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Noncommutativity and discrete physics

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Abstract

This paper is an introduction to discrete physics based on a non-commutative calculus of finite differences. This gives a context for the Feynman-Dyson derivation of non-commutative electromagnetism, and for generalizations of this result. The paper discusses these ideas and their relations with quantum groups and topological quantum field theory.

Keywords: Discrete; Leibnia rule; Commutator; Poisson bracket; Spin network

1. Introduction

The purpose of this paper is to present an introduction to a point of view for discrete foundations of physics. In taking a discrete stance, we find that the initial expression of physical theory will occur in a context of noncommutative algebra and noncommutative vector analysis. In this way the formalism of quantum mechanics occurs first, but not necessarily with the usual interpretations. By following this line carefully we can show how the outlines of the well-known commutative forms of physical theory arise first in noncommutative form. This much, the present paper will make clear with specific examples and mathematical formulations. The exact relation of commutative and noncommutative theories raises a host of problems.

In Section 2 of this paper we discuss the properties of the noncommutative discrete calculus that underlies our work. The section ends with the consequences in our framework for a particle whose position and momentum commutator is equated to a (noncommutative) metric field. In Section 3 we discuss how our discrete stance leads to an inversion of the usual Dirac maxim "replace Poisson brackets with commutators". If we replace commutators with Poisson brackets that obey a Leibniz rule satisfied by our commutators, then the dynamical variables will obey Hamilton's equations. Thus we can take Hamilton's equations as the natural classical version of our theory. Section 4 shows how the noncommutativity is neccessary in this approach and shows how certain representations of the theory lead to chaotic dynamics. Section 5 discusses the relationship of the discrete ordered calculus with q-deformations and quantum groups. We show that in a quantum group with a special group-like element representing the square of the antipode, there is a representation of the discrete ordered calculus. In this calculus on a quantum group the

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square of the antipode represents one tick of the clock. Then follows section (Section 6) on networks and discrete space–time. This section is a general exposition of ideas related to spin networks and topological quantum field theory. It is our speculation that the approaches to discrete physics inherent in discrete calculus and in topological field theory are deeply interrelated. At the end of this section we outline this relationship in the case of a recent model for quantum gravity due to Louis Crane.

2. Discrete ordered calculus

Consider successive measurements of position and velocity. In measurement of position, no time step is required. In measuring velocity, we need positions of two neighboring instants of time.

Thinking discretely, let us assume that the particle has positions

$$X, X', X'', \ldots$$

at successive moments of time. Discrete unit time steps are indicated by the primes appended to the X. A general point in the time series at time t will be denoted by X^t. By convention let the time step between successive points in the series be equal to 1:

$$\Delta t = 1.$$

Then we can define the velocity at time t by the formula:

$$v(t) = X^{t+1} - X^t.$$

More generally, if X denotes the position at a given time, then $X' - X$ denotes the velocity *at that time*, where the phrase "at that time" must involve the next time as well. In a discrete context there is no notion of instantaneous velocity.

Measure position, and you find X. Then measure velocity, and you get $X' - X$. Now measure position, and you get X' because the time has shifted to the next time in order to allow the velocity measurement. In order to measure velocity the position is necessarily shifted to its value at the next time step. In this sense, position and velocity measurements cannot commute in a discrete framework.

Our project is to take this basic noncommutativity at face value and follow out its consequences. To this end we will formulate a calculus of finite differences that takes the order of observations into account. This formalization is explained below. Remarkably, the resulting calculus is actually a discrete version of time evolution in standard quantum mechanics.

We begin by recalling the usual derivative in the calculus of finite differences, generalised to a (possibly) non-commutative context.

Definition 1. Let

$$\mathrm{d}X = X' - X$$

define the finite difference derivative of a variable X whose successive values in discrete time are

$$X, X', X'', \ldots$$

This $\mathrm{d}X$ is a classical derivative in the calculus of finite differences. It is still defined even if the quantities elements of the time series are in a noncommutative algebra. We shall assume that the values of the time series are in a possibly noncommutative ring R with unit. (Thus the values could be real numbers, complex numbers, matrices, linear operators on a Hilbert space, or elements of an appropriate abstract algebra.) This means that for every element A of the ring R there is a well-defined successor element A', the next term in the time series. It is convenient to assume that the ring itself has this temporal structure. In practice, one is concerned with a particular time series and not the structure of the entire ring. Moreover, we shall assume that the next-time operator distributes over both addition and multiplication in the sense that

$$(A + B)' = A' + B'$$

and

$$(AB)' = A'B'.$$

An element c of the ring R is said to be a *constant* if $c' = c$.

Lemma 2.

$$\mathrm{d}(XY) = X'\,\mathrm{d}(Y) + \mathrm{d}(X)Y.$$

Proof.

$$\begin{aligned}\mathrm{d}(XY) &= X'Y' - XY = X'Y' - X'Y + X'Y - XY\\ &= X'(Y' - Y) + (X' - X)Y = X'\,\mathrm{d}(Y) + \mathrm{d}(X)Y. \qquad \square\end{aligned}$$

This formula is *different* from the usual formula in Newtonian calculus by the time shift of X to X' in the first term. We now correct this discrepancy in the calculus of finite differences by taking a *new* derivative D as an *instruction to shift the time to the left of the operator* D. That is, we take $XD(Y)$ quite literally as an instruction to *first find* $\mathrm{d}Y$ *and then find the value of* X. In order to find $\mathrm{d}Y$ the clock must advance one notch. Therefore X has advanced to X' and we have that the evaluation of $XD(Y)$ is

$$X'(Y' - Y).$$

In order to keep track of this noncommutative time-shifting, we will write

$$DX = J(X' - X),$$

where the element J is a special time-shift operator satisfying

$$ZJ = JZ'$$

for any Z in the ring R. The time-shifter, J, acts automatically to evaluate expressions in the resulting noncommutative calculus of finite differences. We call this calculus DOC (for discrete ordered calculus). Note that J formalizes the operational ordering inherent in our initial discussion of velocity and position measurements. An operator containing J causes a time-shift in the variables or operators to the left of J in the sequence order.

Formally, we extend the ring of values R (see the definition of d above) by adding a new symbol J with the property that $AJ = JA'$ for every A in R. It is assumed that the extended ring R is associative and satisfies the distributive law so that $J(A + B) = JA + JB$ and $J(AB) = (JA)B$ for all A and B in the ring. We also assume that J itself is a constant in the sense that $J' = J$.

The key result in DOC is the following adjusted difference formula:

Lemma 3.

$$D(XY) = XD(Y) + D(Y)X.$$

Proof.

$$\begin{aligned} D(XY) &= J(X'Y' - XY) = J(X'Y' - X'Y + X'Y - XY) = J(X'(Y' - Y) + (X' - X)Y) \\ &= JX'(Y' - Y) + J(X' - X)Y = XJ(Y' - Y) + J(X' - X)Y = XD(Y) + D(X)Y. \qquad \square \end{aligned}$$

The upshot is that DOC behaves formally like infinitesimal calculus and can be used as a calculus in this version of discrete physics. In [13] Kauffman and Noyes used this foundation to build a derivation of a noncommutative version of electromagnetism. Another version of this derivation can be found in [12]. In both cases the derivation is a translation to this context of the well-known Feynman–Dyson derivation of electromagnetic formalism from commutation relations of position and velocity.

Note that the definition of the derivative in DOC is actually a commutator:

$$DX = J(X' - X) = JX' - JX = XJ - JX = [X, J].$$

The operator J can be regarded as a discretized time-evolution operator in the Heisenberg formulation of quantum mechanics. In fact we can write formally that

$$X' = J^{-1}XJ$$

since $JX' = XJ$ (assuming for this interpretation that the operator J is invertible). Putting the time variable back into the equation, we get the evolution

$$X^{t+\Delta t} = J^{-1}X^tJ.$$

This aspect can be compared to the formalism of Alain Connes' theory of noncommutative geometry [3].

In Connes' theory there is a notion of quantized differential that takes the form (in his language) $de = [F, e]$ where F is a bounded operator on a Hilbert space H and $[e]$ is a class in the K-theory of a certain algebra A acting on the Hilbert space. In this context Connes' quantized calculus is used to obtain a wide range of connections with various aspects of physics, including a new view of the standard model for fundamental particles. Our approach to aspects of the formalism of the DOC quantized calculus may fit into the context of Connes' theory. This is a topic that derserves futher investigation. In this paper, and in our previous work we have used the most elementary noncommutative algebraic tools to obtain our results. It is our hope that these results will fit into more complex contexts that are directly related to both theory and measurement.

In the discrete ordered calculus, X and DX have no reason to commute:

$$[X, DX] = XJ(X' - X) - J(X' - X)X = J(X'(X' - X) - (X' - X)X).$$

Hence

$$[X, DX] = J(X'X' - 2X'X + XX).$$

This is nonzero even in the case where X and X$'$ commute with one another. Consequently, we can consider physical laws in the form

$$[X^i, DX^j] = g^{ij},$$

where g^{ij} is a function that is suitable to the given application. In [13] we showed how the formalism of electromagnetism arises when g^{ij} is δ^{ij}, the Kronecker delta. In [15] we show how the general case corresponds to a "particle" moving in a noncommutative gauge field coupled with geodesic motion relative to the Levi-Civita connection associated with the g^{ij}. This result can be used to place the work of Tanimura [18] in a discrete context.

It should be emphasized that all physics that we derive in this way is formulated in a context of noncommutative operators and variables. We do not derive electromagnetism, but rather a noncommutative analog. It is not yet clear just what these noncommutative physical theories really mean. Our initial idealization of measurement is not the only model for measurement that corresponds to actual observations. Certainly the idea that we can measure time in a way that "steps between the steps of time" is an idealization. It happens to be an idealisation that fits a model of the universe as a cellular automaton. In a cellular automaton an observation is what an operator of the automaton might be able to do. It is not necessarily what the "inhabitants" of the automaton can perform. Here is the crux of the matter. The inhabitants can have only limited observations of the running of the automaton, due to the fact that they themselves are processes running on the automaton. I believe that the theories we build on the basis of DOC are theories *about* the structure of these automata. They will eventually lead to theories of what the processes that run on such automata can observe. It is quite possible that the well-known phenomena of quantum mechanics will arise naturally in such a context. These points of view should be compared with [9].

In order to illustrate these methods, I will show part of the calculations related to

$$[X^i, \dot{X}^j] = g^{ij},$$

Here $\dot{X}^j$ is the shorthand for DX^j. Along with this commutator equation, we will assume that

$$[X^i, X^j] = 0, \quad [X^i, g^{jk}] = 0, \quad [X^i, g_{jk}] = 0.$$

Here it is assumed that

$$g^{ij} g_{jk} = \delta^i_k$$

and that

$$g_{ij} g^{jk} = \delta^k_i.$$

The first result that is a direct consequence of these assumptions coupled with the discrete ordered calculus is the symmetry of the "metric" coefficients g^{ij}. That is, we shall show that

$$g^{ij} = g^{ji}.$$

Lemma 4. $g^{ij} = g^{ji}$.

Proof.

$$\begin{aligned} g^{ij} &= [X^i, \dot{X}^j] = [X^i, [X^j, J]] \\ &= -[J, [X^i, X^j]] - [X^j, [J, X^i]] = -[J, 0] + [X^j, [X^i, J]] = [X^j, [X^i, J]] = g^{ji}. \end{aligned}$$ □

A stream of consequences then follow by differentiating both sides of the equation

$$g^{ij} = [X^i, \dot{X}^j],$$

where

$$\dot{F} = \dot{X}^j \partial_j F$$

and it is understood that

$$\partial_j F = [F, \dot{X}_j] = [F, g_{jk}\dot{X}^k]$$

for any function F of the variables X^k and their derivatives $\dot{X}^k$. In particular, the Levi-Civita connection

$$\Gamma^{ijk} = (1/2)(\partial^j g^{ki} + \partial^k g^{ij} - \partial^i g^{jk})$$

associated with the g^{ij} comes up almost at once from the differentiation process described above. One finds that

$$D^2 X^i = G^i - g^{ir} g^{js} F_{rs} \dot{X}_j - \Gamma^{ijk} \dot{X}_j \dot{X}_k,$$

where $F_{rs} = [\dot{X}_r, \dot{X}_s]$. It follows from the Jacobi identity that F_{rs} satisfies the equation

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0,$$

identifying F_{ij} as a noncommutative analog of a gauge field. In a more technical sense, G^i is a noncommutative analog of a scalar field, satisfying

$$\langle \partial_i G_j \rangle = \langle \partial_j G_i \rangle,$$

where the brackets around this equation indicate an analog of the Weyl ordering for operator products. The details of this calculation can be found in [15].

This brief technical description of the equations for a noncommutative particle in a metric field illustrates well the role of the background of discrete time in this theory. In terms of the backgound time the metric coefficients are not constant. It is through this variation that the space–time derivatives of the theory are articulated. Thus we are in this way producing the beginning of a theory of space–time based on a background process. The background is a process with its own form of discrete time, but no space–time structure as we know and observe it. Our observation of space–time structure appears as a rough (commutative) approximation to the processes described as consequences of the basic noncommutative equations of the DOC.

3. Poisson brackets and commutator brackets

Dirac [7] introduced a fundamental relationship between quantum mechanics and classical mechanics that is summarized by the maxim *replace Poisson brackets by commutator brackets.* Recall that the Poisson bracket $\{A, B\}$ is defined by the formula

$$\{A, B\} = (\partial A/\partial q)(\partial B/\partial p) - (\partial A/\partial p)(\partial B/\partial q),$$

where q and p denote classical position and momentum variables, respectively.

In our version of discrete physics the noncommuting variables are functions of discrete time, with a DOC derivative D as described in the previous section. Since $DX = XJ - JX = [X, J]$ is itself a commutator, it follows that

$$D([A, B]) = [DA, B] + [A, DB]$$

for any expressions A, B in our ring R. A corresponding Leibniz rule for Poisson brackets would read

$$(\mathrm{d}/\mathrm{d}t)\{A, B\} = \{\mathrm{d}A/\mathrm{d}t, B\} + \{A, \mathrm{d}B/\mathrm{d}t\}.$$

However, here there is an easily verified exact formula:

$$(\mathrm{d}/\mathrm{d}t)\{A, B\} = \{\mathrm{d}A/\mathrm{d}t, B\} + \{A, \mathrm{d}B/\mathrm{d}t\} - \{A, B\}(\partial\dot{q}/\partial q + \partial\dot{p}/\partial p).$$

This means that the Leibniz formula will hold for the Poisson bracket exactly when

$$(\partial \dot{q}/\partial q + \partial \dot{p}/\partial p) = 0.$$

This is an integrability condition that will be satisfied if p and q satisfy Hamilton's equations

$$\dot{q} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial q.$$

This, of course, means that q and p are following a principle of least action with respect to the Hamiltonian H. Thus we can interpret the *fact* $D([A, B]) = [DA, B] + [A, DB]$ in the discrete context as an analog of the principle of least action. Taking the discrete context as fundamental, we say that Hamilton's equations are *motivated* by the presence of the Leibniz rule for the discrete derivative of a commutator. The classical laws are obtained by following Dirac's maxim in the opposite direction! Classical physics is produced by following the correspondence principle upwards from the discrete.

Taking the last paragraph seriously, we must reevaluate the meaning of Dirac's maxim. The meaning of quantization has long been a basic mystery of quantum mechanics. By traversing this territory in reverse, starting from the noncommutative world, we begin these questions anew.

4. Scalar variables, chaos and representations of the discrete ordered calculus

The purpose of this short section is to point out the inherent noncommutativity of the operators in any theory based on the DOC. It is natural to hope for actual scalar variables in the course of articulating a theory based on DOC.

Consider the equation $[X, DX] = Jk$ where k is a constant. This reads

$$J(X'X' - 2X'X + XX) = Jk$$

and hence we may consider solutions to the equation

$$(X'X' - 2X'X + XX) = k.$$

If X and X' commute then this becomes

$$(X - X')^2 = k$$

with the solution

$$X' = X \pm k^{1/2}.$$

For some problems it may be sufficient to consider the situation where the variables are successively incremented or decremented by a constant.

The problems arise when we go to more than one variable. For example, consider the equation

$$[X_i, DX_j] = Jk\delta_{ij},$$

where i and j range from 1 to 3. Then for $i \neq j$ we have

$$[X_i, DX_j] = 0.$$

Let $X_i = A$ and $X_j = B$. Then this equation reads

$$AJ(B - B') - J(B - B')A = 0.$$

Hence

$$A'(B - B') - (B - B')A = 0.$$

Thus if A and B commute, we conclude that $(A' - A)(B' - B) = 0$. Unfortunately, this contradicts the equations $[A, DA] = Jk$ and $[B, DB] = Jk$ that are given by our assumptions, except in the case where $k = 0$. This analysis shows that noncommutativity of the dynamical variables in theories based on the DOC is a part of life.

Example 5. Noncommutativity can have a scalar source. For example, suppose that $X = DT$, where T and T' are commuting scalars. Consider the equation

$$[X, DX] = J^2 k,$$

where k is a commuting scalar constant. Then we have $[DT, DDT] = J^2 k$. Let

$$\Delta = T' - T$$

and note that Δ is also a commuting scalar. Then $DT = J\Delta$, and therefore

$$[DT, DDT] = J^2(\Delta''(\Delta' - \Delta) - (\Delta'' - \Delta')\Delta).$$

Hence the equation $[X, DX] = J^2 k$ translates into

$$\Delta''(\Delta' - \Delta) - (\Delta'' - \Delta')\Delta = k,$$

whence

$$\Delta'' = (k - \Delta\Delta')/(\Delta' - 2\Delta).$$

This recursion relation for Δ and its time series has remarkable properties. For a fixed non-zero value of k, the recursion is highly sensitive to initial conditions, with regions that give rise to bounded oscillations and other regions that give rise to unbounded oscillations. There are boundary values in the initial conditions where the system undergoes chaotic transition between bounded and unbounded behavior.

We are investigating this method (of letting $X_i = D^n T_i$ for some n, where T_i and T_j' are commuting scalars) for producing a system of noncommuting extrinsic dynamical variables with an underlying scalar structure. If this idea is correct, then there will emerge a picture of noncommutative discrete physics based on DOC as a global description occurring over a substrate of discrete chaotic dynamics.

There are other possibilites for the direct representation of the discrete noncommutative dynamics. There may be matrix representations of these theories over finite fields, the simplest cases being modular number systems with prime modulus. This subject will be taken up in a future publication.

5. Discussion on q-deformation

The direct relation between the content of local physical descriptions based on the DOC calculus and more global considerations are a matter of speculation. One strong hint is contained in the properties of the discrete derivative that has the form

$$D_q f(x) = (f(qx) - f(x))/(qx - x).$$

The classical derivative occurs in the limit as q approaches one.

In the setting of q not equal to one, the derivative D_q is directly related to fundamental noncommutativity. Consider variables x and y such that $yx = qxy$ where q is a commuting scalar. Then the expansion of $(x + y)^n$ generates a q-binomial theorem with q-choice coefficients composed in q-factorials of q-integers $[n]_q$, where

$$[n]_q = 1 + q + q^2 + \cdots + q^{(n-1)}.$$

The derivative D_q is directly related to the q-integers via the formula

$$D_q(x^n) = [n]_q x^{n-1}.$$

In the context of this paper, we have considered discrete derivatives in the form

$$d_\Delta f(x) = (f(x + \Delta) - f(x))/\Delta.$$

This will convert to the q-derivative if $x + \Delta = qx$. Thus we need

$$q = (x + \Delta)/x.$$

This means that a direct translation from DOC to q-derivations could be effected if we allowed q to vary as a function of x and introduced the temporal operator J into the calculus of q-derivatives.

In general, many q-deformed structures such as the quantum groups associated with the classical Lie algebras appear to be entwined with the discretization inherent in D_q. The quantum groups have turned out to be deeply connected with topological amplitudes for networks describing knots and three-dimensional spaces. (See Section 6.) The analog for the quantum groups in dimension four is being sought. If there is a connection between the local and the global parts of our essay it may lie in hidden connections between discretization and quantum groups. Clearly there is much work to be done in this field.

There is a clue about the meaning of the operator J ($DF = [F, J]$ in the DOC) in the context of quantum groups. Quantum groups are Hopf algebras. A quantum group such as $G = U_q(SU(2))$ is actually an algebra over a field k with an antipode

$$S : G \longrightarrow G$$

and a coproduct

$$\Delta : G \longrightarrow G \otimes G,$$

a unit 1 and a counit

$$\epsilon : G \longrightarrow k$$

The coproduct is a map of algebras. The antipode is an antimorphism, $S(xy) = S(y)S(x)$, and generalizes the inverse in a group in the sense that $\Sigma S(x_1)x_2 = \epsilon(x)1$ and $\Sigma x_1 S(x_2) = \epsilon(x)1$ where $\Delta(x) = \Sigma x_1 \otimes x_2$.

An element g in a quantum group G is said to be a *group-like element* if $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$. In many quantum groups (such as $G = U_q(SU(2))$) the square of the antipode is represented via conjugation by a special group-like element that we shall denote by J. Thus

$$S^2(x) = J^{-1}xJ$$

for all x in G. This means that it is possible to define the discrete ordered calculus in the context of a quantum group G (as above) by taking J to be the special group-like element. Then we have

$$DX = [X, J] = XJ - JX = J(J^{-1}XJ - X) = J(S^2(X) - X).$$

Conjugation by the special group-like element in the quantum group constitutes the time-evolution operator in this algebra.

There are a number of curious aspects to the use of the DOC in a quantum group. First of all, it is the case that in some quantum groups (for example with undeformed classical Lie algebras) the square of the antipode is equal to the identity mapping. From the point of view of DOC, time does not exist in these algebras. But in the q-deformations such as $U_q(SU(2))$, the square of the antipode is quite nontrivial and can serve well as the tick of the clock. In this way, q-deformations do provide a context for time. In particular, this suggests that the q-deformations of classical spin networks [16] should be able to accommodate time. A suggestion directly related to this remark occurs in [5], and we shall take this up at the end of the next section of this paper.

6. Networks and discrete space–time

One can consider replacing continuous space (such as Euclidean space with the usual topology) by a discrete structure of relationships. The geometry of the Greeks held a discrete web of relationships in the context of continuous space. That space was not coordinated in our way, nor was it held as an infinite aggregate of points. In general topology there is a wide choice for possible spatial structures (where we mean by a space a topology on some set).

Discretization of space and time implicates the replacement of space–time by a network, graph or complex that has nodes for the points and edges to indicate significant relationships among the points.

Euler's work in the 18th century brought forth the use of abstract graphs as holders of spatial structure. After Euler it was possible to find the classification of the Greek regular solids in the (wider) classification of the regular graphs on the surface of the sphere. Metric can disappear into relationship under the topological constraint of Euler's formula $V - E + F = 2$, where V denotes the number of vertices, E the number of edges and F denotes the number of faces for the connected graph G on the sphere.

A network itself can represent an abstract space. Embeddings of that network into a given space (such as graphs on the two-dimensional sphere) correspond to global constraints on the structure of the abstract graph.

Now a new theme arises, motivated by a conjunction of combinatorics and physics. Imagine labeling the edges of the network from some set of "colors". These colors can represent the basic states of a physical system, or they can be an abstract set of distinct markers for purely mathematical purposes. Once the network is labeled, each vertex is an entity with a collection of labels incident on it. Let there be given a function that associates a number (or algebra element) to each such labeled vertex. Call this number the *vertex weight* at that vertex. Let C denote a specific coloring of the network N and consider the product, over all the vertices of N of the values of the vertex weights. Finally let $Z(N)$, the *amplitude* of the network, be defined as the summation of the product of the vertex weights over all colorings of the net. $Z(N)$ is also called the *partition function* of the network.

Amplitudes of this sort are exactly what one computes in finding the partition function of a physical system or the quantum mechanical amplitude for a discrete process. In all these cases the network is interwoven with the algebraic structure of the vertex weights. It is only recently that topological properties of networks in three-dimensional space have come to be understood in this way [1,11,19]. This has led to new information about the topology of low dimensional spaces, and new relationships between physics and topology.

A classical example of such an amplitude was discovered by Penrose [17] in elucidating special colorings of 3-regular graphs in the plane. A 3-regular graph G has three edges incident to each vertex. When embedded in the plane, these edges acquire a specific cyclic order. Three colors are used. One associates to each vertex the weight

$$\sqrt{-1}\,\epsilon_{abc},$$

where a,b and c denote the edges meeting the vertex in this cyclic order, and the epsilon is equal to $1, -1$ accordingly as the edges have distinct labels in the given or reverse cyclic order, or 0 if there is a repetition of labels. The resulting amplitude counts the number of ways to color the network with three colors so that three distinct colors are incident on each vertex. This result is a perspicuous generalization of the classical four color problem of coloring maps in the plane with four colors so that adjacent regions receive different colors.

The Penrose example generalizes to networks whose amplitudes embody geometrical properties of Euclidean three-dimensional space (angles and their dependence). Geometry begins to emerge in terms of the averages of properties of an abstract and discrete network of relationships. Topological properties emerge in the same way. The idea of space can change to the idea of a network with global states and a functor that associates this network and its states to the more familiar properties that a classical observer might see.

6.1. *Remarks on quantum mechanics*

We should remark on the basic formalism for amplitudes in quantum mechanics. The Dirac notation $\langle A|B\rangle$ [7] denotes the probability amplitude for a transition from A to B. Here A and B could be points in space (for the path of a particle), fields (for quantum field theory), or geometries on space–time (for quantum gravity). The probability amplitude is a complex number. The actual probability of an event is the absolute square of the amplitude. If a complete set of intermediate states $C_1, C_2, \ldots, C_n$ is known, then the amplitude can be expanded to a summation

$$\langle A|B\rangle = \Sigma_{i=1}^{n}\langle A|C_i\rangle\langle C_i|B\rangle.$$

This formula follows the formalism of the usual rules for probability, and it allows for the constructive and destructive interference of the amplitudes. It is the simplest case of a quantum network of the form

$$A - - - * - - - C - - - * - - - B$$

where the colors at A and B are fixed and we run through all choices of colors for the middle edge. The vertex weights at the vertices labeled $*$ are $\langle A|C\rangle$ and $\langle C|B\rangle$, respectively. A measurement at the C edge reduces the big summation to a single value.

Consider the generalization of the previous example to the graph

$$A - - - * - - - C^1 - - - * - - - C^2 - - - * - - - \ldots - - - * - - - C^m - - - B$$

With A and B fixed the amplitude for the net is

$$\langle A|B\rangle = \sum_{1\le i_1\le\cdots\le i_m\le n}\langle A|C^1_{i_1}\rangle\langle C^2_{i_2}|C^3_{i_3}\rangle\cdots\langle C^m_{i_m}|B\rangle$$

One can think of this as the sum over all the possible paths from A to B. In fact in the case of a "particle" traveling between two points in space, this is exactly what must be done to compute an amplitude – integrate over all the paths between the two points with appropriate weightings. In the discrete case this sort of summation makes perfect sense. In the case of a continuum there is no known way to make rigorous mathematical sense out of all cases of such integrals. Nevertheless, the principles of quantum mechanics must be held foremost for physical purposes and so such "path integrals" and their generalizations to quantum fields are in constant use by theoretical physicists [9] who take the point of view that the proof of a technique is in the consistency of the results with the experiments. When the observations themselves are mathematical (such as finding invariants of knots and links), the issue acquires a new texture.

Now consider the summation discussed above in the case where $n = 2$. That is, we shall assume that each C^kcan take two values, call these values L and R. Furthermore let us suppose that $\langle L|R\rangle = \langle R|L\rangle = \sqrt{-1}$ while $\langle L|L\rangle = \langle R|R\rangle = 1$. The amplitudes that one computes in this case correspond to solutions to the Dirac equation [7] in one space variable and one time variable. This example is related to an observation of Feynman [9]. In [14] we give a very elementary derivation of this result and we show how these amplitudes give solutions to the discretized Dirac equation, so everything is really quite exact and one can understand just what happens in taking the limit to the continuum. In this example a state of the network consists in a sequence of choices of L or R. These can be interpreted as choices to move left or right along the light-cone in a Minkowski plane. It is in summing over such paths in space–time that the solution to the Dirac equation appears. In this case, time has been introduced into the net by interpreting the sequence of nodes in the network as a temporal direction.

Thus one way to incorporate space–time is to introduce a temporal direction into the net. At a vertex, one must specify labels of *before* and *after* to each edge of the net that is incident to that vertex. If there is a sufficiently coherent assignment of such local times, then a global time direction can emerge for the entire network. Networks endowed with temporal directions have the structure of morphisms in a category where each morphism points from past to future. A category of quantum networks emerges equipped with a functor (via the algebra of the vertex weights) to morphisms of vector spaces and representations of generalized symmetry groups. Appropriate traces of these morphisms produce the amplitudes.

Quantum nonlocality is built into the network picture. Any observer taking a measurement in the net has an effect on the global set of states available for summation and hence affects the possibilities of observations at all other nodes in the network. By replacing space with a network we obtain a precursor to space–time in which much of quantum mechanics is built into the initial structure.

Remark 6. A striking parallel to the views expressed in this section can be found in [8]. Concepts of time and category are discussed by Crane [4,5] in relation to topological quantum field theory. In the case of Crane's work there is a deeper connection with the methods of this paper, as I shall explain below.

6.2. *Temporality and the Crane Model for quantum gravity*

Crane uses a partition function defined for a triangulated four-manifold. Let us denote the partition function by $Z(M^4, A, B) = \langle A|B\rangle_M$ where M^4 is a four-manifold and A and B are (colored – see the next sentence) three-dimensional submanifolds in the boundary of M. The partition function is constructed by summing over all colorings of the edges of a dual complex to this triangulation from a finite set of colors that correspond to certain representations of the quantum group $U_q(SU(2))$, where q is a root of unity. The sum is over products of $15J_q$ symbols (natural generalizations of the $6J$ symbols in angular momentum theory) evaluated with respect to the colorings. The specific form of the partition function (here written in the case where A and B are empty) is

$$Z(M^4) = N^{v-e}\Sigma_\lambda \Pi_\sigma \dim_q(\lambda(\sigma))\Pi_\tau \dim_q^{-1}(\lambda(\tau))\Pi_\zeta 15J_q(\lambda(\zeta)).$$

Here λ denotes the labeling function, assigning colors to the faces and tetrahedra of M^4 and $v-e$ is the difference of the number of vertices and the number of edges in M^4. Faces are denoted by σ, tetrahedra by τ and 4-simplices by ζ. We refer the reader to [6] for further details.

In computing $Z(M^4, A, B) = \langle A|B\rangle_M$ one fixes the choice of coloration on the boundary parts A and B. The analog with quantum gravity is that a colored three manifold A can be regarded as a three manifold with a choice of (combinatorial) metric. The coloring is the combinatorial substitute for the metric. In the three manifold case this is quite specifically so, since the colors can be regarded as affixed to the edges of the simplices. The color on a given edge is interpreted as the generalized distance between the endpoints of the edge. Thus $\langle A|B\rangle_M$ is a summation over "all possible metrics" on M^4 that can extend the given metrics on A and B. $\langle A|B\rangle_M$is an amplitude for the metric (coloring) on A to evolve in the space–time M^4 to the metric (coloring) on B.

The partition function $Z(M^4, A, B) = \langle A|B\rangle_M$ is a topological invariant of the four-manifold M^4. In particular, if A and B are empty (a vacuum-vacuum amplitude), then the Crane–Yetter invariant, $Z(M^4)$, is a function of the signature and Euler characteristic of the four-manifold [6]. On the mathematical side of the picture this is already significant since it provides a new way to express the signature of a four-manifold in terms of local combinatorial data.

From the point of view of a theory of quantum gravity, $Z(M^4, A, B) = \langle A|B\rangle_M$, as we have described it so far, is lacking in a notion of time and dynamical evolution on the four-manifold M^4. One can think of A and B as manifolds at the initial and final times, but we have not yet described a notion of time within M^4 itself.

Crane proposes to introduce time into M^4 and into the partition function $\langle A|B\rangle_M$ by labeling certain three-dimensional submanifolds of M^4 with special group-like elements from the quantum group $U_q(SU(2))$ and extending the partition function to include this labeling. Movement across such a labeled hypersurface is regarded as one tick of the clock. The special group-like elements act on the representations in such a way that the partition function can be extended to include the extra labels. Then one has the project to understand the new partition function and its relationship with discrete dynamics for this model of quantum gravity.

Let us denote the special group-like element in the Hopf algebra $G = U_q(SU(2))$ by the symbol J. Then, as discussed at the end of the previous section, one has that the square of the antipode $S : G \longrightarrow G$ is given by the formula $S^2(x) = J^{-1}xJ$. This is the tick of the clock. The DOC derivative in the quantum group is given by the formula $DX = [X, J] = J(S^2(X) - X)$. I propose to generalize the DOC on the quantum group to a DOC on the four-manifold M^4 with its hyperthreespaces labeled with special group-likes. This generalized calculus will be a useful tool in elucidating the dynamics of the Crane model. Much more work needs to be done in this domain.

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