3(M^7) = H^4(M^7) = 0$. (Integer coefficients are to be understood.) In §2 a general criterion is given for proving that an $n$-manifold is homeomorphic to the sphere $S^n$. Some examples of 7-manifolds are studied in §3 (namely 3-sphere bundles over the 4-sphere). The results of the preceding two sections are used to show that certain of these manifolds are topological 7-spheres, but not differentiable 7-spheres. Several related problems are studied in §4.

All manifolds considered, with or without boundary, are to be differentiable, orientable and compact. The word *differentiable* will mean differentiable of class $C^\infty$. A closed manifold $M^n$ is *oriented* if one generator $\mu \in H_n(M^n)$ is distinguished.

## §1. The invariant $\lambda(M^7)$

For every closed, oriented 7-manifold satisfying (*) we will define a residue class $\lambda(M^7)$ modulo 7. According to Thom [5] every closed 7-manifold $M^7$ is the boundary of an 8-manifold $B^8$. The invariant $\lambda(M^7)$ will be defined as a function of the index $\tau$ and the Pontrjagin class $p_1$ of $B^8$.

An orientation $\nu \in H_8(B^8, M^7)$ is determined by the relation $\partial\nu = \mu$. Define a quadratic form over the group $H^4(B^8, M^7)/(\text{torsion})$ by the formula $\alpha \to \langle \nu, \alpha^2 \rangle$. Let $\tau(B^8)$ be the index of this form (the number of positive terms minus the number of negative terms, when the form is diagonalized over the real numbers).

Let $p_1 \in H^4(B^8)$ be the first Pontrjagin class of the tangent bundle of $B^8$. (For the definition of Pontrjagin classes see [2] or [6].) The hypothesis (*) implies that the inclusion homomorphism

$$i: H^4(B^8, M^7) \to H^4(B^8)$$

is an isomorphism. Therefore we can define a "Pontrjagin number"

$$q(B^8) = \langle \nu, (i^{-1}p_1)^2 \rangle.$$

Theorem 1. *The residue class of $2q(B^8) - \tau(B^8)$ modulo 7 does not depend on the choice of the manifold $B^8$.*

Define $\lambda(M^7)$ as this residue class.[2] As an immediate consequence we have:

Corollary 1. *If $\lambda(M^7) \neq 0$ then $M^7$ is not the boundary of any 8-manifold having fourth Betti number zero.*

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Let $B_1^8$, $B_2^8$ be two manifolds with boundary $M^7$. (We may assume they are disjoint.) Then $C^8 = B_1^8 \cup B_2^8$ is a closed 8-manifold which possesses a differentiable structure compatible with that of $B_1^8$ and $B_2^8$. Choose that orientation $\nu$ for $C^8$ which is consistent with the orientation $\nu_1$ of $B_1^8$ (and therefore consistent with $-\nu_2$). Let $q(C^8)$ denote the Pontrjagin number $\langle \nu, p_1^2(C^8)\rangle$.

According to Thom [5] or Hirzebruch [2] we have

$$\tau(C^8) = \langle \nu, \tfrac{1}{45}(7p_2(C^8) - p_1^2(C^8))\rangle;$$

and therefore

$$45\tau(C^8) + q(C^8) = 7\langle \nu, p_2(C^8)\rangle \equiv 0 \pmod 7.$$

This implies

$$2q(C^8) - \tau(C^8) \equiv 0 \pmod 7. \tag{1}$$

LEMMA 1. *Under the above conditions we have*

$$\tau(C^8) = \tau(B_1^8) - \tau(B_2^8) \tag{2}$$

*and*

$$q(C^8) = q(B_1^8) - q(B_2^8). \tag{3}$$

Formulas 1, 2, 3 clearly imply that

$$2q(B_1^8) - \tau(B_1^8) \equiv 2q(B_2^8) - \tau(B_2^8) \pmod 7;$$

which is just the assertion of Theorem 1.

PROOF OF LEMMA 1. Consider the diagram

$$\begin{array}{ccc} H^n(B_1, M) \oplus H^n(B_2, M) & \xleftarrow[\approx]{h} & H^n(C, M) \\ \Big\downarrow i_1 \oplus i_2 & & \Big\downarrow j \\ H^n(B_1) \oplus H^n(B_2) & \xleftarrow{k} & H^n(C) \end{array}$$

Note that for $n = 4$, these homomorphisms are all isomorphisms. If $\alpha = jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C)$, then

$$\langle \nu, \alpha^2\rangle = \langle \nu, jh^{-1}(\alpha_1^2 \oplus \alpha_2^2)\rangle = \langle \nu_1 \oplus (-\nu_2), \alpha_1^2 \oplus \alpha_2^2\rangle = \langle \nu_1, \alpha_1^2\rangle - \langle \nu_2, \alpha_2^2\rangle. \tag{4}$$

Thus the quadratic form of $C^8$ is the "direct sum" of the quadratic form of $B_1^8$ and the negative of the quadratic form of $B_2^8$. This clearly implies formula (2).

Define $\alpha_1 = i_1^{-1}p_1(B_1)$ and $\alpha_2 = i_2^{-1}p_1(B_2)$. Then the relation

$$k(p_1(C)) = p_1(B_1) \oplus p_1(B_2)$$

implies that

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[2] Similarly for $n = 4k - 1$ a residue class $\lambda(M^n)$ modulo $s_k\mu(L_k)$ could be defined. (See [2] page 14.) For $k = 1, 2, 3, 4$ we have $s_k\mu(L_k) = 1, 7, 62, 381$ respectively.

$$jh^{-1}(\alpha_1 \oplus \alpha_2) = p_1(C).$$

The computation (4) now shows that

$$\langle \nu, p_1^2(C) \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle,$$

which is just formula (3). This completes the proof of Theorem 1.

The following property of the invariant $\lambda$ is clear.

LEMMA 2. *If the orientation of $M^7$ is reversed then $\lambda(M^7)$ is multiplied by $-1$.*

As a consequence we have

COROLLARY 2. *If $\lambda(M^7) \neq 0$ then $M^7$ possesses no orientation reversing diffeomorphism[3] onto itself.*

## §2. A partial characterization of the $n$-sphere

Consider the following hypothesis concerning a closed manifold $M^n$ (where $R$ denotes the real numbers).

(H) *There exists a differentiable function $f: M^n \to R$ having only two critical points $x_0$, $x_1$. Furthermore these critical points are non-degenerate.*

(That is if $u_1, \cdots, u_n$ are local coordinates in a neighborhood of $x_0$ (or $x_1$) then the matrix $(\partial^2 f/\partial u_i \partial u_j)$ is non-singular at $x_0$ (or $x_1$).)

THEOREM 2. *If $M^n$ satisfies the hypothesis* (H) *then these exists a homeomorphism of $M^n$ onto $S^n$ which is a diffeomorphism except possibly at a single point.*

*Added in proof.* This result is essentially due to Reeb [7].

The proof will be based on the orthogonal trajectories of the manifolds $f =$ constant.

Normalize the function $f$ so that $f(x_0) = 0$, $f(x_1) = 1$. According to Morse ([3] Lemma 4) there exist local coordinates $v_1, \cdots, v_n$ in a neighborhood $V$ of $x_0$ so that $f(x) = v_1^2 + \cdots + v_n^2$ for $x \in V$. (Morse assumes that $f$ is of class $C^3$, and constructs coordinates of class $C^1$; but the same proof works in the $C^\infty$ case.) The expression $ds^2 = dv_1^2 + \cdots + dv_n^2$ defines a Riemannian metric in the neighborhood $V$. Choose a differentiable Riemannian metric for $M^n$ which coincides with this one in some neighborhood[4] $V'$ of $x_0$. Now the gradien of $f$ can be considered as a contravariant vector field.

Following Morse we consider the differential equation

$$\frac{dx}{dt} = \text{grad } f / \| \text{grad } f \|^2.$$

In the neighborhood $V'$ this equation has solutions

$$(v_1(t), \cdots, v_n(t)) = (a_1(t)^{\frac{1}{2}}, \cdots, a_n(t)^{\frac{1}{2}})$$

for $0 \leqq t < \varepsilon$, where $a = (a_1, \cdots, a_n)$ is any $n$-tuple with $\sum a_i^2 = 1$. These can be extended uniquely to solutions $x_a(t)$ for $0 \leqq t \leqq 1$. Note that these solutions satisfy the identity

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[3] A *diffeomorphism* $f$ is a homeomorphism onto, such that both $f$ and $f^{-1}$ are differentiable.

[4] This is possible by [4] 6.7 and 12.2.

$$f(x_a(t)) = t.$$

Map the interior of the unit sphere of $R^n$ into $M^n$ by the map

$$(a_1(t)^{\frac{1}{2}}, \cdots, a_n(t)^{\frac{1}{2}}) \rightarrow x_a(t).$$

It is easily verified that this defines a diffeomorphism of the open $n$-cell onto $M^n - (x_1)$. The assertion of Theorem 2 now follows.

Given any diffeomorphism $g: S^{n-1} \rightarrow S^{n-1}$, an $n$-manifold can be obtained as follows.

Construction (C). *Let $M^n(g)$ be the manifold obtained from two copies of $R^n$ by matching the subsets $R^n - (0)$ under the diffeomorphism*

$$u \rightarrow v = \frac{1}{\| u \|} g\left(\frac{u}{\| u \|}\right).$$

(Such a manifold is clearly homeomorphic to $S^n$. If $g$ is the identity map then $M^n(g)$ is diffeomorphic to $S^n$.)

Corollary 3. *A manifold $M^n$ can be obtained by the construction* (C) *if and only if it satisfies the hypothesis* (H).

Proof. If $M^n(g)$ is obtained by the construction (C) then the function

$$f(x) = \| u \|^2/(1 + \| u \|^2) = 1/(1 + \| v \|^2)$$

will satisfy the hypothesis (H). The converse can be established by a slight modification of the proof of Theorem 2.

## §3. Examples of 7-manifolds

Consider 3-sphere bundles over the 4-sphere with the rotation group SO(4) as structural group. The equivalence classes of such bundles are in one-one correspondence[5] with elements of the group $\pi_3(\mathrm{SO}(4)) \approx Z + Z$. A specific isomorphism between these groups is obtained as follows. For each $(h, j) \in Z + Z$ let $f_{hj}: S^3 \rightarrow \mathrm{SO}(4)$ be defined by $f_{hj}(u) \cdot v = u^h v u^j$, for $v \in R^4$. Quaternion multiplication is understood on the right.

Let $\iota$ be the standard generator for $H^4(S^4)$. Let $\xi_{hj}$ denote the sphere bundle corresponding to $(f_{hj}) \in \pi_3(\mathrm{SO}(4))$.

Lemma 3. *The Pontrjagin class $p_1(\xi_{hj})$ equals $\pm 2(h - j)\iota$.*

(The proof will be given later. One can show that the characteristic class $\bar{c}(\xi_{hj})$ (see [4]) is equal to $(h + j)\iota$.)

For each odd integer $k$ let $M_k^7$ be the total space of the bundle $\xi_{hj}$ where $h$ and $j$ are determined by the equations $h + j = 1$, $h - j = k$. This manifold $M_k^7$ has a natural differentiable structure and orientation, which will be described later.

Lemma 4. *The invariant $\lambda(M_k^7)$ is the residue class modulo 7 of $k^2 - 1$.*

Lemma 5. *The manifold $M_k^7$ satisfies the hypothesis* (H).

Combining these we have:

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[5] See [4] §18.

THEOREM 3. *For* $k^2 \not\equiv 1 \bmod 7$ *the manifold* $M_k^7$ *is homeomorphic to* $S^7$ *but not diffeomorphic to* $S^7$.

(For $k = \pm 1$ the manifold $M_k^7$ is diffeomorphic to $S^7$; but it is not known whether this is true for any other $k$.)

Clearly any differentiable structure on $S^7$ can be extended through $R^8 - (0)$. However:

COROLLARY 4. *There exists a differentiable structure on* $S^7$ *which cannot be extended throughout* $R^8$.

This follows immediately from the preceding assertions, together with Corollary 1.

PROOF OF LEMMA 3. It is clear that the Pontrjagin class $p_1(\xi_{hj})$ is a linear function of $h$ and $j$. Furthermore it is known that it is independent of the orientation of the fibre. But if the orientation of $S^3$ is reversed, then $\xi_{hj}$ is replaced by $\xi_{-j-h}$. This shows that $p_1(\xi_{hj})$ is given by an expression of the form $c(h - j)\iota$. Here $c$ is a constant which will be evaluated later.

PROOF OF LEMMA 4. Associated with each 3-sphere bundle $M_k^7 \to S^4$ there is a 4-cell bundle $\rho_k: B_k^8 \to S^4$. The total space $B_k^8$ of this bundle is a differentiable manifold with boundary $M_k^7$. The cohomology group $H^4(B_k^8)$ is generated by the element $\alpha = \rho_k^*(\iota)$. Choose orientations $\mu$, $\nu$ for $M_k^7$ and $B_k^8$ so that

$$\langle \nu, (i^{-1}\alpha)^2 \rangle = +1.$$

Then the index $\tau(B_k^8)$ will be $+1$.

The tangent bundle of $B_k^8$ is the "Whitney sum" of (1) the bundle of vectors tangent to the fibre, and (2) the bundle of vectors normal to the fibre. The first bundle (1) is induced (under $\rho_k$) from the bundle $\xi_{hj}$, and therefore has Pontrjagin class $p_1 = \rho_k^*(c(h - j)\iota) = ck\alpha$. The second is induced from the tangent bundle of $S^4$, and therefore has first Pontrjagin class zero. Now by the Whitney product theorem ([2] or [6])

$$p_1(B_k^8) = ck\alpha + 0.$$

For the special case $k = 1$ it is easily verified that $B_1^8$ is the quaternion projective plane $P_2(K)$ with an 8-cell removed. But the Pontrjagin class $p_1(P_2(K))$ is known to be twice a generator of $H^4(P_2(K))$. (See Hirzebruch [1].) Therefore the constant $c$ must be $\pm 2$, which completes the proof of Lemma 3.

Now $q(B_k^8) = \langle \nu, (i^{-1}(\pm 2k\alpha))^2 \rangle = 4k^2$; and $2q - \tau = 8k^2 - 1 \equiv k^2 - 1$ (mod 7). This completes the proof of Lemma 4.

PROOF OF LEMMA 5. As coordinate neighborhoods in the base space $S^4$ take the complement of the north pole, and the complement of the south pole. These can be identified with euclidean space $R^4$ under stereographic projection. Then a point which corresponds to $u \in R^4$ under one projection will correspond to $u' = u/\| u \|^2$ under the other.

The total space $M_k^7$ can now be obtained as follows.[5] Take two copies of $R^4 \times S^3$ and identify the subsets $(R^4 - (0)) \times S^3$ under the diffeomorphism

$$(u, v) \to (u', v') = (u/\| u \|^2, u^h v u^j / \| u \|)$$

(using quaternion multiplication). This makes the differentiable structure of $M_k^7$ precise.

Replace the coordinates $(u', v')$ by $(u'', v')$ where $u'' = u'(v')^{-1}$. Consider the function $f: M_k^7 \to R$ defined by

$$f(x) = \Re(v)/(1 + \| u \|^2)^{\frac{1}{2}} = \Re(u'')/(1 + \| u'' \|^2)^{\frac{1}{2}};$$

where $\Re(v)$ denotes the real part of the quaternion $v$. It is easily verified that $f$ has only two critical points (namely $(u, v) = (0, \pm 1)$) and that these are non-degenerate. This completes the proof.

## §4. Miscellaneous results

THEOREM 4. *Either* (a) *there exists a closed topological* 8-*manifold which does not possess any differentiable structure; or* (b) *the Pontrjagin class* $p_1$ *of an open* 8-*manifold is not a topological invariant.*

(The author has no idea which alternative holds.)

PROOF. Let $X_k^8$ be the topological 8-manifold obtained from $B_k^8$ by collapsing its boundary (a topological 7-sphere) to a point $x_0$. Let $\bar{\alpha} \in H^4(X_k^8)$ correspond to the generator $\alpha \in H^4(B_k^8)$. Suppose that $X_k^8$, possesses a differentiable structure, and that $p_1(X_k^8 - (x_0))$ is a topological invariant. Then $p_1(X_k^8)$ must equal $\pm 2k\bar{\alpha}$, hence

$$2q(X_k^8) - \tau(X_k^8) = 8k^2 - 1 \equiv k^2 - 1 \qquad (\text{mod } 7).$$

But for $k^2 \not\equiv 1$ (mod 7) this is impossible.

Two diffeomorphisms $f, g: M_1^n \to M_2^n$ will be called *differentiably isotopic* if there exists a diffeomorphism $M_1^n \times R \to M_2^n \times R$ of the form $(x, t) \to (h(x, t), t)$ such that

$$h(x, t) = \begin{cases} f(x) & (t \leqq 0) \\ g(x) & (t \geqq 1). \end{cases}$$

LEMMA 6. *If the diffeomorphisms* $f, g: S^{n-1} \to S^{n-1}$ *are differentiably isotopic, then the manifolds* $M^n(f)$, $M^n(g)$ *obtained by the construction* (C) *are diffeomorphic.*

The proof is straightforward.

THEOREM 5. *There exists a diffeomorphism* $f: S^6 \to S^6$ *of degree* $+1$ *which is not differentiably isotopic to the identity.*

Proof. By Lemma 5 and Corollary 3 the manifold $M_3^7$ is diffeomorphic to $M^7(f)$ for some $f$. If $f$ were differentiably isotopic to the identity then Lemma 6 would imply that $M_3^7$ was diffeomorphic to $S^7$. But this is false by Lemma 4.

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