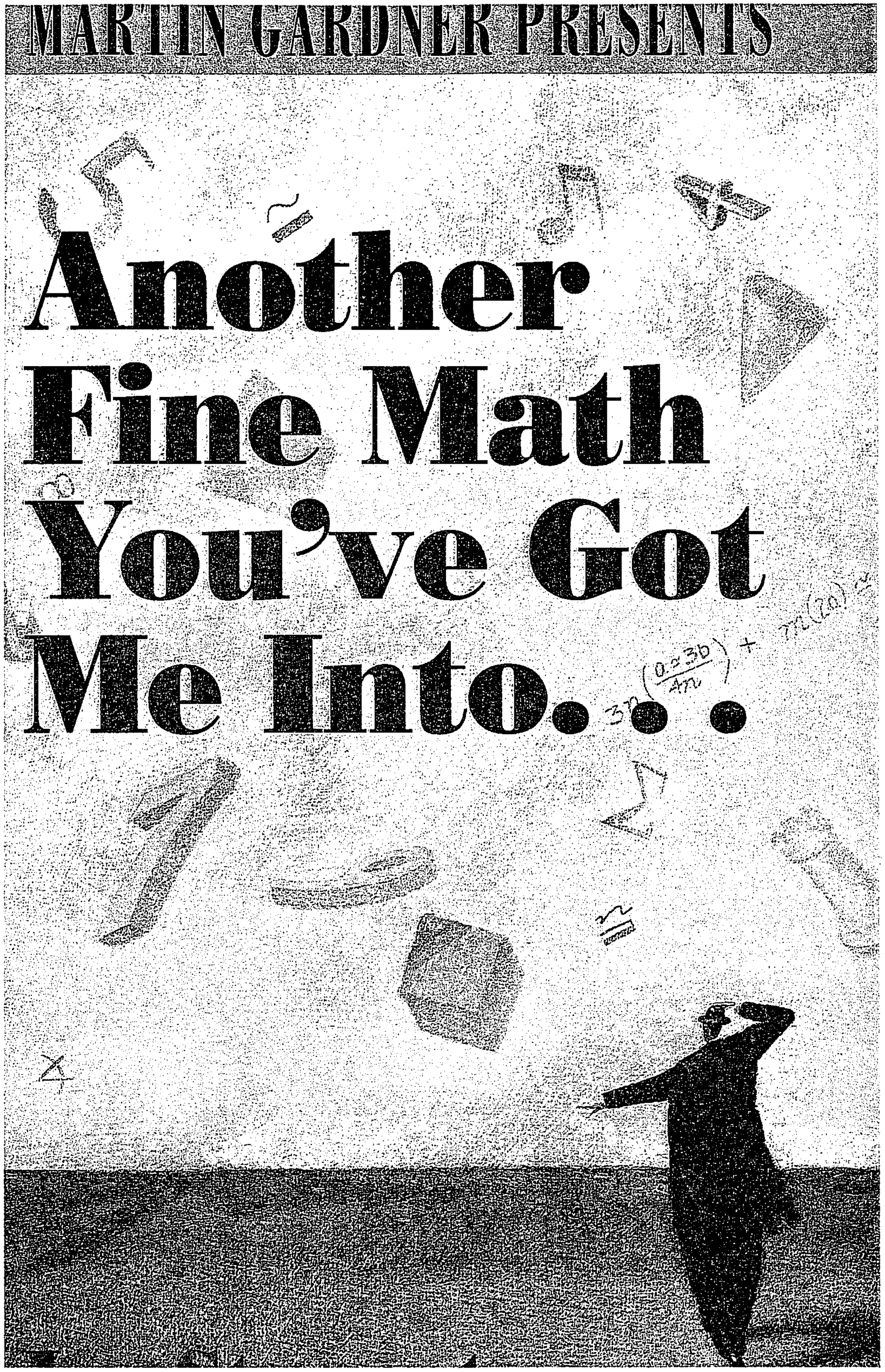
MARTIN GARDNER PRESENTS
Another
Fine Math
You've Got
Me Into. . .

CHAPTER 15

The Well-Tempered Calculator

"O blessed silence!" said Oliver Gurney.

"Don't be such a fussy old killjoy," said Deirdre. "*I* thought the 'Pheasant-Plucking Song' was rather good."

"I suppose," said Oliver grumpily. "But it doesn't suit the atmosphere of the Potted Dormouse. I knew that new landlord was going to cause trouble."

Oliver Gurney is eccentric, rotund, and he invents things—most of which lead to disaster. The *Potted Dormouse* is an extremely old pub east of Manchester: from the outside it's a gray stone box with a sign that closely resembles a pig in a fur coat. Oliver discovered it seven years ago while he was inventing a bacterium that converts oil into treacle, and he acts as if he owns it. Inside, for six days of the week, venerable Lancastrians eye each other stonily across their beer mugs and play shove-ha'penny, just as they have always done. But after the new landlord took over, Friday night is music night, and a bunch of local lads and lasses sing and play the guitar for a few hours.

"Anyway," Oliver went on, "that guitarist keeps getting his fingers all mixed up. At the very—"

"He only did it once," I interrupted. "He's got rather big fingers and he has trouble on the high notes when the frets get close together."

"Then he should get a guitar with the frets spaced farther apart."

"I don't think that would work," said Deirdre.

"No, it wouldn't," I confirmed. "There's a very good reason why the frets have to be spaced the way they are."

"To make the notes sound right, I suppose," said Deirdre, and I nodded. "But I don't see why the spacing gets smaller as the notes get higher," she added.

"Elementary physics of vibrating strings," said Oliver pompously. At the same time I said "Mathematics." Probably equally pompously, but I'm a poor judge of my own failings so I can't tell you for sure.

"I'm not very keen on mathematics *or* physics," said Deirdre. "Too impersonal. I prefer human things, like history and the arts."

"What fascinates me about music," I said, "is that it combines the lot: science, arts, culture, history . . . in fact, music is one of the oldest sciences. It was music, more than anything else, that led the Pythagoreans to believe that the universe is a harmonious place governed by numbers."

"Music a *science?*" said Deirdre in astonishment. Oliver perked up at once: he *loves* anything scientific, but there are huge gaps in his knowledge. So for the next hour or so I gave them a guided tour of music as a mathematical endeavor.

Today's Western music is based upon a scale of notes, generally referred to by the letters A–G, together with symbols # (sharp) and ♭ (flat). Starting from C, for example, successive notes are

	$C^\#$		$D^\#$			$F^\#$		$G^\#$		$A^\#$	
C		D		E	F		G		A		B
	$D^\flat$		$E^\flat$			$G^\flat$		$A^\flat$		$B^\flat$	

and then it all repeats with C, but one octave higher. On a piano the white keys are C D E F G A B, and the black keys are the sharps and flats. This is a very curious system: some notes seem to have two names, while others, such as $B^{\#}$, are not represented at all.

Of course there's more to it than that, and appearances are rather deceptive.

Today's system evolved over a long period of time, and it's a compromise between conflicting requirements, all of which trace back to the Pythagorean cult of ancient Greece. For convenience, I'll use the modern notation when giving examples, but purists will rightly object that I'm confusing slightly different ideas.

Claudius Ptolemy, who flourished around A.D. 150, is best known for his astronomical and geographical work, but he also wrote a book on the theory of music, called the *Harmonics.* Here Ptolemy reports the Pythagorean contention that the intervals between musical notes can be represented by whole number ratios. They demonstrated this experimentally using a rather clumsy device known as a canon (Figure 104*A*), a sort of one-string guitar.

If you slide the movable bridge along a canon, certain positions seem to produce notes that are more harmonious than others when compared with the note sounded by a full string. The most basic such interval is the octave: on a piano it is a gap of eight white notes. On a canon, it is the interval between the note played by a full string (Figure 104*A*) and that played by one of exactly half the length (Figure 104*B*). Thus the ratio of the length of string that produces a given note, to the length that produces its octave, is 2/1. This is true independently of the pitch of the original note. Other whole-number ratios produce harmonious intervals as well. The main ones are the fourth, a ratio of 4/3 (Figure 104*C*) and the fifth, a ratio of 3/2 (Figure 104*D*). Starting at a base note of C these are

```
  C   D E   F    G  A B   C
  |         |    |        |
base     fourth fifth   octave
```

and you can probably see where the names came from. Other intervals are formed by combining these building blocks.

It is thought that, in order to create a harmonious scale, the Pythagoreans began at a base note and ascended in fifths. This yields a series of notes played by strings whose lengths have the ratios

$$1\ \left(\frac{3}{2}\right)\ \left(\frac{3}{2}\right)^2\ \left(\frac{3}{2}\right)^3\ \left(\frac{3}{2}\right)^4\ \left(\frac{3}{2}\right)^5$$

or

$$1 \quad \frac{3}{2} \quad \frac{9}{4} \quad \frac{27}{8} \quad \frac{81}{16} \quad \frac{243}{32}$$

Most of these notes lie outside a single octave, that is, the ratios are greater than 2/1. But we can descend from them in octaves (dividing successively by 2) until the ratios lie between 1/1 and 2/1. Then we rearrange the ratios in numerical order, to get

$$1 \quad \frac{9}{8} \quad \frac{81}{64} \quad \frac{3}{2} \quad \frac{27}{16} \quad \frac{23}{128}$$

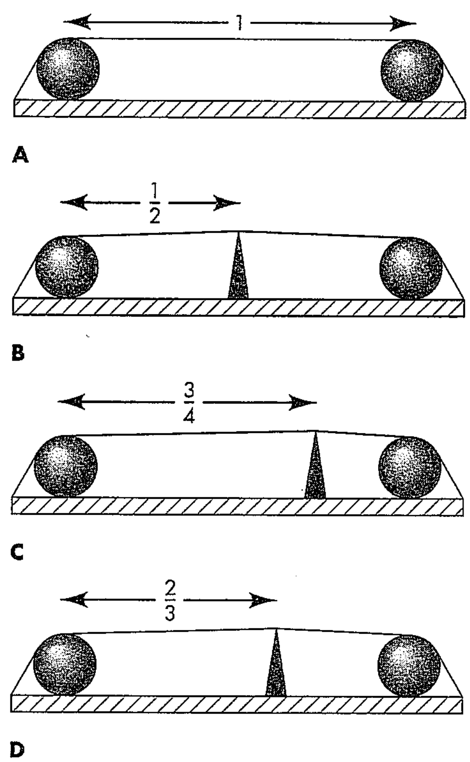

FIGURE 104 The canon, an experimental device used by the ancient Greeks to study musical ratios. *A.* Full string sounds base note. *B.* String 1/2 the length (ratio 2/1) sounds note an octave above base note. *C.* String 3/4 the length (ratio 4/3) sounds note a fourth above base note. *D.* String 2/3 the length (ratio 3/2) sounds note a fifth above base note.

On a piano, these correspond approximately to the notes

C D E G A B

and, as the notation suggests, there's something missing! The gap between 81/64 and 3/2 sounds "bigger" than the others. We can plug the gap neatly by adding in the fourth, a ratio of 4/3, which is F on the piano. In fact, we could have incorporated it from the start if we had *descended* from the base note by a fifth, adding the ratio 2/3 to the front of the sequence, and then ascended by an octave to get $2 \times (2/3) = (4/3)$.

The resulting scale corresponds approximately to the white notes on the piano, shown in Figure 105. The last line shows the intervals between successive notes, also expressed as ratios. There are exactly two different ratios: the tone 9/8 and the semitone 256/243.

It is here that the black notes of the piano, the sharps and flats, come in. An interval of two semitones is $(256/243)^2$, or 65,536/59,049, which is approximately 1.11. A tone is a ratio of $9/8 = 1.125$. These are not quite the same, but nevertheless it looks as if two semitones make a tone. This means that there are gaps in the scale: each tone interval must be divided up into two intervals, each being as close as is feasible to a semitone.

There are various schemes for doing this. The so-called chromatic scale starts with the fractions $(3/2)^n$ for $n = -6, -5, \ldots, 5, 6$. It reduces them to the same octave by repeatedly multiplying or dividing by two, and then places them in order. The result is shown in Figure 106. Each sharp bears a ratio 2,187/2,048 to the note below it, and from which it takes its name; each flat bears a ratio 2,048/2,187 to the note

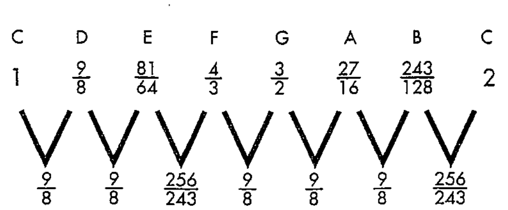

FIGURE 105 Scale formed purely from fifths and octaves approximates the white notes on a piano.

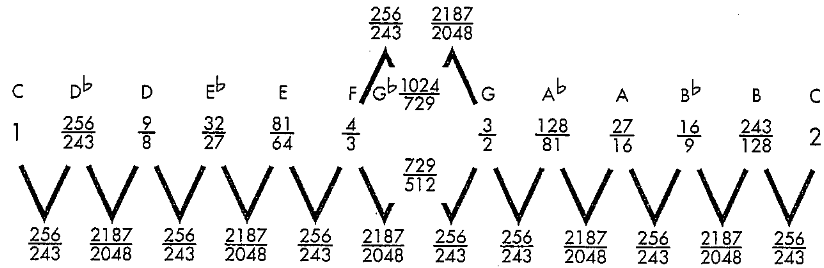

FIGURE 106 Chromatic scale of twelve notes, incorporating the black notes (sharps ♯ and flats ♭) as well. Two notes, F♯ and G♭, are trying to occupy the same slot.

above. There's a glitch in the middle: two notes, F♯ and G♭, are trying to occupy the same slot, but they differ very slightly from each other. There are many other schemes, also leading to distinctions between sharps and flats, but they all involve a 12-note scale that is very close to that formed by the white *and* black notes of the piano.

"Ah," said Oliver. "There's a good physical reason for all this, you know."

"You mean the waveforms—"

"Look, Ian, you've been talking for half an hour non-stop. It's my turn!" I apologized and he took up the tale.

"Y'see, Deirdre, when a string vibrates it does so as a standing wave [Figure 107]. And you have to fit a whole number of wavelengths in between the two ends, so that's where the Pythagorean whole numbers come in. When you sound a note on a guitar you don't just get a single wave along the string: you get harmonics with two waves, three, four, and so on. They all combine to give a richer sound.

"Now, if you combine two waves of slightly different wavelength, you get beats where they reinforce each other [Figure 108]. Those sound rather unpleasant to the ear."

"I think it may have something to do with the non-linear response of the eardrum," I put in. "There's probably a physiological rea—"

"The same problem occurs if *harmonics* of notes beat. The simplest way to avoid that is to use notes whose wavelengths are related by simple numerical ratios, say 3/2 or 4/3. So that's where the Pythagorean ratios come from, too."

"That makes sense," said Deirdre.

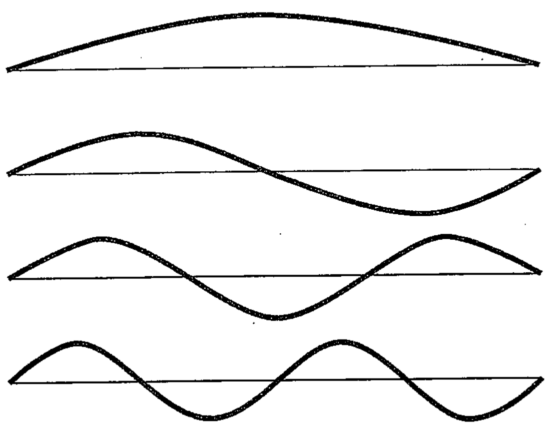

FIGURE 107 Vibrations of a string form a standing sinusoidal wave. The length of the string is an integer number of wavelengths.

"Yes," I said, "but I still think you have to consider the physiol—"

"A good test of that theory," said Oliver, kicking me under the table, "was performed by Hermann von Helmholtz in 1877. He studied beats between harmonics, and used them to predict how the degree of dissonance between two notes should vary with the ratio between them. It agrees very well with psychological judgments made by human volunteers" (Figure 109).

"I *said* you have to consider the way an actual human ear—"

"I'll have another glass of wine, please, Oliver," said Deirdre tactfully, and sent him off to get one while I still had some shin left. That gave me time to pick up the story again.

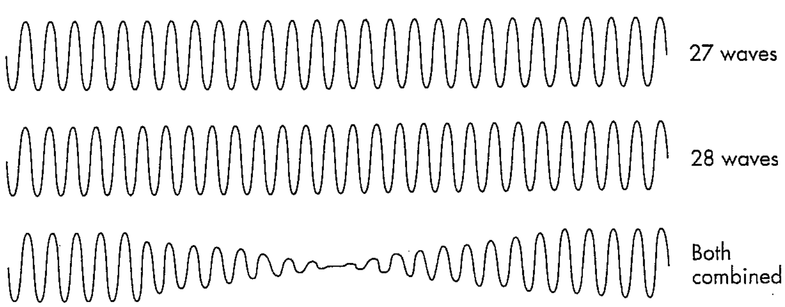

FIGURE 108 Combining waves of slightly different wavelengths leads to unpleasant beats.

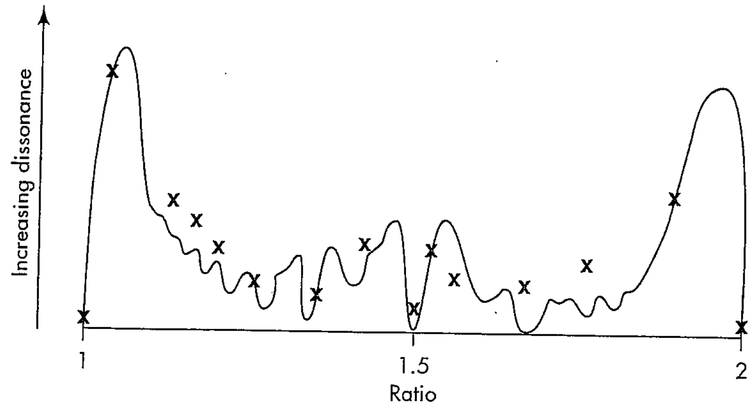

FIGURE 109 Helmholtz's theoretical curve of dissonance against musical interval (solid curve) compared with assessments by human observers (crosses).

As I explained to Deirdre, the reason for the glitch in the chromatic scale, and the reason that there are many different schemes for building scales, is that no "perfect" 12-note scale can be based on the Pythagorean ratios of 3/2 and 4/3. By a perfect scale I mean one where the ratios are

$$1 \quad r \quad r^2 \quad r^3 \quad r^4 \ldots r^{12} = 2$$

for a fixed number r. The Pythagorean ratios involve only the primes 2 and 3: every ratio is of the form 2^a3^b for various integers a and b. For instance $243/128 = 2^{-7}3^5$. Suppose that $r = 2^a3^b$ and $r^{12} = 2$. Then $2^{12a}3^{12b} = 2$, so $2^{12a-1} = 3^{-12b}$. But an integer power of 2 cannot equal an integer power of 3, by uniqueness of prime factorization.

PROBLEM

Does it make any difference to this argument if the scale has a number of notes that differ from 12? Or if we allow other primes to occur in the ratios?

This property of primes puts paid to a musical scale based on Pythagorean principles of the harmony of whole numbers; but it doesn't mean we can't find a suitable number r. The equation $r^{12} = 2$ has a perfectly good solution, namely

$$r = \sqrt[12]{2} = 1.059463094. . .$$

The resulting scale is said to be equitempered.

If you start playing a Pythagorean scale somewhere in the middle—in other words; if you change key—then the sequence of intervals changes slightly. Equitempered scales don't have this problem, so they are useful if you want to play the same instrument in different keys. Musical instruments that must play fixed intervals, such as pianos and guitars, generally use the equitempered scale. The Pythagorean semitone interval is $256/243 = 1.05349. . .$, which is close to $\sqrt[12]{2}$, so the name semitone is used for the basic interval of the equitempered scale.

Deirdre thought about that for a moment. "You said that for vibrating strings, the musical interval is given by the ratio of the lengths. So how does that lead to the positions of the frets on a guitar?"

"Well," I said, "think about the first fret along, corresponding to an increase in pitch of one semitone. The length of string that is allowed to vibrate has to be $1/r$ times the length of the complete string. So the distance to the first fret is $1 - 1/r$ times the length of the complete string. To get the next distance, you just observe that everything has shrunk by a factor of r, so the spaces between successive frets are in the proportions

$$1 \qquad 1/r \qquad 1/r^2 \qquad 1/r^3$$

and so on. Now r is bigger than 1, so $1/r$ is less than 1, and that means that the distances between successive frets are *smaller*" (Figure 110).

Oliver returned with a dry white wine, plus two pints of Fosdick's Best Bitter ("The beer that refreshes parts you don't even have") and three bags of tripe-and-onion flavored crisps, which he adores and Deirdre and I detest. He always buys them because then he gets to eat the lot. By now he was his usual jovial self, and he launched into an animated explanation of how embarrassing it must have been for the Pythagoreans to discover that their beautiful numerical schemes had practical flaws.

I pointed out that when the Greeks were faced with irrational numbers such as $\sqrt[12]{2}$, which cannot be written as exact fractions, they

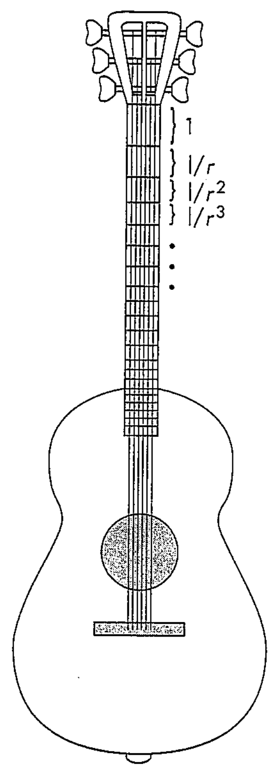

FIGURE 110 Distances between guitar frets shrink for the higher notes.

usually resorted to geometry. According to tradition, Greek geometry placed considerable emphasis on those lengths that can be constructed using only a ruler and a pair of compasses. For example, squares and square roots can be so constructed (Figure 111).

PROBLEM ❷

The ancient problem of "duplicating the cube" asks for such a construction for $\sqrt[3]{2}$*. We now know that no such construction exists. (We also suspect that the emphasis on ruler and compasses was less strong than many history books try to claim, but that's another matter.) Deduce that there is no ruler-and-compass construction for* $\sqrt[12]{2}$ *either.*

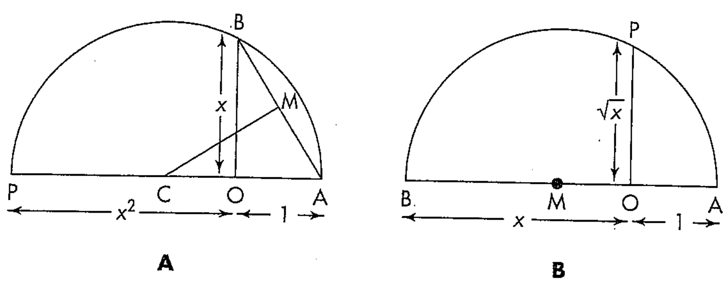

FIGURE 111 Constructing squares and square roots with ruler and compasses, given a line of unit length. *A*. Squares: Draw a right triangle AOB with OA = 1, OB = x. Find the midpoint M of AB and draw MC perpendicular to AB to meet the extension of AO at C. Draw a semicircle with center C through A, to meet the extension of AO at P. Then OP has length x^2. *B*. Square roots: Draw a line AOB with OA = 1, OB = x. Find the midpoint M of AB and draw the semicircle center M through B and A. Draw OP perpendicular to AB to cut the semicircle at P. Then OP has length $\sqrt{x}$.

"Yes, but," Deirdre pointed out, "you've said that the equitempered scale is a compromise, an approximation. And the true fourth, at an interval of 4/3, does in fact sound more harmonious than the equitempered fourth. Singers find it more natural, for example."

There was a stunned silence. "You know about this," I said.

"Yes, I studied musical theory at Huddersfield Poly. But you were explaining it all so nicely, I hated to interrupt."

Oliver started laughing and then stopped again when he realized she must also have heard about von Helmholtz. Deirdre picked up the thread of her remarks. "What I wanted to say was, since the equitempered scale is just a compromise, isn't there some *approximate* geometric construction that tells you where to put the frets on a guitar?"

That set me off again. You see, not only is there just such an approximate construction, but it has a very curious history indeed. The story illustrates the deep elegance of mathematics, but it is also a humbling tale: an outstanding triumph of a practical man nullified by a professional mathematician's carelessness.

"Oooh goody!" said Deirdre, and Oliver's eyes lit up—though that could have been the beer. "*Do* tell."

In the sixteenth and seventeenth centuries, finding geometrical methods for placing frets upon musical instruments—lute and viol rather than guitar—was a serious practical question. In 1581 Vincenzo Galilei, the father of the great Galileo Galilei, advocated the approximation

$$18/17 = 1.05882\ldots$$

This led to a perfectly practical method that was in common use for several centuries. In 1636 Marin Mersenne, a monk better known for his prime numbers of the form 2^p-1, approximated an interval of four semitones by the ratio

$$\frac{2}{3-\sqrt{2}}$$

Taking square roots twice, he could then obtain a better approximation to the interval for one semitone:

$$\sqrt{\sqrt{\left(\frac{2}{3-\sqrt{2}}\right)}} = 1.05973\ldots$$

which is certainly close enough for practical purposes. The formula involves only square roots and thus can be constructed geometrically. However, it is difficult to implement this construction in practice, because errors tend to build up. Something more accurate than Galilei's approximation, but easier to use than Mersenne's, was needed.

In 1743 Daniel Strähle, a craftsman with no mathematical training, published an article in the *Proceedings of the Swedish Academy* presenting a simple and practical construction (Figure 112). You might like to try it out, and compare with measurements from an actual instrument. But how accurate is it? The geometer and economist Jacob Faggot performed a trigonometric calculation to find out, and appended it to Strähle's article, concluding that the maximum error is 1.7%. This is about five times more than a musician would consider acceptable.

Faggot was a founding member of the Swedish Academy, served for three years as its secretary, and published eighteen articles in its *Proceedings*. In 1776 he was ranked as number four in the Academy: Carl Linnaeus, the botanist who set up the basic principles for classifying animals and plants into families and genera, then ranked just ahead of him in second place. So when Faggot declared that Strähle's method was

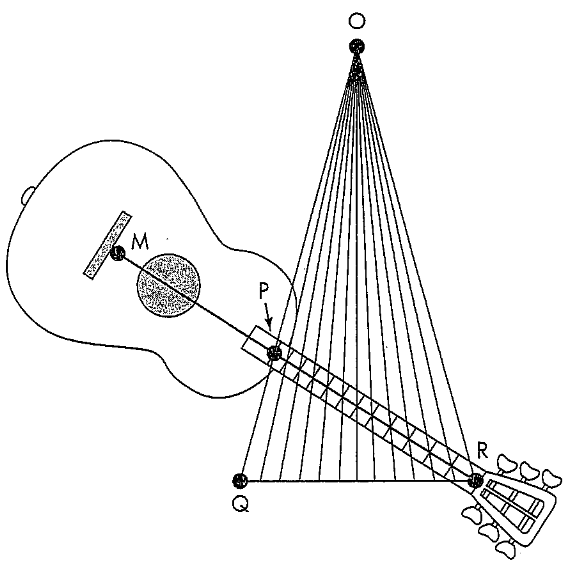

FIGURE 112 Strähle's construction. Let QR be 12 units long, divided into 12 equal intervals of length 1. Find O such that OQ = OR = 24. Join O to the equally spaced points along QR. Let P lie on OQ with PQ 7 units long. Draw RP and extend it to M so that PM = RP. If RM is the fundamental pitch and PM its octave, then the points of intersection of RP with the 11 successive rays from O are successive semitones within the octave, that is, the positions of the 11 frets between R and M.

insufficiently accurate, that was that. For example, F. W. Marpurg's *Treatise on Musical Temperament* of 1776 lists Faggot's conclusion without describing Strähle's method.

It was not until 1957 that J. M. Barbour of Michigan State University discovered that Faggot had made a mistake.

Faggot began by finding the base angle ∠OPQ of the main triangle: it is 75°31′. From this he could find the length RP and the angle ∠PRQ. Each of the 11 angles formed at the top of the main triangle by the rays from the base could also be calculated without difficulty: it was then simple enough to find the lengths cut off along the line RPM.

However, Faggot had computed ∠PRQ as 49°14′, when in fact it is 33°32′. This error, as Barbour puts it, "was fatal, since ∠PRQ was used in the solution of each of the other triangles, and exerted its baleful influence impartially upon them all." The mistake was equivalent to making PQ equal to 8.6 instead of 7. The maximum error reduces from 1.7% to 0.15%, which is perfectly acceptable.

Thus far the story puts mathematicians, if not mathematics itself, in something of a bad light. If only Faggot had bothered to *measure* ∠PRQ! But Barbour went further, asking *why* Strähle's method is so accurate; and what he found is a beautiful illustration of the ability of mathematics to lay bare the reasons behind apparent coincidences. (I should say immediately that there is no suggestion that Strähle himself adopted a similar line of reasoning: as far as anyone knows his method was based upon the intuition of the craftsman rather than any specific mathematical principles. We shall see that his intuition was extremely good!)

The spacing of the nth fret along the line MPR can be represented on a graph (Figure 113*A*). We take the x-axis of the graph to be the line QR in Figure 112, with Q at the origin and R at 1. We move MPR so that it forms the y-axis of the graph, with M at the origin, P at 1, and R at 2. The successive frets are placed along the y-axis at the points 1, r, r^2, . . ., r^{11}, $r^{12} = 2$. (Note that this differs from the ratios $1/r$, $1/r^2$, . . . mentioned above, because we are working from the opposite end of the string.)

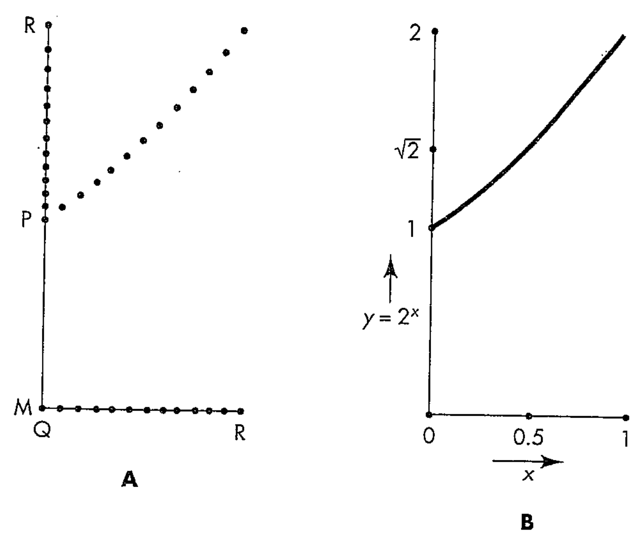

FIGURE 113 *A*. Graph representing Strähle's construction as a function. *B*. Finding the best-fitting fractional linear function by fitting it to the points $x = 0, \frac{1}{2}, 1$.

A mathematician would call Strähle's construction a projection with center O from a set of equally spaced points along QR to the desired points along MPR. It can be shown by simple geometric arguments that such a projection always has the algebraic form $y = (ax + b)/(cx + d)$ where a, b, c, d are constants. This is called a fractional linear function. For Strähle's method, the constants are a = 10, b = 24, c = −7, d = 24: the projection takes a given point x on QR to the point $y = (10x + 24)/(-7x + 24)$ on MPR. I'll call this formula Strähle's function, but we must remember that Strähle himself didn't write it down—it's just an algebraic version of his geometric construction. However, it is the key to the problem.

If the construction were exact, we would have $y = 2^x$. Then the 13 equally spaced points $x = n/12$ on QR, where $n = 0,1,2,\ldots,12$, would be transformed to the points $2^{n/12} = (2^{1/12})^n = r^n$ on MPR, as desired for exact equal temperament.

But it's *not* exact, even though Barbour's calculations show that it's very accurate. Why? The clue is to find the *best possible approximation* to 2^x, valid in the range $0 \le x \le 1$, and of the form $(ax + b)/(cx + d)$. One way to do this is to require the two expressions to agree when $x = 0, \frac{1}{2}$, and 1 (Figure 113*B*). That gives three equations to solve for a, b, c, d; namely

$$b/d = 1$$

$$(\tfrac{1}{2}a + b)/(\tfrac{1}{2}c + d) = \sqrt{2}$$

$$(a + b)/(c + d) = 2$$

At first sight we seem to need one more equation to find four unknowns, but really we only need the *ratios* b/a, c/a, and d/a, so three equations are enough.

This approach leads to the values

$$a = 2 - \sqrt{2}$$

$$b = \sqrt{2}$$

$$c = 1 - \sqrt{2}$$

$$d = \sqrt{2}$$

so that the best possible approximation to 2^x by a fractional linear function takes the form

$$y = \frac{(2-\sqrt{2})x+\sqrt{2}}{(1-\sqrt{2})x+\sqrt{2}} \quad (1)$$

"That doesn't look much like Strähle's function," said Deirdre.

"No," I agreed. "But now comes a final bit of nifty footwork."

"You change $\sqrt{2}$ to some approximation," suggested Oliver.

"Well, I wasn't actually going to do *quite* that. What Barbour did was to estimate the *error* in terms of the approximation 58/41 to $\sqrt{2}$. And Isaac Schoenberg did the same when he wrote about the problem in 1982. You see, if you substitute 58/41 for $\sqrt{2}$ in (1) then you get $(24x+58)/(-17x+58)$, which is different from Strähle's function.

"But now you come to mention it, that does seem a more natural thing to try. I doubt it will work, but let's see." I grabbed a napkin, borrowed a pen from Oliver, and started scribbling. The tools of the mathematician's trade are pencil and paper: in consequence, no mathematician ever carries either, and they *always* have to borrow a pen and scribble on a napkin.

Light began to dawn.

There is a series of rational numbers that approximate $\sqrt{2}$. One way to get them is to start from the equation $p/q = \sqrt{2}$ and square to get $p^2 = 2q^2$. Because $\sqrt{2}$ is irrational, you can't find integers p and q that satisfy this equation (or, more accurately, because you can't find integers p and q that satisfy this equation, $\sqrt{2}$ must be irrational!). But you can come close by looking for integers p and q such that p^2 is close to $2q^2$. The best approximations are those for which the error is smallest; that is, solutions of the equation $p^2 = 2q^2 \pm 1$. For example, $3^2 = 2.2^2 + 1$, and $3/2 = 1.5$ is moderately close to $\sqrt{2}$. The next case is $7^2 = 2.5^2 - 1$, leading to $7/5 = 1.4$, which is closer. Next comes $17^2 = 2.12^2 + 1$, yielding the approximation $17/12 = 1.4166\ldots$, closer still. You can go on forever, and there's a beautiful theory that leads into continued fractions and Pell's equation and things like that.

What my scribbles had revealed was this. Divide the numerator and denominator of formula (1) by 2 and rewrite it as the equivalent formula

$$y = \frac{x + \frac{1}{\sqrt{2}}(1 - x)}{\frac{x}{2} + \frac{1}{\sqrt{2}}(1 - x)} \qquad (2)$$

Then replace $\sqrt{2}$ by the approximation 17/12, so that $1/\sqrt{2}$ becomes 12/17. This gives

$$y = \frac{x + \frac{12}{17}(1 - x)}{\frac{x}{2} + \frac{12}{17}(1 - x)} \qquad (3)$$

Finally, this simplifies to give $y = (10x + 24)/(-7x + 24)$, which is *precisely* Strähle's formula!

So Strähle's construction is very accurate because it effectively combines *two* good approximations:

- The best fractional linear approximation to 2^x is formula (1) above.
- Strähle's function is obtained from formula (1) by replacing $\sqrt{2}$ by the excellent approximation 17/12.

The errors corresponding to the various approximations discussed above are compared in Figure 114. The biggest errors are Faggot's!

"So," I finished up, "thanks to the mathematico-historical detective work of Barbour, we now know not only that Strähle's method is extremely accurate: we also have a very good idea of *why* it's so accurate. It's related to basic ideas in approximation theory and in number theory."

Which left just one question unanswered—and I fear forever unanswerable. It was raised by Oliver Gurney after he had digested an hour of mathematics in addition to his three packets of tripe-and-onion crisps.

"That's fascinating," he said. "Absolutely remarkable. The expert confounded and the practical man vindicated after a mere 218 years. Let it not be said that there's no justice in this world! If I ever meet Strähle in the afterlife I'll tell him; I'm sure he'll be pleased to have his name

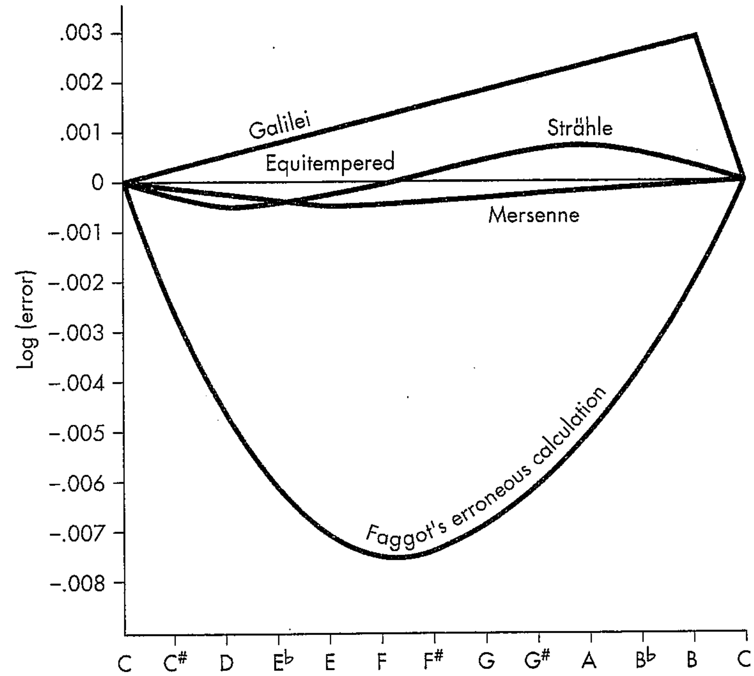

FIGURE 114 Errors in various constructions. The size of the error is measured by taking the logarithm of the ratio of the approximate value to the true value.

cleared. But what I'd really like to ask him is this: *How on earth did he think of his construction in the first place?*"

ANSWERS

1. No scale of finite length, with a constant ratio between notes that is a rational number, can ascend an exact octave—with the trivial exception of a scale that just goes up in octaves. Equivalently, the equation $r^n = 2$ can have no rational solution r when n is an integer greater than or equal to 2. To prove this, write r as a product of primes: $r = 2^a 3^b \ldots p^c$. Then $r^n = 2$ implies that $2^{na-1} 3^{nb} \ldots p^{nc} = 1$. By uniqueness of prime factorization, we must have $b = 0, \ldots, c = 0$. Also $na = 1$ so $n = 1$, $a = 1$, and the scale must go up in octaves.

2. If there were a ruler-and-compass construction for $\sqrt[12]{2}$, then by squaring twice (using ruler and compasses as in Figure 111) we could construct $\sqrt[3]{2}$, which we know is impossible. So there can be no ruler-and-compass construction for $\sqrt[12]{2}$.

FURTHER READING

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