Non-commutative Calculus and Discrete Physics

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1 Introduction

This paper is an expanded version of [28] and [30] where there is presented an introduction to a point of view for discrete foundations of physics. In taking a discrete stance, we find that the initial expression of physical observation naturally occurs in a context of non-commutative algebra. In this way a formalism similar to quantum mechanics occurs first, but not necessarily with the usual interpretations. By following this line we show how the outlines of the well-known forms of physical theory arise first in non-commutative form. The exact relation of commutative and non-commutative theories raises a host of problems.

The starting point for this investigation is the representation of calculus in a non-commutative framework. In such a framework derivatives are represented by commutators, or more generally by products that satisfy the Jacobi identity and the Leibniz rule. If we take commutators $[A, B] = AB - BA$ in an abstract algebra and define $DA = [A, J]$ for a fixed element J, then D acts like a derivative in the sense that $D(AB) = D(A)B + AD(B)$ (the Leibniz rule). As soon as we have calculus in such a framework, concepts of geometry are immediately available. For example, if we have two derivatives $\nabla_J A = [A, J]$ and $\nabla_K A = [A, K]$, then we can consider the commutator of these derivatives $[\nabla_J, \nabla_K]A = \nabla_J \nabla_K A - \nabla_K \nabla_J A = [[J,K],A]$. The non-commutation of derivations corresponds to curvature in geometry, and indeed we shall see that the earliest emergence of curvature in this context is the formal analog of the curvature of a gauge connection!

For multivariable calculus we need variables $X_1, X_2, \cdots, X_n$ and elements $P_1, P_2, ..., P_n$ such that $\partial_i A = \partial A/\partial X_i = [A, P_i]$. For a simplest representation we shall assume the the X_i commute with one another, and that the P_j commute with one another. Since we want $\partial_i X_j = \delta_{ij}$ (the Kronecker delta δ_{ij} is equal to one if i and j are equal and is zero otherwise), we must have the commutator equation $[X_i, P_j] = \delta_{ij}$. Thus multivariable calculus in this non-commutative representation demands the commutation relations

$$[X_i, X_j] = 0$$

$$[P_i, P_j] = 0$$

$$[X_i, P_j] = \delta_{ij}$$

These equations are the "flat background" for our non-commutative calculus. The reader will note that this flat background has the same pattern of commutation relations as a bare form of quantum mechanics when the X variables are interpreted as position and the P variables are interpreted as momenta. In a certain sense this means that our considerations start in the quantum domain. Note that flat is a correct adjective, since the derivatives ∂_i all commute with one another.

Let A_i be a collection of elements of this algebra. Define "covariant derivatives" with $\Lambda_i = P_i - A_i$ by the formula

$$\nabla_i Z = [Z, \Lambda_i] = \partial_i Z - [Z, A_i].$$

Computing the curvature, one finds

$$[\nabla_i, \nabla_j]Z = [[\Lambda_i, \Lambda_j], Z]$$

and

$$[\Lambda_i, \Lambda_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

The reader will recognize this last expression as the formula for the curvature of a gauge connection.

In interfacing this formalism with physics we adopt the coupling equation

$$dX_i/dt = \Lambda_i = P_i - A_i.$$

The reader will recognise this as the minimal coupling postulate in the context of Poisson brackets. Here we take it in the context of commutators or Poisson brackets, or a more general product satisfying the Jacobi identity and the Leibniz rule as described above. One retrieves the physics of a gauge field in this formalism. This is the essence of the pattern behind the Feynman-Dyson derivation of electromagnetism from commutation relations [24, 27], and its import is more general. Because the brackets can be interpreted as commutators or as Poisson brackets with special structure, the formalism can be seen in a multiplicity of contexts. Deeper relationships with curvature and metric are related to this shifting of contexts as are relationships with quantum mechanics where the quantum formalism is obtained by the Dirac prescription of replacing Poisson brackets by commutators. We will discuss these issues in Section 5 of this paper. The organization of the paper is as follows.

Section 2 of this paper we discuss the properties of the non-commutative discrete calculus that underlies our work. Here we begin with the consideration of a temporal operator J with the property that $YJ = JY'$ for a "time series" $X, X', X'', \cdots$. Thus $XJ = JX', X'J = JX'', \cdots$. This formalism for time series gives rise to the time derivative $DA = [A, J] = AJ - JA = JA' - JA = J(A' - A)$, a commutator representing a discrete derivative. Note that DA satisfies the Leibniz rule, a privilege not shared by the usual commutative discrete derivative. This section discusses the discrete ordered calculus (DOC) that arises from this idea and applies these ideas to a number of situations. In particular, we consider the one variable case of the commutator equation $[X, DX] = Jk$ and show that it leads to a Brownian walk, and that if we take the size of the time step into account, then the diffusion constant for a Brownian process arises naturally as $k/2$. We compare this with the usual derivation of the diffusion constant and the diffusion differential equation. We then compare this situation with the one dimensional Schrödinger equation, modeling it in relation to a diffusion process with complex amplitudes. In this viewpoint one sees that the step length of the diffusion process is the Compton wavelength associated with the mass for the particle, and the time is the Compton time. For the Planck mass this gives a step equal to the Planck length and a time interval equal to the Planck time. We speculate on the relationship of this result to joint work with Pierre Noyes and others [37]. We consider other time series that can be regarded as solutions to this Heisenberg relation, the problem of using more variables and a model that is related to a discrete version of the Feynman-Dyson derivation of electromagnetic formalism.

Section 3 examines the consequences for a particle whose position - momentum commutator is equated to a metric field. Here we see how the Levi-Civita connection (and implicitly differential geometric structure) comes naturally from the non-commutative calculus. This is a very general result and in section 4 we discuss it in a more axiomatic context as described in this introduction. This section discusses the intimate relationship between that Levi-Civita connection and the Jacobi and Leibniz identities that is revealed by our non-commutative calculus. In section 5 our stance leads to an inversion of the usual Dirac maxim "replace Poisson brackets with commutators". If we replace commutators with Poisson brackets that obey a Leibniz rule satisfied by the commutators, then the dynamical variables will obey Hamilton's equations. Thus we can take Hamilton's equations as a classicization of our theory. Among other things, this point of view explains the appearance of the Levi-Civita connection in the abstract formalism. Interpreting with Poisson brackets, we obtain a new proof (via Jacobi identity) of the classical result that a Newtonian particle moving in generalized coordinates according to Lagrange's equations describes a geodesic in the Levi-Civita connection. Section 6 discusses the relationship of the discrete ordered calculus with q-deformations and quantum groups. We show that in a quantum group with a special grouplike element representing the square of the antipode, there is a representation of the discrete ordered calculus. In this calculus on a quantum group the square of the antipode can represent one tick of the clock. Then follows section 7 on networks and discrete spacetime. This section is an exposition of ideas related to spin networks and topological quantum field theory. As an early example we discuss the discretization of the Dirac equation in $1 + 1$ dimensional spacetime. It is our speculation that the approaches to discrete physics inherent in discrete calculus and in topological field theory are deeply interrelated. At the end of this section we outline this relationship in the case of a model for quantum gravity due to Louis Crane. Section 8 is an appendix on the iterant approach to matrix algebra. We include this appendix to show how one can conceptualize matrix algebra from point of view of the discrete. Section 9 is a philosophical appendix discussing the nature of foundations in mathematics and in physics.

Remark. The following references in relation to non-commutative calculus are useful in comparing with our approach [7, 10, 13, 35]. Much of the present work is the fruit of a long series of discussions with Pierre Noyes, and we will be preparing collaborative papers on it. The present paper is a summary for the proceedings of the ANPA Conference held in Cambridge, England in the summer of 2002. I particularly thank Eddie Oshins for pointing out the relevance of minimal coupling. The paper [34] also works with minimal coupling for the Feynman-Dyson derivation. The first remark about the minimal coupling occurs in the original paper by Dyson [4], in the context of Poisson brackets. The paper [17] is worth reading as a companion to Dyson. In the present paper we generalize the minimal coupling to contexts including both commutators and Poisson brackets. The reader can see the full generality of our approach by first reading this introduction and then going directly to sections 4 and 5. It is the purpose of this paper to indicate how non-commutative calculus can be used in foundations.

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2 Discrete Ordered Calculus

In this section we recall the construction of an ordered version of the calculus of finite differences *DOC* [24], [28]. In this calculus the Leibniz rule is satisfied, and so the calculus can be used in a variety of applications.

In the abstract framework of this calculus, there are variables X, each of which connotes a time series

$$X, X', X'', ...$$

Discrete unit time steps are indicated by the primes appended to the X. A general point in the time series at time t will be denoted by X^t. By convention let the time step between successive points in the series be equal to 1 :

$$\Delta t = 1.$$

Then we can define the velocity at time t by the formula:

$$v(t) = X^{t+1} - X^t.$$

More generally, if X denotes position at a given time, then $X' - X$ denotes the velocity *at that time*, where the phrase "at that time" must involve the next time as well. In a discrete context there is no notion of instantaneous velocity.

Measure position, and you find X. Then measure velocity, and you get $X' - X$. Now measure position, and you get X' because the time has shifted to the next time in order to allow the velocity measurement. In order to measure velocity the position is necessarily shifted to its value at the next time step. In this sense, *position and velocity measurements cannot commute in a discrete framework.* This is the key physical idea that motivates our constructions. It was this idea, told to the author by Pierre Noyes, that led to our papers and particularly to [24].

The simplest interpretation of the variable X is that the time series values are numerical values, commuting with one another and with any operators that might be present in the associated mathematics or physics. In fact, we will often deal with situations where the X and the elements of the time series are in fact operators, not necessarily commuting with one another. At the very least we will construct an algebra that mirrors the discrete non-commutativity of the operations of position and velocity measurement.

Our project is to take this basic noncommutativity at face value and follow out its consequences. To this end we will formulate a calculus of finite differences that takes the order of observations into account. This formalization is explained below.

To see most clearly the non-commutativity that is at the base of our considerations, let J denote the operation of shifting time by one increment. Thus we can envisage an algebra of operations that consists in commands like JX (measure X, then tick the clock). *Note that we will agree to take the sequence of operations from right to left.* Let $|JX|$ denote the "spatial evaluation" of this sequence of operations, obtained in general by performing all the instructions and then evaluating the spatial position. Thus

$$|JX| = X$$

while

$$|XJ| = X'$$

since when the clock ticks, the position shifts to the position at the next time. We see therefore, that $XJ \neq JX$. This is the first instance of non-commutativity in the physics of discrete space and time. From the point of view of spatial evaluation it is most convenient to declare the equation

$$XJ = JX'$$

since these two expressions have identical spatial evaluations.

We can then define the DOC derivative by the equation

$$DX = [X, J] = XJ - JX = JX' - JX = J(X' - X) = JdX$$

where dX denotes the classical discrete derivative with unit time step. The key point about the DOC derivative is that it is a commutator, and consequently satisfies the Leibniz rule

$$D(XY) = D(X)Y + XD(Y).$$

This makes it possible to do discrete calculus in a way that is formally similar to classical calculus. We will repeat this structure more slowly now, first recalling the properties of classical discrete derivatives.

We begin by recalling the usual derivative in the calculus of finite differences, generalised to a (possibly) non-commutative context.

Definition. Let

$$dX = X' - X$$

define the finite difference derivative of a variable X whose successive values in discrete time are

$$X, X', X'', ...$$

This dX is a classical derivative in the calculus of finite differences. It is still defined even if the quantities elements of the time series are in a non-commutative algebra. We shall assume that the values of the time series are in a possibly non-commutative ring R with unit. (Thus the values could be real numbers, complex numbers, matrices, linear operators on a Hilbert space, or elements of an appropriate abstract algebra.) This means that for every element A of the ring R there is a well-defined successor element A', the next term in the time series. It is convenient to assume that the ring itself has this temporal structure. In practice, one is concerned with a particular time series and not the structure of the entire ring. Moreover, we shall assume that the next-time operator distributes over both addition and multiplication in the sense that

$$(A + B)' = A' + B'$$

and

$$(AB)' = A'B'.$$

An element c of the ring R is said to be a *constant* if $c' = c$.

Lemma 1.

$$d(XY) = X'd(Y) + d(X)Y.$$

Proof.

$$d(XY) = X'Y' - XY$$

$$= X'Y' - X'Y + X'Y - XY$$

$$= X'(Y' - Y) + (X' - X)Y$$

$$= X'd(Y) + d(X)Y.$$

This formula is *different* from the usual formula in Newtonian calculus by the time shift of X to X' in the first term. We now correct this discrepancy in the calculus of finite differences by taking a *new* derivative D as an *instruction to shift the time to the left of the operator D.* That is, we take $XD(Y)$ quite literally as an instruction to *first find dY and then find the value of X.* In order to find dY the clock must advance one notch. Therefore X has advanced to X' and we have that the evaluation of $XD(Y)$ is

$$X'(Y' - Y).$$

In order to keep track of this non-commutative time-shifting, we will write

$$DX = J(X' - X)$$

where the element J is a special time-shift operator satisfying

$$ZJ = JZ'$$

for any Z in the ring R. The time-shifter, J, acts to automatically evaluate expressions in the resulting non-commutative calculus of finite differences. We call this calculus DOC (for discrete ordered calculus). Note that J formalizes the operational ordering inherent in our initial discussion of velocity and position measurements. An operator containing J causes a time shift in the variables or operators to the left of J in the sequence order.

Formally, we extend the ring of values R (see the definition of d above) by adding a new symbol J with the property that $AJ = JA'$ for every A in R. It is assumed that the extended ring R is associative and satisfies the distributive law so that $J(A + B) = JA + JB$ and $J(AB) = (JA)B$ for all A and B in the ring. We also assume that J itself is a constant in the sense that $J' = J$.

The key result in DOC is the following adjusted difference formula:

Lemma 2.

$$D(XY) = XD(Y) + D(Y)X.$$

Proof.

$$\begin{aligned} & D(XY) \\ &= J(X'Y' - XY) \\ &= J(X'Y' - X'Y + X'Y - XY) \\ &= J(X'(Y' - Y) + (X' - X)Y \\ &= JX'(Y' - Y) + J(X' - X)Y \\ &= XJ(Y' - Y) + J(X' - X)Y \\ &= XD(Y) + D(X)Y. \end{aligned}$$

The upshot is that DOC behaves formally like infinitesimal calculus and can be used as a calculus in this version of discrete physics. In [24] Pierre Noyes and the author use this foundation to build a derivation of a non-commutative version of electromagnetism. Another version of this derivation can be found in [27]. In both cases the derivation is a translation to this context of the well-known Feynman-Dyson derivation of electromagnetic formalism from commutation relations of position and velocity.

Note that the definition of the derivative in DOC is actually a commutator:

$$DX = J(X' - X) = JX' - JX = XJ - JX = [X, J].$$

The operator J can be regarded as a discretised time-evolution operator in the Heisenberg formulation of quantum mechanics. In fact we can write formally that

$$X' = J^{-1}XJ$$

since $JX' = XJ$ (assuming for this interpretation that the operator J is invertible). Putting the time variable back into the equation, we get the evolution

$$X^{t+\Delta t} = J^{-1}X^tJ.$$

This aspect can be compared to the formalism of Alain Connes' theory of non-commutative geometry [7].

In DOC, X and DX have no reason to commute:

$$[X, DX] = XJ(X' - X) - J(X' - X)X = J(X'(X' - X) - (X' - X)X)$$

Hence

$$[X, DX] = J(X'X' - 2X'X + XX).$$

This is non-zero even in the case where X and X' commute with one another. Consequently, we can consider physical laws in the form

$$[X_i, DX_j] = g_{ij}$$

where g_{ij} is a function that is suitable to the given application. In [24] we show how the formalism of electromagnetism arises when g^{ij} is δ^{ij}, the Kronecker delta. In [26] we will show how the general case corresponds to a "particle" moving in a non-commutative gauge field coupled with geodesic motion relative to the Levi-Civita connection associated with the g_{ij}. This result can be used to place the work of Tanimura [42] in a discrete context.

It should be emphasized that all physics that we derive in this way is formulated in a context of non-commutative operators and variables. We do not derive electromagnetism, but rather a non-commutative analog. It is not yet clear just what these non-commutative physical theories really mean. Our initial idealisation of measurement is not the only model for measurement that corresponds to actual observations. Certainly the idea that we can measure time in a way that has "steps between the steps of time" is an idealisation. It happens to be an idealisation that fits a model of the universe as a cellular automaton. In a cellular automaton an observation is what an operator of the automaton might be able to do. It is not necessarily what the "inhabitants" of the automaton can perform. Here is the crux of the matter. The inhabitants can have only limited observations of the running of the automaton, due to the fact that they themselves are processes running on the automaton. The theories we build on the basis of *DOC* can be theories *about* the structure of these automata. They will eventually lead to theories of what can be observed by the processes that run on such automata. It is possible that the well known phenomena of quantum mechanics will arise naturally in such a context. These points of view should be compared with [14].

2.1 Brownian Walks and the Diffusion Equation

To return to basics, consider the commutator equation in one space variable X.

$$[X, DX] = Jk$$

for a single variable X. Written out, this equation becomes

$$Jk = [X, J(X'-X)] = XJ(X'-X) - J(X'-X)X = J(X'(X'-X) - (X'-X)X).$$

If k and the elements of the time series $\{X, X', X'', ...\}$ are all commuting scalars then this equation reduces to

$$k = (X' - X)^2.$$

Thus

$$X' = X \pm k^{1/2},$$

a Brownian random walk, is a solution to the simplest one-dimensional commutator equation.

Now lets examine this Brownian walk more closely by quantifying the time step as well as the space step. We take

$$\Delta t = \tau$$

so that

$$DX = J(X' - X)/\tau$$

where it is assumed that τ is a scalar, commuting with all elements of the time series and commuting with the operator J (that is, τ does not change with time). Now examine once again the equation

$$[X, DX] = Jk.$$

Let $|X' - X| = \Delta$. Then, repeating the calculation, we find

$$k = (X' - X)^2/\tau = \Delta^2/\tau.$$

Hence

$$\Delta^2/\tau = k.$$

This tells us that if k is to be constant then there must be a constant relationship between the square of the space interval for the Brownian walk and the size of the time interval. The remarkable point here is that it is just this constant relationship that is required for a Brownian process to be described by the diffusion equation

$$\partial P(x,t)/\partial t = C\partial^2 P(x,t)/\partial x^2$$

where the diffusion constant C is given by the formula

$$C = \Delta^2/2\tau = k/2.$$

The diffusion constant comes directly from our consideration involving the DOC commutator without any of the usual conceptual apparatus about approximating a differential equation.

To make this comparison, lets recall how the diffusion equation usually arises in discussing Brownian motion. We are given a Brownian process where

$$x(t+\tau) = x(t) \pm \Delta$$

so that the time step is τ and the space step is of absolute value Δ. We regard the probability of left or right steps as equal, so that if $P(x,t)$ denotes the probability that the Brownian particle is at point x at time t then

$$P(x,t+\tau) = P(x-\Delta,t)/2 + P(x+\Delta)/2.$$

From this equation for the probability we can write a difference equation for the partial derivative of the probability with respect to time:

$$[(P(x,t+\tau) - P(x,t))/\tau] = (h^2/2\tau)[(P(x-\Delta,t) - 2P(x,t) + P(x+\Delta))/\Delta^2]$$

The expression in brackets on the right hand side is a discrete approximation to the second partial of $P(x,t)$ with respect to x. Thus if the ratio $C = \Delta^2/2\tau$ remains constant as the space and time intervals approach zero, then this equation goes in the limit to the diffusion equation

$$\partial P(x,t)/\partial t = C\partial^2 P(x,t)/\partial x^2.$$

It is most curious how the diffusion constant comes up in these two contexts. Lets try to think about the comparison between the non-commutative observational starting point and the more standard differential approximation. In the non-commutative context we get Δ^2 from the appearance of the square of the difference of X' and X in the calculation of the commutator of X and DX. In the differential approximation, we get the Δ^2 from the approximation of the second derivative of the probability $P(x,t)$ with respect to x. The concept of probability does not appear in the non-commutative context. Clearly this subject needs more thought.

2.2 Planck's Numbers, Schrödinger's Equation and the Diffusion Equation

First recall the Planck Numbers. $\hbar$ is Planck's constant divided by 2π. c is the speed of light. G is Newton's gravitational constant. The Planck length will be denoted by L, the Planck time by T and the Planck mass by M. Their formulas are

$$M = \sqrt{\hbar c/G}$$

$$L = \hbar/Mc$$

$$T = \hbar/Mc^2.$$

These amounts of mass, length and time have just these dimensions and are constructed from the values of fundamental physical constants. They have roles in physics that point to deeper reasons than the formal for introducing them. Here we shall see how they are related to the Schrödinger equation.

Recall that Schrödinger's equation can be regarded as the diffusion equation with an imaginary diffusion constant. Recall how this works. The Schrödinger equation is

$$i\hbar\partial\psi/\partial t = H\psi$$

where the Hamiltonian H is given by the equation $H = p^2/2m + V$ where $V(x,t)$ is the potential energy and $p = \hbar/i\partial/\partial x$ is the momentum operator. With this we have $p^2/2m = (-\hbar^2/2m)\partial^2/\partial x^2$. Thus with $V(x,t) = 0$, the equation becomes $i\hbar\partial\psi/\partial t = (-\hbar^2/2m)\partial^2\psi/\partial x^2$ which simplifies to

$$\partial\psi/\partial t = (i\hbar/2m)\partial^2\psi/\partial x^2.$$

Thus we have arrived at the form of the diffusion equation with an imaginary constant, and it is possible to make the identification with the diffusion equation by setting

$$\hbar/m = \Delta^2/\tau$$

where Δ denotes a space interval, and τ denotes a time interval as explained in the last section about the Brownian walk. With this we can ask what space interval and time interval will satisfy this relationship with a mass and Planck's constant? *Remarkably, the answer is that this equation is satisfied when m is the Planck mass, Δ is the Planck length and τ is the Planck time!!* For note that

$$L^2/T = (\hbar/Mc)^2/(\hbar/Mc^2) = \hbar/M.$$

I now quote an email comment of Pierre Noyes: "With regard to your DOC derivation of the diffusion equation, and with an imaginary diffusion coefficient, the Schrödinger equation, note that the relation $\hbar/m = L^2/T$ is satisfied for *any* mass m provided we take $L =$ Compton wavelength $= \hbar/mc$ and $T =$Compton time$= \hbar/mc^2$ — which is simply the time of a step length of this length taken at the velocity of light. I have a vague idea that I heard of this relation when I was a graduate student. In any case I am sure Feynman had it in mind when he used a random walk on the light cone to derive the 1+1 Dirac equation, and counted steps using i! So, in a sense, your DOC derivation of the diffusion equation does connect the Maxwell equations derivation via DOC, to the Dirac equation derivation — which in a vague sense was what I hoped we would be able to do this spring (2002). Of course this general result applies in particular to the Planck mass, which was your first observation. It is intriguing that if the mass scale is m [the Planck mass], then we can use either the Compton wavelength or the Schwarzchild radius at that mass scale as the step length in DOC. This reinforces my conviction (expressed long ago) that elementary particles are small black holes."

The last part of Noyes' remark about the Schwarzchild radius refers to our work [37] explaining Ed Jones' microcosmology. Jones observed that if, for a particle of mass m we set the Schwarzchild radius ($R_S = 2mG/c^2$) equal to the Compton radius ($R_C = \hbar/2mc$), then the resulting mass m is equal to one half the Planck mass!

$$R_S = R_C$$
$$2mG/c^2 = \hbar/2mc$$
$$m = (1/2)\sqrt{\hbar c/G} = M/2$$

This is highly suggestive of limiting conditions on matter ("Plancktonic matter") prior to the Big Bang and leads in this way to specific cosmological predictions. It also gives an intriguing physical meaning to the Planck mass.

What does all this say about the nature of the Schrödinger equation itself? Interpreting it as a diffusion equation with imaginary constant suggests comparing with the DOC equation

$$[X, DX] = JiC$$

for a real constant C. This equation implicates a Brownian process where $X' = X \pm Z$ where $Z^2/\tau = iC$. We can take $Z = \sqrt{i}L$ where L is a real step-length. This gives a Brownian walk in the complex plane with the correct

DOC diffusion constant. However, the relationship of this walk with the Schrödinger equation is less clear because the ψ in that equation is not the probability for the Brownian process. To see a closer relationship we will take a different tack.

Consider a discrete function $\psi(x,t)$ defined (recursively) by the following equation

$$\psi(x,t+\tau) = (i/2)\psi(x-\Delta,t) + (1-i)\psi(x,t) + (i/2)\psi(x+\Delta,t)$$

In other words, we are thinking here of a random "quantum walk" where the amplitude for stepping right or stepping left is proportional to i while the amplitude for not moving at all is proportional to $(1-i)$. It is then easy to see that ψ is a discretization of

$$\partial\psi/\partial t = (i\Delta^2/2\tau)\partial^2\psi/\partial x^2.$$

Just note that ψ satisfies the difference equation

$$(\psi(x,t+\tau) - \psi(x,t))/\tau = (i\Delta^2/2\tau)(\psi(x-\Delta,t) - 2\psi(x,t) + \psi(x+\Delta,t))/\Delta^2$$

This gives a direct interpretation of the solution to the Schrödinger equation as a limit of a sum over generalized Brownian paths with complex amplitudes. We can then reinterpret this in DOC terms by the equation $[X,DX] = J(\Delta^2/\tau)$ or $[X,DX] = 0$, each of these contingincies happening probabilistically. It remains to be seen whether there is further insight to be gained into the Schrödinger equation via this combination of the DOC approach and the stochastic approach.

2.3 DOC Chaos

Along with the simple Brownian motion solution to the one dimensional commutator equation, there is a heirarchy of time series that solve this equation, with periodic and chaotic behaviour. These solutions can be obtained by taking

$$X = J^n Y$$

where Y is a numerical scalar, and taking the commutator equation to be

$$[X,DX] = J^{2n+1}k$$

where k is a scalar. Expanding this equation, we find

$$XJ(X' - X) - J(X' - X)X = J^{2n+1}k$$

$$J^nYJ(J^nY' - J^nY) - J(J^nY' - J^nY)J^nY = J^{2n+1}k$$
$$J^{2n+1}Y^{n+1}(Y' - Y) - J^{2n+1}(Y^{n+1} - Y^n)Y = J^{2n+1}k$$
$$Y^{n+1}(Y' - Y) - (Y^{n+1} - Y^n)Y = k$$
$$Y^{n+1}(Y' - 2Y) = k - Y^nY$$
$$Y^{n+1} = (k - Y^nY)/(Y' - 2Y).$$

This last equation expresses the time series recursively where Y refers to the value of the series that is n time steps back from Y^n. The first case of this recursion is

$$Y'' = (k - Y'Y)/(Y' - 2Y).$$

Next case is

$$Y''' = (k - Y''Y)/(Y' - 2Y).$$

These recursions depend critically on the value of the parameter k. In the first case one sees periodic oscillations that (for appropriate values of k) destabilize and blow up, alternating between an unbounded phase and a bounded semi-periodic phase. We will investigate these time series in a separate paper.

2.4 More Variables

In the Feynman-Dyson derivation of electromagnetic formalism from commutation relations [24] one uses the relations

$$[X_i, X_j] = 0$$

$$[X_i, DX_j] = k\delta_{ij}$$

where k is a scalar. Here we shall use

$$[X_i, X_j] = 0$$

$$[X_i, DX_j] = Jk\delta_{ij}$$

as we did in analyzing the one-dimensional case. This allows us to have scalar evolution of the time series, but changes some of the issues in the

Feynman-Dyson derivation. These are in fact handled by the more general formalism that we discuss in the next two sections. Thus we shall aim in this section to see to what extent one can make simple models for this version of the Feynman-Dyson relations. Models of this sort will be another level of approximation to discrete electromagnetism.

Writing out the commutation relation $[X, DX] = Jk$, and not making any assumption that X' commutes with X, we find

$$J^{-1}[X, DX] = X'(X' - X) - (X' - X)X$$
$$= X'(X' - X) - X(X' - X) + X(X' - X) - (X' - X)X$$
$$= (X' - X)^2 + (XX' - X'X) = (X' - X)^2 + [X, X'].$$

Thus the commutation relation $[X, DX] = Jk$ becomes the equation

$$(X' - X)^2 + [X, X'] = k.$$

By a similar calculation, the equation $[X, DY] = 0$ becomes the equation

$$(X' - X)(Y' - Y) + [X, Y'] = 0.$$

These equations are impossible to satisfy simultaneously for $k \neq 0$ if we assume that X and X' commute and that X and Y' commute and that $[Y, DY] = Jk$. For then we would need to solve:

$$(X' - X)^2 = k.$$
$$(Y' - Y)^2 = k.$$
$$(X' - X)(Y' - Y) = 0.$$

with the first two equations implying that $(X - X')$ and $(Y - Y')$ are each non-zero, and the third implying that their product is equal to zero. In other words, the equations below cannot be satisfied if the time series are composed of commuting scalars.

$$[X, DX] = Jk$$
$$[Y, DY] = Jk$$
$$[X, Y] = 0$$

In order to make such models we shall have to introduce non-commutativity into the time series themselves. In a certain sense this is analoguous to the introduction of non-commutative algebra in the Dirac equation in $3 + 1$ dimensions, and to the introduction of non-commutative fields in gauge theory.

Here is an example of such a model.

Return to the equations

$$(X' - X)^2 + [X, X'] = k.$$

$$(X' - X)(Y' - Y) + [X, Y'] = 0$$

expressing the behaviour for two distinct variables X and Y. If $[X, X'] = 0$, then we have $(X' - X)^2 = k$ so that

$$X' = X \pm \sqrt{k}.$$

In order for the second equation to be satisfied, we need that

$$[X, Y'] = \pm k$$

where the ambiguity of sign is linked with the varying signs in the temporal behaviour of X and Y. We will make the sign more precise in a moment, but the radical part of this suggestion is that for two distinct spatial variables X and Y, there will be a commutation relation between one and a time shift of the other.

If the space variables are labeled X_i, then we can write

$$X_i^{t+1} = X_i^t + \epsilon_i^t k$$

where ϵ_i^n is plus one or minus one. Thus each space variable performs a walk with the fixed step-length k. We shall write informally

$$X_i' = X_i + \epsilon_i k$$

where it is understood that the epsilon without the superscript connotes the sign change that occurs in this juncture of the process. We then demand the commutation relations

$$[X_i', X_j] = [X_j', X_i] = \epsilon_i \epsilon_j k.$$

Each X_i is a scalar in its own domain, but does not commute with the time shifts of the other directions. We then can have the full set of commutation relations:

$$[X_i', X_j] = [X_j', X_i] = \epsilon_i \epsilon_j k.$$

$$[X_i, X_j] = 0$$
$$[X_i, DX_j] = Jk\delta_{ij}$$

so that the system will satisfy the assumptions supporting the Feynman-Dyson derivation. In this system,the elements of a given time series $X_i, X_i', X_i'', \cdots$ commute with one another. The basic field element in the Feynman-Dyson set up is the magnetic field B defined by the (non-commutative) vector cross product

$$B = (1/k)DX \times DX.$$

Here we have

$$DX_i = J(X_i' - X_i) = J\epsilon_i\sqrt{k}.$$

Thus

$$B = J^2\ \epsilon' \times \epsilon$$

where $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ (assuming three spatial coordinates) and ϵ' denotes this vector of signs at the next time step. In this way we see that we can think of each spatial coordinate as providing a long temporal bit string and the three coordinates together give the field in terms of the vector cross product of their temporal cross sections at neighboring instants. It is interesting to compare this model with the color algebra in the following paper by Wene [43].

2.5 Discrete Classical Electromagnetism

It is of interest to compare these results with a direct discretization of classical electromagnetism. Suppose that $X, X', X'', X''', \cdots$ is a time series of vectors in R^3 (where R denotes the real numbers). Let $dX = X' - X$ be the usual discrete derivative (with time step equal to one for convenience). Let $A \bullet B$ denote the usual inner product of vectors in three dimesions. Assume that there are fields E and B such that

$$d^2X = E + dX \times B$$

(the Lorentz force law). Assume also that E and B are perpendicular to the velocity vector dX, and that E is perpendicular to B. Then we have

$$dX' \times dX = (dX' - dX) \times dX = (d^2X) \times (dX)$$

$$= E \times dX + (dX \times B) \times dX$$

$$= E \times dX - dX(B \bullet dX) + (dX \bullet dX)B.$$

Since E is perpendicular to dX we know there is a λ such that $E \times dX = \lambda B$ and we have $B \bullet dX = 0$ since B is perpendicular to dX. Therefore

$$dX' \times dX = \lambda B + ||dX||^2 B$$

so that

$$B = dX' \times dX/(\lambda + ||dX||^2).$$

Up to the factor in the denominator, this formula is in exactly the same pattern as the formula in our discrete model for DOC electromagnetism as described in the previous subsection. To see this, note that the B field in the DOC model is proportional to $DX \times DX$ and that $DX = JdX$ so that $DX \times DX = JdX \times JdX = J^2 dX' \times dX$. Up to the time-shifting algebra and a proportionality constant, the expressions are the same! Clearly more work is needed in comparing classical discrete electromagnetism with the results of a discrete analysis of the Feynman-Dyson derivation.

3 Gauge Fields and Differential Geometry

Letting X_i ($i = 1, 2, ..., d$) denote a set of spatial variables (non-commutative time series in the sense of our discrete ordered calculus), we will look at a collection of basic assumptions about the commutation of these variables and of their derivatives. It is natural from the point of view of the discrete ordered calculus to have

$$[X_i, X_j] = 0$$

for all i and j. There are no other natural commutations from the point of view of this calculus.

We shall define g_{ij} by the equation

$$[X_i, \dot{X}_j] = g_{ij}.$$

Here $\dot{X}_j$ is shorthand forDX_j and

$$[A, B] = AB - BA.$$

Along with this commutator equation, we will assume that

$$[X_i, X_j] = 0,$$

$$[X_i, g_{jk}] = 0$$

and

$$[g_{rs}, g_{jk}] = 0.$$

Here it is assumed that g_{ij} is non-degenerate in the sense that there exists g^{ij} so that

$$g^{ij}g_{jk} = \delta^i_k$$

and that

$$g_{ij}g^{jk} = \delta^k_i.$$

Here we are using the Einstein summation convention that implicitly assumes that we sum over repeated indices in an expression. Symbol δ^i_j is a Kronecker delta, equal to 1 when i equals j and 0 otherwise.

The first result that is a direct consequence of these assumptions is the symmetry of the "metric" coefficients g^{ij}. That is, we shall show that

$$g^{ij} = g^{ji}.$$

Lemma 3. $g_{ij} = g_{ji}$.

Proof.

$$\begin{aligned} & g_{ij} - g_{ji} \\ &= [X_i, \dot{X}_j] - [X_j, \dot{X}_i] \\ &= [X_i, \dot{X}_j] + [\dot{X}_i, X_j] \\ &= D[X_i, X_j] \\ &= 0. \end{aligned}$$

For the purpose of doing calculus in this situation we define $\dot{X}^i$ by the equation

$$\dot{X}^i = g^{ik}\dot{X}_k.$$

The operator $\dot{X}^i$ is simply the index shift of the corresponding $\dot{X}_i$. We do not define a corresponding X^i. It is easy to check the equation

$$[X_i, \dot{X}^j] = \delta_i^j.$$

Consequently, we define the derivative of an operator F with respect to X_i by the equation

$$\partial^i F = [F, \dot{X}^i]$$

and the corresponding lowered derivative by the formula

$$\partial_i F = [F, \dot{X}_i].$$

Note that we have

$$\partial_i X_j = g_{ij}.$$

We also define

$$\hat{\partial}_i F = [X_i, F],$$

the derivative of F with respect to the conjugate variable $\dot{X}^i$.

With these partial derivatives in hand, we define $\dot{F}$ by the formula

$$\dot{F} = \partial^k F \dot{X}_k.$$

If F commutes with g^{ij} then it is easy to see that

$$\dot{F} = \partial_k F \dot{X}^k.$$

These formulas extend (implicitly) the definition of the time series to entities other than the operators X_i since

$$\dot{F} = DF = J(F' - F).$$

A stream of consequences then follows by differentiating both sides of the equation

$$g_{ij} = [X_i, \dot{X}_j].$$

Note that

$$\dot{g_{ij}} = [\dot{X}_i, \dot{X}_j] + [X_i, D^2 X_j]$$

by the Leibniz rule

$$D[A, B] = [DA.B] + [A, DB].$$

Note also that we can freely use the Jacobi identity

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

In particular, the Levi-Civita connection

$$\Gamma_{ijk} = (1/2)(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

associated with the g_{ij} comes up almost at once from the differentiation process described above. To see how this happens, view the following calculation where

$$\hat{\partial}_i \hat{\partial}_j F = [X_i, [X_j, F]].$$

We apply the operator $\hat{\partial}_i \hat{\partial}_j$ to the second DOC derivative of X_k.

Lemma 4. $\Gamma_{ijk} = (1/2)\hat{\partial}_i \hat{\partial}_j D^2 X_k$

Proof.

$$\hat{\partial}_i \hat{\partial}_j D^2 X_k = [X_i, [X_j, D^2 X_k]]$$

$$= [X_i, \dot{g_{jk}} - [\dot{X}_j, \dot{X}_k]]$$

$$= [X_i, \dot{g_{jk}}] - [X_i, [\dot{X}_j, \dot{X}_k]]$$

$$= [X_i, \dot{g_{jk}}] + [\dot{X}_k, [X_i, \dot{X}_j]] + [\dot{X}_j, [\dot{X}_k, X_i]$$

$$= [g_{jk}, \dot{X}_i] + [\dot{X}_k, g_{ij}] + [\dot{X}_j, -g_{ik}]$$

$$= \partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}$$

$$= 2\Gamma_{kij}.$$

It is remarkable that the form of the Levi-Civita connection comes up directly from this non-commutative calculus without any apriori geometric interpretation. We shall discuss the context of this result in the next two sections of the paper.

One finds that

$$D^2 X_i = G_i + g_{ir} g_{js} F^{rs} \dot{X}^j + \Gamma_{ijk} \dot{X}^j \dot{X}^k$$

where

$$F^{rs} = [\dot{X}^r, \dot{X}^s].$$

It follows from the Jacobi identity that

$$F_{ij} = g_{ir} g_{js} F^{rs}$$

satisfies the equation

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0,$$

identifying F_{ij} as a non-commutative analog of a gauge field. G_i is a non-commutative analog of a scalar field. The details of these calculations will be found in [26].

This description of the equations for a non-commutative particle in a metric field illustrates the role of the background discrete time in this theory. In terms of the background time the metric coefficients are not constant. It is through this variation that the spacetime derivatives of the theory are articulated. The background is a process with its own form of discrete time, but no spacetime structure as we know and observe it. Our observation of spacetime structure appears as a rough (commutative) approximation to the processes described as consequences of the basic non-commutative equations of the discrete ordered calculus.

4 Curvature, Jacobi Identity and the Levi-Civita Connection

In this section, we go back to basics and examine the context of calculus defined via commutators. We shall use a partially index-free notation. In this notation, we avoid nested subscripts by using different variable names and then using these names as subscripts to refer to the relevant variables. Thus we write X and Y instead of X_i and X_j, and we write g_{XY} instead of g_{ij}. It is assumed that the derivation DX has the form $DX = [X, J]$ for some J.

The bracket $[A, B]$ is not assumed to be a commutator. It is assumed to satisfy the Jacobi identity, bilinearity in each variable, and the Leibniz rule for all functions of the form $\delta_K(A) = [A, K]$. That is we assume that

$$\delta_K(AB) = \delta_K(A)B + A\delta_K(B).$$

Recall that in classical differential geometry one has the notion of a covariant dervative, defined by taking a difference quotient using parallel translation via a connection. Covariant derivatives in different directions do not necessarily commute. The commutator of covariant derivatives gives rise to the curvature tensor in the form

$$[\nabla_i, \nabla_j]X^k = R^k_{lij}X^l.$$

If derivatives do not commute then we regard their commutator as expressing a curvature. In our non-commutative context this means that curvature arises *prior* to any notion of covariant derivatives since *even the basic derivatives do not commute.*

We shall consider derivatives in the form

$$\nabla_X(A) = [A, \Lambda_X].$$

Examine the following computation:

$$\nabla_X\nabla_Y F = [[F, \Lambda_Y], \Lambda_X] = -[[\Lambda_X, F], \Lambda_Y] - [[\Lambda_Y, \Lambda_X], F]$$

$$= [[F, \Lambda_X], \Lambda_Y] + [[\Lambda_X, \Lambda_Y], F]$$

$$= \nabla_Y \nabla_X F + [[\Lambda_X, \Lambda_Y], F].$$

Thus

$$[\nabla_X, \nabla_Y]F = R_{XY}F$$

where

$$R_{XY}F = [[\Lambda_X, \Lambda_Y], F].$$

We can regard R_{XY} as a curvature operator.

The analog in this context of flat space is abstract quantum mechanics! That is, we assume position variables (operators) X, Y, $\cdots$ and momentum variables (operators) P_X, P_Y, $\cdots$ satisfying the equations below.

$$[X, Y] = 0$$

$$[P_X, P_Y] = 0$$

$$[X, P_Y] = \delta_{XY}$$

where δ_{XY} is equal to one if X equals Y and is zero otherwise. We define

$$\partial_X F = [F, P_X]$$

and

$$\partial_{P_X} F = [X, F].$$

In the context of the above commutation relations, note that these derivatives behave correctly in that

$$\partial_X(Y) = \delta_{XY}$$

and

$$\partial_{P_X}(P_Y) = \delta_{XY}$$

$$\partial_{P_X}(Y) = 0 = \partial_X(P_Y)$$

with the last equations valid even if $X = Y$. Note also that iterated partial derivatives such as $\partial_X \partial_Y$ commute. Hence the curvature R_{XY} is equal to zero. We shall regard these position and momentum operators and the corresponding partial derivatives as an abstract algebraic substitute for flat space.

With this reference point of (algebraic, quantum) flat space we can define

$$\hat{P_X} = P_X - A_X$$

for an arbitrary algebra-valued function of the variable X. In indices this would read

$$\hat{P}_i = P_i - A_i,$$

and with respect to this deformed momentum we have the covariant derivative

$$\nabla_X F = [F, \hat{P_Y}] = [F, P_Y + A_Y] = \partial_Y F + [F, A_Y].$$

The curvature for this covariant derivative is given by the formula

$$R_{XY}F = [\nabla_X, \nabla_Y]F = [[\lambda_X, \lambda_Y], F]$$

where $\lambda_X = P_X - A_X$. Hence

$$R_{XY} = [P_X - A_X, P_Y - A_Y] = -[P_X, A_Y] - [A_X, P_Y] + [A_X, A_Y]$$

$$= \partial_X A_Y - \partial_Y A_X + [A_X, A_Y].$$

With indices this reads

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

and the reader will note that this has the abstract form of the curvature of a Yang-Mills gauge field, and specifically the form of the electromagnetic field when the potentials A_i and A_j commute with one another.

Continuing with this example, we compute

$$[X, \hat{P_Y}] = [X, P_Y - A_Y] = \delta_{XY} - [X, A_Y].$$

Let

$$g_{XY} = \delta_{XY} - [X, A_Y]$$

so that

$$[X, \hat{P_Y}] = g_{XY}.$$

We will shortly consider the form of this general case, but first it is useful to restrict to the case where $[X, A_Y] = 0$ so that $g_{XY} = \delta_{XY}$. This is the domain to which the original Feynman-Dyson derivation applies. In order to enter this domain, we set

$$\dot{X} = DX = \hat{P_X} = P_X - A_X.$$

We then have

$$[X_i, X_j] = 0$$
$$[X_i, \dot{X}_j] = \delta_{ij}$$

and

$$R_{ij} = [\dot{X}_i, \dot{X}_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

Note that even under these restrictions we are still looking at the possibility of a non-abelian gauge field. The pure electromagnetic case is when the commutator of A_i and A_j vanishes. But why do we set $\dot{X} = \hat{P_X}$? The answer to this is the key to the gauge interpretation of electromagnetism, for with this interpretation we find that $\dot{X}$ satisfies the Lorentz force law $\ddot{X} = E + \dot{X} \times B$ where B represents the magnetic field and E the electric field (in the case of three space variables X_i with $i = 1, 2, 3$.) To see how this works, suppose that $\ddot{X}_i = E_i + F_{ij}\dot{X}_j$ and suppose that E_i and F_{ij} commute with X_k. Then we can compute

$$[X_i, \ddot{X}_j] = [X_i, E_j + F_{jk}\dot{X}_k]$$
$$= F_{jk}[X_i, \dot{X}_k] = F_{jk}\delta_{ik} = F_{ji}.$$

This implies that

$$F_{ij} = [\dot{X}_i, \dot{X}_j] = R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

since $[X_i, \ddot{X}_j] + [\dot{X}_i, \dot{X}_j] = D[X_i, \dot{X}_j] = 0$. It is then easy to verify that the Lorentz force equation is satisfied with $B_k = \epsilon_{ijk} R_{ij}$ and that in the case of $[A_i, A_j] = 0$ this leads directly to standard electromagnetic theory when the bracket is a Poisson bracket (see the next section for a discussion of Poisson brackets). When this bracket is not zero but the potentials A_i are functions only of the X_j we can look at a generalization of gauge theory where the non-commutativity comes from internal Lie algebra parameters. This shows that the Feynman-Dyson derivation supports certain generalizations of classical electromagnetism, and this will be the subject of a more expanded version of this paper.

In regard to this last remark, the reader should note that in our [28, 27] algebraic and discrete version of the Feynman-Dyson derivation it was actually an additional assumption that $B \times B = 0$ where $B \times B$ denotes the (non-commutative) vector cross product of B with itself. (Note that $B = (1/2)\dot{X} \times \dot{X}$.) In the original Dyson paper this cross product vanished because of assumptions about the operators and their Hilbert space representations. With $B \times B$ as an extra term, the Feynman-Dyson derivation is indeed a non-commutative generalization of electromagnetism and includes forms of gauge theories among its models.

Generalizing, we wish to examine the structure of the following special axioms for a bracket.

$$[X, DY] = g_{XY}$$

$$[X, Y] = 0$$

$$[Z, g_{XY}] = 0$$

$$[g_{XY}, g_{ZW}] = 0$$

Note that

$$Dg_{YZ} = D[Y, DZ] = [DY, DZ] + [Y, D^2 Z].$$

and that $D[X, g_{XY}] = 0$ implies that

$$[g_{XY}, DZ] = [Z, Dg_{XY}].$$

Define two types of derivations as follows

$$\nabla_X(F) = [F, DX]$$

and

$$\nabla_{DX}(F) = [X, F].$$

These are dual with respect to g_{XY} and will act like partials with respect to these variables in the special case when g_{XY} is a Kronecker delta, δ_{XY}. If the form g_{XY} is invertible, then we can rewrite these derivations by contracting the inverse of g to obtain standard formal partials.

$$\nabla_{DX}\nabla_{DY}D^2Z = [X,[Y,D^2Z]]$$
$$= [X, Dg_{YZ} - [DY, DZ]] = [X, Dg_{YZ}] - [X,[DY,DZ]]$$
$$= [g_{YZ}, DX] - [X,[DY,DZ]]$$
$$= \nabla_X(g_{YZ}) - [X,[DY,DZ]].$$

Now use the Jacobi identity on the second term and obtain

$$\nabla_{DX}\nabla_{DY}D^2Z = \nabla_X(g_{YZ}) + [DZ,[X,DY]] + [DY,[DZ,X]]$$

$$= \nabla_X(g_{YZ}) - \nabla_Z(g_{XY}) + \nabla_Y(g_{XZ}).$$

This is the formal Levi-Civita connection.

At this stage we face once again the mystery of the appearance of the Levi-Civita connection. There is a way to see that the appearance of this connection is not an accident, but rather quite natural. We shall explain this point of view in the next section where we discuss Poisson brackets and the connection of this formalism with classical physics. On the other hand, we have seen in this section that it is quite natural for curvature in the form of the non-commutativity of derivations to appear at the outset in a non-commutative formalism. We have also see that this curvature and connection can be understood as a measurement of the deviation of the theory from the "flat" commutation relations of ordinary quantum mechanics. Electromagnetism and Yang-Mills theory can be seen as the theory of the curvature introduced by such a deviation. On the other hand, from the point of view of metric differential geometry, the Levi-Civita connection is the unique connection that preserves the inner product defined by the metric under the parallel translation defined by the connection. We would like to see that the formal Levi-Civita connection produced here has this property as well.

To this end lets recall the formalism of parallel translation. The infinitesimal parallel translate of A is denoted by $A' = A + \delta A$ where

$$\delta A^k = -\Gamma^k_{ij} A^i dX^j$$

where here we are writing in the usual language of vectors and differentials with the Einstein summation convention for repeated indices. We assume that the Christoffel symbols satisfy the symmetry condition $\Gamma^k_{ij} = \Gamma^k_{ji}$. The inner product is given by the formula

$$< A, B >= g_{ij} A^i B^j$$

Note that here the bare symbols denote vectors whose coordinates may be indicated by indices. The requirement that this inner product be invariant under parallel displacement is the requirement that $\delta(g_{ij} A^i A^j) = 0$. Calculating, one finds

$$\delta(g_{ij} A^i A^j) = (\partial_k g_{ij}) A^i A^j dX^k + g_{ij} \delta(A^i) A^j + g_{ij} A^i \delta(A^j)$$

$$= (\partial_k g_{ij}) A^i A^j dX^k - g_{ij} \Gamma^i_{rs} A^r dX^s A^j - g_{ij} A^i \Gamma^j_{rs} A^r dX^s$$

$$= (\partial_k g_{ij}) A^i A^j dX^k - g_{ij} \Gamma^i_{rs} A^r A^j dX^s - g_{ij} \Gamma^j_{rs} A^i A^r dX^s$$

$$= (\partial_k g_{ij}) A^i A^j dX^k - g_{sj} \Gamma^s_{ik} A^i A^j dX^k - g_{is} \Gamma^s_{jk} A^i A^j dX^k$$

Hence

$$(\partial_k g_{ij}) = g_{sj} \Gamma^s_{ik} + g_{is} \Gamma^s_{jk}.$$

From this it follows that

$$\Gamma_{ijk} = g_{is} \Gamma^s_{jk} = (1/2)(\partial_k g_{ij} - \partial_i g_{jk} + \partial_j (g_{ik})).$$

Certainly these notions of variation can be imported into our abstract context. The question remains how to interpret the new connection that arises. We now have a new covariant derivative in the form

$$\hat{\nabla}_i X^j = \partial_i X^j + \Gamma^j_{ki} X^k.$$

The question is how the curvature of this connection interfaces with the gauge potentials that gave rise to the metric in the first place. The theme of this investigation has the flavor of gravity theories with a qauge theoretic background. We will investigate these relationships in detail in a sequel to this paper.

5 Poisson Brackets and Commutator Brackets

Dirac [11] introduced a fundamental relationship between quantum mechanics and classical mechanics that is summarized by the maxim *replace Poisson brackets by commutator brackets.* Recall that the Poisson bracket $\{A, B\}$ is defined by the formula

$$\{A, B\} = (\partial A/\partial q)(\partial B/\partial p) - (\partial A/\partial p)(\partial B/\partial q),$$

where q and p denote classical position and momentum variables respectively.

In our version of discrete physics the noncommuting variables are functions of discrete time, with a DOC derivative D as described in the first section. Since $DX = XJ - JX = [X, J]$ is itself a commutator, it follows that

$$D([A, B]) = [DA, B] + [A, DB]$$

for any expressions A, B in our ring R. A corresponding Leibniz rule for Poisson brackets would read

$$(d/dt)\{A, B\} = \{dA/dt, B\} + \{A, dB/dt\}.$$

However, here there is an easily verified exact formula:

$$(d/dt)\{A, B\} = \{dA/dt, B\} + \{A, dB/dt\} - \{A, B\}(\partial\dot{q}/\partial q + \partial\dot{p}/\partial p).$$

This means that the Leibniz formula will hold for the Poisson bracket exactly when

$$(\partial\dot{q}/\partial q + \partial\dot{p}/\partial p) = 0.$$

This is an integrability condition that will be satisfied if p and q satisfy Hamilton's equations

$$\dot{q} = \partial H/\partial p,$$

$$\dot{p} = -\partial H/\partial q.$$

This, of course, means that q and p are following a principle of least action with respect to the Hamiltonian H. Thus we can interpret the *fact* $D([A,B]) = [DA,B] + [A,DB]$ in the discrete (commutator) context as an analog of the principle of least action. Taking the discrete context as fundamental, we say that Hamilton's equations are *motivated* by the presence of the Leibniz rule for the discrete derivative of a commutator. The classical laws are obtained by following Dirac's maxim in the opposite direction! Classical physics is produced by following the correspondence principle upwards from the discrete.

Taking the last paragraph seriously, we must reevaluate the meaning of Dirac's maxim. The meaning of quantization has long been a basic mystery of quantum mechanics. By traversing this territory in reverse, starting from the non-commutative world, we begin these questions anew.

In making this backwards journey to classical physics we see how our earlier assertion that bare quantum mechanics of commutators can be regarded as the background for the coupling with other fields (as in the description of formal gauge theory in the last section), fits with Poisson brackets. The bare Poisson brackets satisfy

$$\{q_i, q_j\} = 0$$

$$\{p_i, p_j\} = 0$$

$$\{q_i, p_j\} = \delta_{ij}.$$

In our previous formalism, we would identify X_i as the correspondent with q_i and P_j as the correspondent of p_j. And, given a classical vector potential A, we could write the coupling $dq_i/dt = p_i - A_i$ to describe the motion of a particle in the presence of an electromagnetic field. The analog of the Feynman Dyson derivation is then expressed classically in terms of the Poisson brackets. Similar remarks apply to the analogs for gauge theory and curvature. In particular it is of interest to see that our derivation of the Levi-Civita connection corresponds to the motion of a particle in generalized coordinates that satisfies Hamilton's equations. The fact that such a particle moves in a geodesic according to the Levi-Civita connection is a classical fact that was surely one of the motivations for the development of differential geometry. Our derivation of the Levi-Civita connection, interpreted in Poisson brackets, reproduces this result.

To see how this works, let $ds^2 = g^{ij}dx_i dx_j$ denote the metric in the generalized coordinates x_k. Then the velocity of the particle has square $v^2 = (ds/dt)^2 = g^{ij}\dot{x}_i\dot{x}_j$. The Lagrangian for the system is the kinetic energy $L = mv^2/2 = mg^{ij}\dot{x}_i\dot{x}_j/2$. Then the canonical momentum is $p_j = \partial L/\partial \dot{x}_j$, and with $q_i = x_i$ we have the Poisson brakets

$$\delta_{ij} = \{q_i, p_j\} = \{x_i, \partial L/\partial \dot{x}_j\} = \{x_i, mg^{jk}\dot{x}_k\}.$$

Taking $m = 1$ for simplicity, we can rewrite this bracket as

$$\{x_i, \dot{x}_j\} = g_{ij}.$$

This, in Poisson brackets, is our generalized equation of motion.

The classical derivation applies Lagrange's equation of motion to the system. Lagrange's equation reads

$$d/dt(\partial L/\partial \dot{x}_i) = \partial L/\partial x_i.$$

Since this equation is equivalent to Hamilton's equation of motion, it follows that the Poisson brackets satisfy the Leibniz rule. With this, we can proceed with our derivation of the Levi-Civita connection in relation to the acceleration of the particle. In the classical derivation, one writes out the Lagrange equation and solves for the acceleration. The advantage of using only the Poisson brackets is that it shows the relationship of the connection with the Jacobi identity and the Leibniz rule.

This discussion raises further questions about the nature of the generalization that we have made. Originally Hermann Weyl [44] generalized classical differential geometry and discovered gauge theory by allowing changes of length as well as changes of angle to appear in the holonomy. Here we arrive at a very similar situation via the properties of a non-commutative discrete calculus of observations. A closer comparison with the geometry of gauge theories is called for.

6 Discussion on q-Deformation

The direct relation between the content of local physical descriptions based on the *DOC* calculus and more global considerations are a matter of speculation.

One strong hint is contained in the properties of the discrete derivative that has the form

$$D_q f(x) = (f(qx) - f(x))/(qx - x).$$

The classical derivative occurs in the limit as q approaches one.

In the setting of q not equal to one, the derivative D_q is directly related to fundamental noncommutativity. Consider variables x and y such that $yx = qxy$ where q is a commuting scalar. Then the expansion of $(x + y)^n$ generates a q-binomial theorem with q-choice coefficients composed in q-factorials of q-integers $[n]_q$ where

$$[n]_q = 1 + q + q^2 + ... + q^{(n-1)}.$$

The derivative D_q is directly related to the q-integers via the formula

$$D_q(x^n) = [n]_q x^{n-1}.$$

In the context of this paper, we have considered discrete derivatives in the form

$$d_\Delta f(x) = (f(x + \Delta) - f(x))/\Delta.$$

This will convert to the q-derivative if $x + \Delta = qx$. Thus we need

$$q = (x + \Delta)/x.$$

This means that a direct translation from DOC to q-derivations could be effected if we allowed q to vary as a function of x and introduced the temporal operator J into the calculus of q-derivatives.

In general, many q-deformed structures such as the quantum groups associated with the classical Lie algebras appear to be entwined with the discretization inherent in D_q. The quantum groups have turned out to be deeply connected with topological amplitudes for networks describing knots and three dimensional spaces. (See the next section of this paper.) The analog for the quantum groups in dimension four is being sought. If there is a connection between the local and the global parts of our essay it may lie in hidden connections between discretization and quantum groups. Clearly there is much work to be done in this field.

There is a clue about the meaning of the operator J ($DF = [F, J]$ in the discrete ordered calculus) in the context of quantum groups. Quantum groups are Hopf algebras. A quantum group such as $G = U_q(SU(2))$ is actually an algebra over a field k with an antipode

$$S : G \longrightarrow G$$

and a coproduct

$$\Delta : G \longrightarrow G \otimes G,$$

a unit 1 and a couinit

$$\epsilon : G \longrightarrow k.$$

The coproduct is a map of algebras. The antipode is an antimorphism, $S(xy) = S(y)S(x)$, and generalizes the inverse in a group in the sense that $\Sigma S(x_1)x_2 = \epsilon(x)1$ and $\Sigma x_1 S(x_2) = \epsilon(x)1$ where $\Delta(x) = \Sigma x_1 \otimes x_2$.

An element g in a quantum group G is said to be a *grouplike element* if $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$. In many quantum groups (such as $G = U_q(SU(2))$) the square of the antipode is represented via conjugation by a special grouplike element that we shall denote by J. Thus

$$S^2(x) = J^{-1}xJ$$

for all x in G. This means that it is possible to define the discrete ordered calculus in the context of a quantum group G (as above) by taking J to be the special grouplike element. Then we have

$$DX = [X, J] = XJ - JX = J(J^{-1}XJ - X) = J(S^2(X) - X).$$

Conjugation by the special grouplike element in the quantum group constitutes the time evolution operator in this algebra.

There are a number of curious aspects to this use of the discrete ordered calculus in a quantum group. First of all, it is the case that in some quantum groups (for example with undeformed classical Lie algebras) the square of the antipode is equal to the identity mapping. From the point of view of DOC, time does not exist in these algebras. But in the q-deformations such as $U_q(SU(2))$, the square of the antipode is quite non-trivial and can serve well as the tick of the clock. In this way, q-deformations do provide a context for time. In particular, this suggests that the q-deformations of classical spin networks [38] should be able to accommodate time. A suggestion directly related to this remark occurs in [9], and we shall take this up at the end of the next section of this paper.

7 Networks, Discrete Spacetime and the Dirac Equation

One can consider replacing continuous space (such as Euclidean space with the usual topology) by a discrete structure of relationships. The geometry of the Greeks held a discrete web of relationships in the context of continuous space. That space was not coordinatized in our way, nor was it held as an infinite aggregate of points. In general topology there is a wide choice for possible spatial structures (where we mean by a space a topology on some set).

Discretization of space and time implicates the replacement of spacetime by a network, graph or complex that has nodes for the points and edges to indicate significant relationships among the points.

Euler's work in the eighteenth century brought forth the use of abstract graphs as holders of spatial structure. After Euler it was possible to find the classification of the Greek regular solids in the the (wider) classification of the regular graphs on the surface of the sphere. Metric can disappear into relationship under the topological constraint of Euler's formula $V-E+F=2$, where V denotes the number of vertices, E the number of edges and F the number of faces for the connected graph G on the sphere.

A network itself can represent an abstract space. Embeddings of that network into a given space (such as graphs on the two dimensional sphere) correspond to global constraints on the structure of the abstract graph.

Now a new theme arises, motivated by a conjunction of combinatorics and physics. Imagine labelling the edges of the network from some set of “colors”. These colors can represent the basic states of a physical system, or they can be an abstract set of distinct markers for purely mathematical purposes. Once the network is labelled, each vertex is an entity with a collection of labels incident to it. Let there be given a function that associates a number (or algebra element) to each such labelled vertex. Call this number the *vertex* *weight* at that vertex. Let C denote a specific coloring of the network N and consider the product, over all the vertices of N of the values of the vertex weights. Finally let $Z(N)$, the *amplitude* of the network, be defined as the summation of the product of the vertex weights over all colorings of the net. $Z(N)$ is also called the *partition function* of the network.

Amplitudes of this sort are exactly what one computes in finding the partition function of a physical system or the quantum mechanical amplitude for a discrete process. In all these cases the network is interwoven with the algebraic structure of the vertex weights. It is only recently that topological properties of networks in three dimensional space have come to be understood in this way [22], [1],[45]. This has led to new information about the topology of low dimensional spaces, and new relationships between physics and topology.

A classical example of such an amplitude was discovered by Roger Penrose [5] in elucidating special colorings of 3-regular graphs in the plane. A 3-regular graph G has three edges incident to each vertex. When embedded in the plane, these edges acquire a specific cyclic order. Three colors are used. One associates to each vertex the weight

$$\sqrt{-1}\ \epsilon_{abc}$$

where a,b,c denote the edges meeting the vertex in this cyclic order, and the epsilon is equal to 1, -1 according as the edges have distinct labels in the given or reverse cyclic order, or 0 if there is a repetition of labels. The resulting amplitude counts the number of ways to color the network with three colors so that three distinct colors are incident to each vertex. This result is a perspicuous generalization of the classical four color problem of coloring maps in the plane with four colors so that adjacent regions receive different colors.

The Penrose example generalizes to networks whose amplitudes embody geometrical properties of Euclidean three dimensional space (angles and their dependence). Geometry begins to emerge in terms of the averages of properties of an abstract and discrete network of relationships. Topological properties emerge in the same way. The idea of space may change to the idea of a network with global states and a functor that associates this network and its states to the more familiar properties that a classical observer might see.

7.1 Remarks on Quantum Mechanics

We should remark on the basic formalism for amplitudes in quantum mechanics. The Dirac notation $\langle A|B\rangle$ [11] denotes the probability amplitude for a transition from A to B. Here A and B could be points in space (for the path of a particle), fields (for quantum field theory), or geometries on spacetime (for quantum gravity). The probability amplitude is a complex number. The actual probability of an event is the absolute square of the amplitude. If a complete set of intermediate states $C_1, C_2, ...C_n$ is known, then the amplitude can be expanded to a summation

$$\langle A|B\rangle = \Sigma_{i=1}^{n} \langle A|C_i\rangle\langle C_i|B\rangle.$$

This formula follows the formalism of the usual rules for probability, and it allows for the constructive and destructive interference of the amplitudes. It is the simplest case of a quantum network of the form

$$A - - - * - - - C - - - * - - - B$$

where the colors at A and B are fixed and we run through all choices of colors for for the middle edge. The vertex weights at the vertices labelled $*$ are $\langle A|C\rangle$ and $\langle C|B\rangle$ respectively. A measurement at the C edge reduces the big summation to a single value.

Consider the generalization of the previous example to the graph

$$A - - - * - - - C^1 - - - * - - - C^2 - - - * - - - ... - - - * - - - C^m - - - B$$

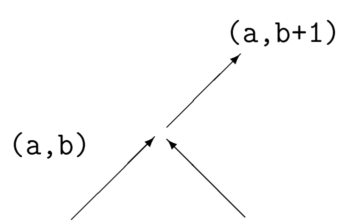
With A and B fixed the amplitude for the net is

$$< A|B >= \Sigma_{1 \le i_1 \le ... \le i_m \le n} < A|C^1_{i_1} >< C^2_{i_2}|C^3_{i_3} > ... < C^m_{i_m}|B >$$

One can think of this as the sum over all the possible paths from A to B. In fact in the case of a "particle" travelling between two points in space, this is exactly what must be done to compute an amplitude - integrate over all the paths between the two points with appropriate weightings. In the discrete case this sort of summation makes perfect sense. In the case of a continuum there is no known way to make rigorous mathematical sense out of all cases of such integrals. Nevertheless, the principles of quantum mechanics must be held foremost for physical purposes and so such "path integrals" and their generalizations to quantum fields are in constant use by theoretical physicists [16] who take the point of view that the proof of a technique is in the consistency of the results with the experiments. When the observations themselves are mathematical (such as finding invariants of knots and links), the issue acquires a new texture.

Now consider the summation discussed above in the case where $n = 2$. That is, we shall assume that each C^kcan take two values, call these values L and R. Furthermore let us suppose that $< L|R >=< R|L >= \sqrt{-1}$ while $< L|L >=< R|R >= 1$. The amplitudes that one computes in this case correspond to solutions to the Dirac equation [11] in one space variable and one time variable. This example is related to an observation of Richard Feynman [16]. In [25] we give a very elementary derivation of this result and we show how these amplitudes give solutions to the discretized Dirac equation, so everything is really quite exact and one can understand just what happens in taking the limit to the continuum. In this example a state of the network consists in a sequence of choices of L or R. These can be interpreted as choices to move left or right along the light-cone in a Minkowski plane. It is in summing over such paths in spacetime that the solution to the Dirac equation appears. In this case, time has been introduced into the net by interpreting the sequence of nodes in the network as a temporal direction.

More specifically, let (a, b) denote a point in discrete Minkowski spacetime in lightcone coordinates. This means that a denotes the number of steps taken to the left and b denotes the number of steps taken to the right. We let $\psi_L(a, b)$ denote the sum over the paths that enter the point (a, b) from the left and $\psi_R(a, b)$ the sum over the paths that enter (a, b) from the right. Each path P contributes $i^{c(P}$ where $c(P)$ denotes the number of corners in the path. View the diagram below.



It is clear from the diagram that

$$\psi_L(a, b+1) = \psi_L(a, b) + i\psi_R(a, b).$$

Thus we have that

$$\partial\psi_L/\partial R = i\psi_R$$

and similarly

$$\partial\psi_R/\partial L = i\psi_L.$$

This pair of equations is the Dirac equation in light cone coordinates.

This discrete derivation of the Dirac equation is simpler than the method used in [25]. I am indebted to Charles Bloom [3] for pointing this out to me. In fact, this form of the discretization is essentially Feynman's original method as is evident from the reproduction of Feynman's handwritten notes in Figure 8 of the review paper [39] by Schweber. For one approach, very close in spirit, that generalizes this exercise of Feynman to four dimensional discrete spacetime see [40].

As in the Dirac equation example, one way to incorporate spacetime is to introduce a temporal direction into the net. At a vertex, one must specify labels of *before* and *after* to each edge of the net that is incident to that vertex. If there is a sufficiently coherent assignment of such local times, then a global time direction can emerge for the entire network. Networks endowed with temporal directions have the structure of morphisms in a category where each morphism points from past to future. A category of quantum networks emerges equipped with a functor (via the algebra of the vertex weights) to morphisms of vector spaces and representations of generalized symmetry groups. Appropriate traces of these morphisms produce the amplitudes.

Quantum non-locality is built into the network picture. Any observer taking a measurement in the net has an effect on the global set of states available for summation and hence affects the possibilities of observations at all other nodes in the network. By replacing space with a network we obtain a precursor to spacetime in which quantum mechanics is built into the initial structure.

Remark. A striking parallel to the views expressed in this section can be found in [12]. Concepts of time and category are discussed by Louis Crane [8], [9] in relation to topological quantum field theory. In the case of Crane's work there is a deeper connection with the methods of this paper, as I shall explain below.

7.2 Temporality and the Crane Model for Quantum Gravity

Crane uses a partition function defined for a triangulated four-manifold. Let us denote the partition function by $Z(M^4, A, B) =< A|B >_M$ where M^4 is a four-manifold and A and B are (colored - see the next sentence) three dimensional submanifolds in the boundary of M. The partition function is constructed by summing over all colorings of the edges of a dual complex to this triangulation from a finite set of colors that correspond to certain representations of the the quantum group $U_q(SU(2))$ where q is a root of unity. The sum is over products of $15J_q$ symbols (natural generalizations of the $6J$ symbols in angular momentum theory) evaluated with respect to the colorings. The specific form of the partition function (here written in the case where A and B are empty) is

$$Z(M^4) = N^{v-e}\Sigma_\lambda \Pi_\sigma dim_q(\lambda(\sigma))\Pi_\tau dim_q^{-1}(\lambda(\tau))\Pi_\zeta 15J_q(\lambda(\zeta)).$$

Here λ denotes the labelling function, assigning colors to the faces and tetrahedra of M^4 and $v-e$ is the difference of the number of vertices and the number of edges in M^4. Faces are denoted by σ, tetrahedra by τ and 4-simplices by ζ. We refer the reader to [6] for further details.

In computing $Z(M^4, A, B) =< A|B >_M$ one fixes the choice of coloration on the boundary parts A and B. The analog with quantum gravity is that a colored three manifold A can be regarded as a three manifold with a choice of (combinatorial) metric. The coloring is the combinatorial substitute for the metric. In the three manifold case this is quite specifically so, since the colors can be regarded as affixed to the edges of the simplices. The color on a given edge is interpreted as the generalized distance between the endpoints of the edge. Thus $< A|B >_M$ is a summation over "all possible metrics" on M^4 that can extend the given metrics on A and B. $< A|B >_M$ is an amplitude for the metric (coloring) on A to evolve in the spacetime M^4 to the metric (coloring) on B.

The partition function $Z(M^4, A, B) =< A|B >_M$ is a topological invariant of the four manifold M^4. In particular, if A and B are empty (a vacuum-vacuum amplitude), then the Crane-Yetter invariant, $Z(M^4)$, is a function of the signature and Euler characteristic of the four-manifold [6]. On the mathematical side of the picture this is already significant since it provides a new way to express the signature of a four-manifold in terms of local combinatorial data.

From the point of view of a theory of quantum gravity, $Z(M^4, A, B) =< A|B >_M$, as we have described it so far, is lacking in a notion of time and dynamical evolution on the four manifold M^4. One can think of A and B as manifolds at the initial and final times, but we have not yet described a notion of time within M^4 itself.

Crane proposes to introduce time into M^4 and into the partition function $< A|B >_M$ by labelling certain three dimensional submanifolds of M^4 with special grouplike elements from the quantum group $U_q(SU(2))$ and extending the partition function to include this labelling. Movement across such a labelled hypersurface is regarded as one tick of the clock. The special grouplike elements act on the representations in such a way that the partition function can be extended to include the extra labels. Then one has the project to understand the new partition function and its relationship with discrete dynamics for this model of quantum gravity.

Lets denote the special grouplike element in the Hopf algebra $G = U_q(SU(2))$ by the symbol J. Then, as discussed at the end of the previous section, one has that the square of the antipode $S : G \longrightarrow G$ is given by the formula $S^2(x) = J^{-1}xJ$. This is the tick of the clock. The DOC derivative in the quantum group is given by the formula $DX = [X, J] = J(S^2(X) - X)$. I propose to generalize the discrete ordered calculus on the quantum group to a discrete ordered calculus on the four manifold M^4 with its hyperthreespaces labelled with special grouplikes. This generalised calculus will be a useful tool in elucidating the dynamics of Crane's model. Much more work needs to be done in this domain.

8 Appendix on Iterants

The primitive idea behind an iterant is a periodic time series or "waveform"

$$\cdots abababababab \cdots .$$

The elements of the waveform can be any mathematically or empirically well-defined objects. We can regard the ordered pairs $[a, b]$ and $[b, a]$ as abbreviations for the waveform or as two points of view about the waveform (a first or b first). Call $[a, b]$ an *iterant*. One has the collection of transformations of the form $T[a, b] = [ka, k^{-1}b]$ leaving the product ab invariant. This tiny model contains the seeds of special relativity, and the iterants contain the seeds of general matrix algebra! Since this paper has been a combination of discussions of non-commutativity and time series, we include this appendix on iterants. A more complete discussion will appear elsewhere. For related discussion see [18, 19, 20, 21, 23, 31, 32, 41].

Define products and sums of iterants as follows

$$[a, b][c, d] = [ac, bd]$$

and

$$[a, b] + [c, d] = [a + c, b + d].$$

The operation of juxtapostion is multiplication while + denotes ordinary addition in a category appropriate to these entities. These operations are natural with respect to the structural juxtaposition of iterants:

$$...abababababab...$$

$$...cdcdcdcdcdcd...$$

Structures combine at the points where they correspond. Waveforms combine at the times where they correspond. Iterants conmbine in juxtaposition.

If $\bullet$ denotes any form of binary compositon for the ingredients (a,b,...) of iterants, then we can extend $\bullet$ to the iterants themselves by the definition $[a, b] \bullet [c, d] = [a \bullet c, b \bullet d]$. In this section we shall first apply this idea to Lorentz transformations, and then generalize it to other contexts.

So, to work: We have

$$[t - x, t + x] = [t, t] + [-x, x] = t[1, 1] + x[-1, 1].$$

Since $[1,1][a,b] = [1a,1b] = [a,b]$ and $[0,0][a,b] = [0,0]$, we shall write

$$1 = [1,1]$$

and

$$0 = [0,0].$$

Let

$$\sigma = [-1,1].$$

σ is a significant iterant that we shall refer to as a *polarity.* Note that

$$\sigma\sigma = 1.$$

Note also that

$$[t-x, t+x] = t + x\sigma.$$

Thus the points of spacetime form an algebra analogous to the complex numbers whose elements are of the form $t + x\sigma$ with $\sigma\sigma = 1$ so that

$$(t + x\sigma)(t' + x'\sigma) = tt' + xx' + (tx' + xt')\sigma.$$

In the case of the Lorentz transformation it is easy to see the elements of the form $[k, k^{-1}]$ translate into elements of the form

$$T(v) = [(1+v)/\sqrt{(1-v^2)}, (1-v)/\sqrt{(1-v^2)}] = [k, k^{-1}].$$

Further analysis shows that v is the relative velocity of the two reference frames in the physical context. Multiplication now yields the usual form of the Lorentz transform

$$T_k(t + x\sigma) = T(v)(t + x\sigma)$$
$$= (1/\sqrt{(1-v^2)} - v\sigma/\sqrt{(1-v^2)})(t + x\sigma)$$
$$= (t - xv)/\sqrt{(1-v^2)} + (x - vt)\sigma/\sqrt{(1-v^2)}$$
$$= t' + x'\sigma.$$

The algebra that underlies this iterant presentation of special relativity is a relative of the complex numbers with a special element σ of square one rather than minus one ($i^2 = -1$ in the complex numbers).

The appearance of a square root of minus one unfolds naturally from iterant considerations. Define the "shift" operator D on iterants by the equation

$$D[a, b] = [b, a].$$

Sometimes it is convenient to think of D as a delay opeator, since it shifts the waveform *...ababab...* by one internal time step. Now define

$$i[a, b] = \sigma D[a, b] = [-1, 1][b, a] = [-b, a].$$

We see at once that

$$ii[a, b] = [-a, -b] = [-1, -1][a, b] = (-1)[a, b].$$

Thus

$$ii = -1.$$

This is the traditional construction of the square root of minus one in terms of operations on ordered pairs. Here we have described $i[a, b]$ in a *new* way as the superposition of the waveforms $\sigma = [-1, 1]$ and $D[a, b]$ where $D[a, b]$ is the delay shift of the waveform $[a, b]$.

8.1 MATRIX ALGEBRA VIA ITERANTS

Matrix algebra has some strange wisdom built into its very bones. Consider a two dimensional periodic pattern or "waveform."

....

...abababababababab...

...cdcdcdcdcdcdcdcd...

...abababababababab...

...cdcdcdcdcdcdcdcd...

...abababababababab...

....

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} b & a \\ d & c \end{pmatrix}, \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

Above are some of the matrices apparent in this array. Compare the matrix with the "two dimensional waveform" shown above. A given matrix freezes out a way to view the infinite waveform. In order to keep track of this patterning, lets write

$$[a, d] + [b, c]\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

where

$$[x, y] = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

and

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The four matrices that can be framed in the two-dimensional wave form are all obtained from the two iterants $[a, d]$ and $[b, c]$ via the delay shift operation $D[x, y] = [y, x]$ which we shall denote by an overbar as shown below

$$D[x, y] = \overline{[x, y]} = [y, x].$$

Letting $A = [a, d]$ and $B = [b, c]$, we see that the four matrices seen in the grid are

$$A + B\eta, B + A\eta, \overline{B} + \overline{A}\eta, \overline{A} + \overline{B}\eta.$$

The operator η has the effect of rotating an iterant by ninety degrees in the formal plane. Ordinary matrix multiplication can be written in a concise form using the following rules:

$$\eta\eta = 1$$

$$\eta Q = \overline{Q}\eta$$

where Q is any two element iterant.
For example, let $\epsilon = [-1, 1]$ so that $\overline{\epsilon} = -\epsilon$ and $\epsilon\epsilon = [1, 1] = 1$. Let

$$i = \epsilon\eta.$$

Then

$$ii = \epsilon\eta\epsilon\eta = \epsilon\overline{\epsilon}\eta\eta = \epsilon(-\epsilon) = -\epsilon\epsilon = -1.$$

We have reconstructed the square root of minus one in the form of the matrix

$$i = \epsilon\eta = [-1, 1]\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

More generally, we see that

$$(A + B\eta)(C + D\eta) = (AC + B\overline{D}) + (AD + B\overline{C})\eta$$

writing the 2×2 matrix algebra as a system of hypercomplex numbers. Note that

$$(A + B\eta)(\overline{A} - B\eta) = A\overline{A} - B\overline{B}$$

The formula on the right corresponds to the determinant of the matrix. Thus we define the *conjugate* of $A + B\eta$ by the formula

$$\overline{A + B\eta} = \overline{A} - B\eta.$$

These patterns generalize to higher dimensional matrix algebra.

It is worth pointing out the first precursor to the quaternions: This precursor is the system

$$\{\pm 1, \pm\epsilon, \pm\eta, \pm i\}.$$

Here $\epsilon\epsilon = 1 = \eta\eta$ while $i = \epsilon\eta$ so that $ii = -1$. The basic operations in this algebra are those of epsilon and eta. Eta is the delay shift operator that reverses the components of the iterant. Epsilon negates one of the components, and leaves the order unchanged. The quaternions arise directly from these two operations once we construct an extra square root of minus one that commutes with them. Call this extra root of minus one $\sqrt{-1}$. Then the quaternions are generated by

$$\{i = \epsilon\eta, j = \sqrt{-1}\epsilon, k = \sqrt{-1}\eta\}$$

with

$$i^2 = j^2 = k^2 = ijk = -1.$$

The "right" way to generate the quaternions is to start at the bottom iterant level with boolean values of 0 and 1 and the operation EXOR (exclusive or). Build iterants on this, and matrix algebra from these iterants. This gives the square root of negation. Now take pairs of values from this new algebra and build 2×2 matrices again. The coefficients include square roots of negation that commute with constructions at the next level and so quaternions appear in the third level of this hierarchy.

8.2 Matrix Algebra in General

Construction of matrix algebra in general proceeds as follows. Let M be an $n \times n$ matrix over a ring R. Let $M = (m_{ij})$ denote the matrix entries. Let π be an element of the symmetric group S_n so that $\pi_1, \pi_2, \cdots, \pi_n$ is a permuation of $1, 2, \cdots, n$. Let $v = (v_1, v_2, \cdots, v_n)$ denote a vector with these components. Let $\Delta(v)$ denote the diagonal matrix whose $i - th$ diagonal entry is v_i. Let $v^\pi = (v_{\pi_1}, \cdots, v_{\pi_n})$. Let $\Delta^\pi(v) = \Delta(v^\pi)$. Let Δ denote any diagonal matrix and Δ^π denote the corresponding permuted diagonal matrix as just described. Let $[\pi]$ denote the permutation matrix obtained by taking the $i - th$ row of $[\pi]$ to be the $\pi_i - th$ row of the identity matrix. Note that $[\pi]\Delta = \Delta^\pi[\pi]$. For each element π of S_n define the vector $v(M, \pi) = (m_{1\pi_1}, \cdots, m_{n\pi_n})$ and the diagonal matrix $\Delta[M]_\pi = \Delta(v(M, \pi))$.

Theorem. $M = (1/(n-1)!)\Sigma_{\pi \in S_n} \Delta[M]_\pi[\pi]$.

The proof of this theorem is omitted here. Note that the theorem expresses any square matrix as a sum of products of diagonal matrices and permutation matrices. Diagonal matrices add and multiply by adding and multiplying their corresponding entries. They are acted upon by permutations as described above. This means that any matrix algebra can be embedded in an algebra that has the structure of a group ring of the permutation group with coefficients Δ in an algebra (here the diagonal matrices) that are acted upon by the permutation group, and following the rule $[\pi]\Delta = \Delta^\pi[\pi]$. This is a full generalization of the case $n = 2$ described in the last section.

It is amusing to note that this theorem tells us that up to the factor of $1/(n-1)!$ a unitary matrix that has unit complex numbers as its entries is a sum of simpler unitary transformations factored into diagonal and permutation matrices. In quantum computing parlance, such a unitary matrix is a sum of products of phase gates and products of swap gates (forming the permutations).

A reason for discussing these formulations of matrix algebra in the present context is that one sees that matrix algebra is generated by the simple operations of juxtaposed addition and multiplication, and by the use of permutations as operators. These are unavoidable discrete elements, and so the operations of matrix algebra can be motivated on the basis of discrete physical ideas and non-commutativity. The richness of continuum formulations, infinite matrix algebra, and symmetry grows naturally out of finite matrix algebra and hence out of the discrete.

9 Philosophical Appendix

The purpose of this appendix is to point to a way of thinking about the relationship of mathematics, physics, persons, and observations that underlies the approach taken in this paper. We began constructions motivating non-commutativity by considering sequences of actions $\cdots DCBA$ written from right to left so that they could be applied to an actant X in the order $\cdots DCBAX = \cdots(D(C(B(AX)))\cdots$. The sequence of events $A, B, C, D, \cdots$ was conceptualized as a temporal order, with the events themselves happening at levels or frames of successive "space". *There is no ambient coordinate space, nor is there any continuum of time.* All that is given is the possibility of structure at any given moment, and the possibility of distinguishing structures from one moment to the next. In this light the formula $DX = [X, J] = XJ - JX = J(X' - X)$ connotes a symbolic representation of the measurment of a difference across one time interval, nothing more. In other words DX represents a difference taken across a background difference (the time step). Once the pandora's box of measuring such differences has been opened, we are subject to the multiplicities of forms of difference $\nabla_K X = [X, K]$, their non-commutativity among themselves, the notion of a flat background that has the formal appearance of quantum mechanics, the emergence of abstract curvature and formal gauge fields. All this occurs in these calculi of differences *prior* to the emergence of differential geometry or topology or even the notion of linear superposition of states (so important to quantum mechanics). Note that in this algebraic patterning each algebra element X is an actant (can be acted upon) and an actor (via the operator ∇_X). In Lie algebras, this is the relationship between the algebra and its adjoint representation that makes each element of the algebra into a representor for that algebra by exactly the formula $adj_A(X) = [A, X] = -\nabla_A(X)$ that we have identified as a formal difference or derivative, a generator for a calculus of differences.

The precursor and conceptual background of our particular formalism is therefore the concept of discrimination, the idea of a distinction. A key work in relation to that concept is the book “Laws of Form” by G. Spencer-Brown [41] in which is set out a calculus of distinction of maximal simplicity and generality. In that calculus a mark (denoted here by a bracket $< \quad >$) represents a distinction and is seen to be a distinction between inside and outside. In this elemental mathematics there is no distinction except the one that we draw between the mathematician and the operator in the formal system as sign/symbol/interpretant. This gives full responsibility to the mathematician to draw the boundaries between the formal system as physical interaction and the formal system as symbolic entity and the formal system as Platonic conceptual form. In making a mathematics of distinction, the mathematician tells a story to himself/herself about the creation of a world. Spencer-Brown’s iconic mathematics can be extended to contact any mathematics, and when this happens that mathematics is transformed into a personal creation of the mathematician who uses it. In a similar (but to a mathematician) darker way, the physicist is intimately bound to the physical reality that he studies.

We could have begun this paper with the the Spencer-Brown mark as bracket: $< \quad >$. This empty bracket is seen to make a distinction between inside and outside. In order for that to occur the bracket has to become a process in the perception of someone. It has to leave whatever objective existence or potentiality it has alone (all one) and become the locus or nexus of an idea in a perceiving mind. As such it is stabilized by that perception/creation and becomes really a solution to $\{< \quad >\} =< \quad >$ where the curly bracket (the form of perception) is in the first place identical to the mark $< \quad >$, and then distinguished from it by the act of distinguishing world and perceiver. It is within this cleft of the infinite recursive and the finite

$$< \quad >= \{< \quad >\} = \{\{< \quad >\}\} = \{\{\{< \quad >\}\}\} = \cdots = \{\{\{\{\{\{\cdots\}\}\}\}\}\}$$

that the objectivity of mathematics/physics (they are not different in the cleft) arises. All the rest of mathematics or calculus of brackets needs come forth for the observer in the same way. Through that interaction there is the possibility of a deep dialogue of many levels, a dialogue where it is seen that mathematics and physics develop in parallel, each describing the same boundary from opposite sides. That boundary is the imaginary boundary between the inner and outer worlds of an individual.

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