

Oriented Quantum Algebras, Categories and Invariants of Knots and Links

Louis Kauffman* and David E. Radford†

Department of Mathematics, Statistics
and Computer Science (m/c 249)
851 South Morgan Street
University of Illinois at Chicago
Chicago, Illinois 60607-7045

Abstract

This paper defines the concept of an oriented quantum algebra and develops its application to the construction of quantum link invariants. We show, in fact, that all known quantum link invariants can be put into this framework.

1 Introduction

An oriented quantum algebra (A, ρ, D, U) is an abstract model for an oriented quantum link invariant. This model is based on a solution to the Yang-Baxter equation and some extra structure that serves to make an invariant possible to construct. The definition of an oriented quantum algebra is as follows: We are given an algebra A over a base ring k , an invertible solution ρ in $A \otimes A$ of the Yang-Baxter equation (in the algebra formulation of this equation – see the Remark below), and automorphisms $U : A \rightarrow A$ and $D : A \rightarrow A$ of the algebra. It is assumed that D and U commute and that

*Research supported in part by NSF Grant DMS 920-5227

†Research supported in part by NSF Grant DMS 980 2178

$$(U \otimes U)\rho = \rho$$

and

$$(D \otimes D)\rho = \rho$$

and that

$$[(1_A \otimes U)\rho][(D \otimes 1_{A^{op}})\rho^{-1}] = 1_{A \otimes A^{op}}$$

and

$$[(D \otimes 1_{A^{op}})\rho^{-1}][(1_A \otimes U)\rho] = 1_{A \otimes A^{op}}.$$

In other words, $[(1_A \otimes U)\rho]$ and $[(D \otimes 1_{A^{op}})\rho^{-1}]$ are inverses in the algebra $A \otimes A^{op}$.

Here A^{op} denotes the opposite algebra. The first equation is formulated in the tensor product of A with itself, while the second equation is formulated in the tensor product of A with its opposite algebra.

When $U = D = T$, then A is said to be *balanced*. In this case

$$(T \otimes T)\rho = \rho,$$

$$[(1_A \otimes T)\rho][(T \otimes 1_{A^{op}})\rho^{-1}] = 1_{A \otimes A^{op}}$$

and

$$[(T \otimes 1_{A^{op}})\rho^{-1}][(1_A \otimes T)\rho] = 1_{A \otimes A^{op}}.$$

In the case where D is the identity mapping, we call the oriented quantum algebra *standard*. As we shall see in section 6, the invariants defined by Reshetikhin and Turaev (associated with a quasi-triangular Hopf algebra) arise from standard oriented quantum algebras. The quantum link invariants defined by Turaev in [24] arise from balanced oriented quantum algebras, as we show at the end of section 4. It is an interesting structural feature of algebras that we have elsewhere [5] called *quantum algebras* (generalizations of quasi-triangular Hopf algebras) that they give rise to standard oriented quantum algebras. Note that the term quantum algebra as used here is more specific than the QA quantum algebra designation that is used in archiving papers related to Hopf algebras and quantum groups.

We shall see that appropriate matrix representations of oriented quantum algebras or the existence of certain traces on these algebras allow the construction of oriented invariants of knots and links. These invariants include all the known quantum link invariants at the time of this writing.

Remark. Note that we have the Yang-Baxter elements ρ and ρ^{-1} in $A \otimes A$. We assume that ρ and ρ^{-1} satisfy the algebraic Yang-Baxter equation. This equation (for ρ) states

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}$$

where ρ_{ij} denotes the placement of the tensor factors of ρ in the i -th and j -th tensor factors of the triple tensor product $A \otimes A \otimes A$.

We write $\rho = \sum e \otimes e'$ and $\rho^{-1} = \sum E \otimes E'$ to indicate that these elements are sums of tensor products of elements of A . The expression $e \otimes e'$ is thus a generic element of the tensor product. However, we often abbreviate and write $\rho = e \otimes e'$ and $\rho^{-1} = E \otimes E'$ where the summation is implicit. We refer to e and e' as the *signifiers* of ρ , and E and E' as the *signifiers* of ρ^{-1} . For example, $\rho_{13} = e \otimes 1 \otimes e'$ in $A \otimes A \otimes A$.

Braiding operators, as they appear in knot theory, differ from the algebraic Yang-Baxter elements by a permutation of tensor factors. This point is crucial to the relationship of oriented quantum algebras and invariants of knots and links, and will be dealt with in the body of the paper (specifically in Section 3).

Remark. In writing a function $F(x)$ we often abbreviate this expression to Fx when this entails no ambiguity. Thus $F(G(H(J(x)))) = FGHJx$ in this notation.

Remark. If A is an $n \times n$ matrix and v is a row vector of dimension n , then vA is a row vector of dimension n . If matrices are regarded as functions on the left ($(v)A = vA$), then the order of composition proceeds from right to left as in $((v)A)B = vAB$. In some of our applications it is useful to think of the matrices as acting on the left in this fashion so that AB denotes both matrix product and composition of functions. The role of this remark will become clear in the sections on matrix models, state models and quasi-triangular Hopf algebras.

The paper is organized as follows. Section 2 describes the tangle category $Tang$ and the flat tangle category $Flat$. Section 3 describes the category $Cat(A)$ of an oriented quantum algebra A and the functor $F : Tang \rightarrow Cat(A)$. In fact, this section motivates the definition of an oriented quantum algebra by showing just what algebraic conditions are necessary for the functor F to be an invariant of regular isotopy of tangles. This analysis is how we discovered the definition of these algebras. Section 4 discusses matrix models for the invariants of knots and links associated with oriented quantum algebras. Included in Section 4 is a discussion of bead sliding for reformulating the evaluations of the matrix models and a specific example of a matrix oriented quantum algebra that gives rise to specializations of the $HOMFLYPT$ polynomial. We end this section with a description of the conditions on the R matrix for a general balanced oriented quantum algebra with diagonalized form of automorphism T . This gives the same conditions on the R matrix as in [24] and in [7]. Section 5 is a discussion of combinatorial state sum invariants of knots and links. The contents of this section can be used to verify the relationship of the algebra of the previous section with the $HOMFLYPT$ polynomial. Section 6 shows that representations of quasitriangular ribbon Hopf algebras have the structure of oriented quantum algebras. We use this section to show explicitly how the link invariants of Reshtikhin and Turaev fit into our framework.

2 The Tangle Category

We recall the *oriented tangle category* denoted here by $Tang$. This is a category that formalizes the structure of knot and link diagrams in a manner suitable for the construction of quantum link invariants. (The reader should note that $Tang$ refers, as explained below, to tangles with all possible multiplicities of input and output.)

We shall refer to embeddings of disjoint unions of circles and arcs into three dimensional space as *string*. In this paper, all strings will be *oriented*. This means that each string is equipped with a preferred direction, usually indicated in a diagram by an arrowhead drawn on the string. When we speak of matching strings in tangles to compose them (see below and Figure 0), we assume that the strings have compatible orientations so that the composition

is also oriented.

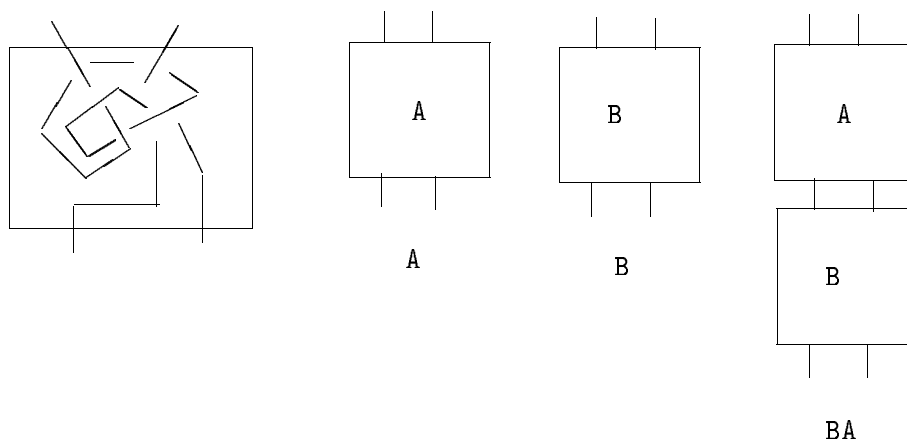


Figure 0 - Tangles and Composition of Tangles

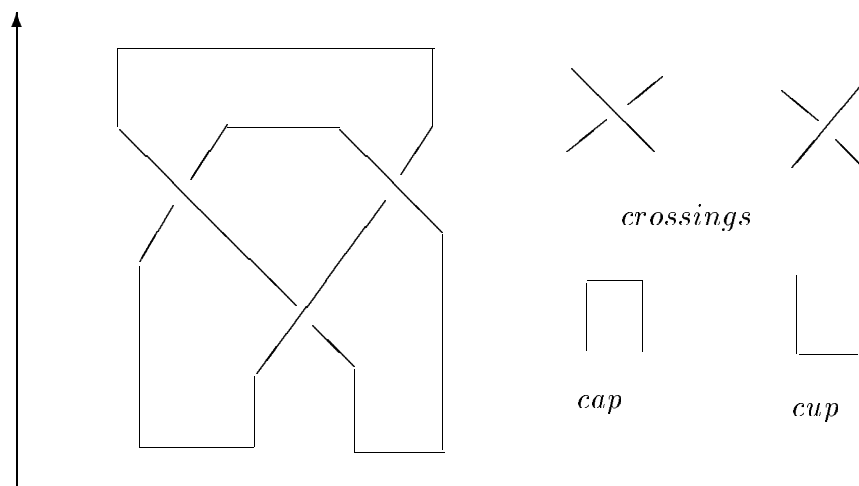


Figure 1 - Knot as Tangle with Cup and Cap Decomposition

Intuitively, a *tangle* is a box in three dimensional space with knotted and linked string embedded within it and a certain number of strands of that string emanating from the surface of the box. There are no open ends of string inside the box. We usually think of some subset of the strands as *inputs* to the tangle and the remaining strands as the *outputs* from the tangle.

Usually the inputs are arranged to be drawn vertically and so that they enter the tangle from below, while the outputs leave the tangle from above. The tangle itself (within the box) is arranged as nicely as possible with respect to a vertical direction. This means that a definite vertical direction is chosen, and that the tangle intersects planes perpendicular to this direction transversely except for a finite collection of critical points. These basic critical points are local maxima and local minima for the space curves inside the tangle. Two tangles configured with respect to the same box are *ambient isotopic* if there is an isotopy in three space carrying one to the other that fixes the input and output strands of each tangle. We can *compose* two tangles A and B where the number of output strand of A is equal to the number of input strands of B . Composition is accomplished by joining each output strand of A to a corresponding input strand of B . Of course this can be done (in space) in more than one way. When we use diagrammatic tangles, as we will in this paper, then the composition operation is naturally well-defined. To have composition well-defined for spatial tangles we can choose an ordering of the input and of the output strands of each tangle and then match them according to this ordering.

Note that the box associated with a tangle or tangle diagram is only a delineation of the location of the tangle. The tangle itself consists in the woven pattern of strings.

We have just given the basic three dimensional description of a tangle. For the purposes of combinatorial topology and algebra, it is useful to give a modified tangle definition that uses diagrams instead of embeddings of the tangle into three dimensions. (A diagram does specify an embedding, but the diagram is not itself an embedding.) A *tangle diagram* is a box in the plane, arranged parallel to a chosen vertical direction with a left-right ordered sequence of input strands entering the bottom of the box, and a left-right ordered sequence of output strands emanating from the top of the box. Inside the box is a diagram of the tangle represented with crossings (a broken arc indicating the undercrossing line) in the usual way for knots and links. We assume, as above, that the tangle is represented so that it is transverse to lines perpendicular to the vertical except for a finite number of points in the vertical direction along the tangle. We shall say that the tangle is *well arranged* or *Morse* with respect to the vertical direction when these

transversality conditions are met. At the critical points we will see a local maximum, a local minimum or a crossing in the diagram. Tangle composition is well-defined (for matching input/output counts) since the input and output strands have an ordering (from left to right for the reader facing the plane on which the tangle diagram is drawn). Note that the cardinality of the set of input strands or output strands can be equal to zero. If they are both zero, then the tangle is simply a knot or link diagram arranged well with respect to the vertical direction.

The *Reidemeister moves* illustrated in Figure 2 are a set of moves on diagrams that combinatorially generate isotopy for knots, links and and tangles [20]. If two tangles are equivalent in three dimensional space, then corresponding diagrams of these tangles can be obtained one from another, by a sequence of Reidemeister moves. Each move is confined to the tangle box and keeps the input and output strands of the tangle diagram fixed. In illustrating the Reidemeister moves in Figure 2 we have shown samples of each type of move. The move 0 is a graphical equivalence in the plane that does not change any diagrammatic relations. The move 1 adds or removes a twist in the diagram. We have shown one of the two basic examples; the other is obtained by switching the crossing in this illustration. Similar remarks apply to obtain other cases of the moves 2 and 3.

Two (tangle) diagrams are said to be *regularly isotopic* if one can be obtained from the other by a sequence of Reidemeister moves of type 0,2,3 (move number 1 is not used in regular isotopy). From now on, all tangles will be tangle diagrams, and we shall say that two tangles are *equivalent* when they are regularly isotopic. If we did not insist on arranging our tangle diagrams as Morse diagrams with respect to vertical direction, the Reidemeister moves on diagrams would suffice to describe equivalence of tangles. In order to describe how to move Morse diagrams to one another that are regularly isotopic, we must add extra moves and rewrite move 0 with respect to the vertical. This is illustrated in Figure 3 and Figure 5 and discussed in more detail below.

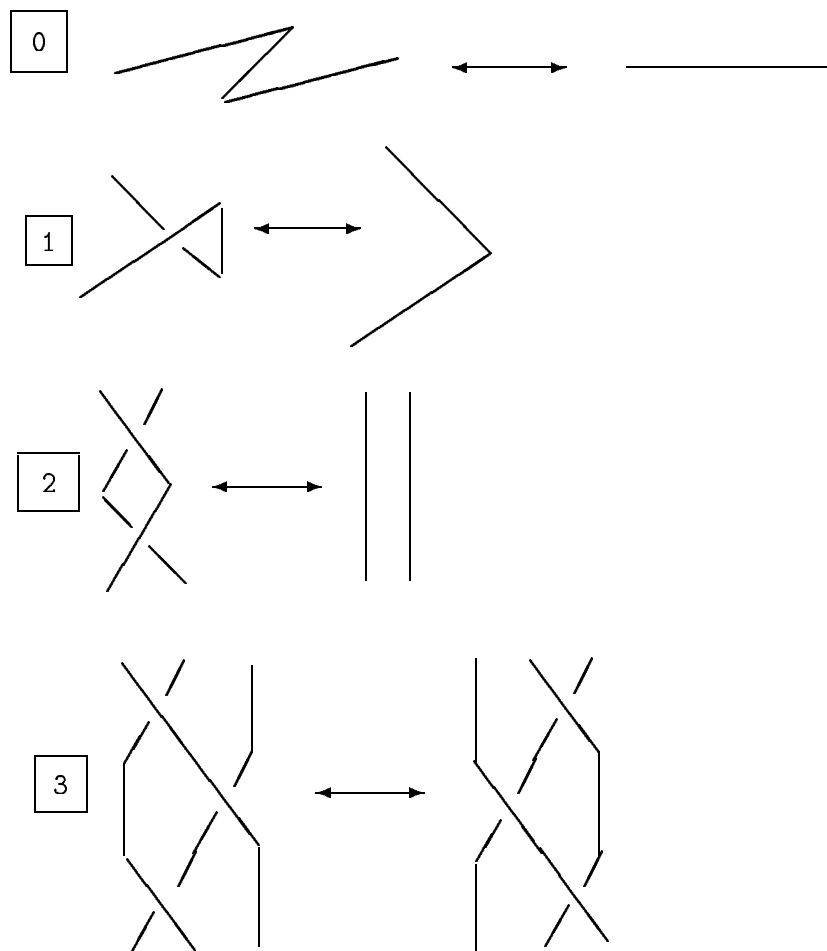


Figure 2 - Reidemeister Moves

If A and B are given tangles, we denote the composition of A and B by AB where the diagram of A is placed below the diagram of B and the output strands of A are connected to the input strands of B . If the cardinalities of the sets of input and output strands are zero, then we simply place one tangle below the other to form the product. Note that the reference here to input and output strands identifies the inputs as the strands connecting to the tangle from below, and the outputs as the strands connecting to the tangle from above. This does not assume any particular orientation. If the tangles are oriented, then we require that the strands connected to one another in

the process of composition have compatible orientations.

Along with tangle composition, as defined in the previous paragraph, we also have an operation of the *product* or *juxtaposition* of tangles. To juxtapose two tangles A and B , simply place their diagrams side by side with A to the left of B and regard this new diagram as a new tangle whose inputs are the inputs of A followed by the inputs of B , and whose outputs are the outputs of A followed by the outputs of B . We denote the tangle product of A and B by $A \otimes B$.

It remains to describe the equivalence relation on tangles that makes them represent regular isotopy classes of embedded string. For this purpose it is useful to note that it follows from our description of the tangle diagrams (see Figure 1) that every tangle is obtained by composition and product from *elementary tangles* where an elementary tangle is one of the following list: a *cup* (a single minimum – zero inputs, two outputs), a *cap* (a single maximum – two inputs, zero outputs), a *crossing* (a single local crossing diagram – two inputs and two outputs). Figure 5 illustrates these elementary tangles and the moves on tangles involving certain compositions of them. These moves include the usual Reidemeister moves configured with respect to a vertical direction plus *switchback* moves involving the passage of a bit of string across a maximum or a minimum. We have illustrated these moves first with the unoriented and then with the different *oriented* elementary tangles. Since we consider here *regular isotopy* of tangles, we do not illustrate the first Reidemeister move which consists in adding or removing a curl (a curl is a crossing with two of its endpoints connected directly to each other) from the diagram. Each elementary tangle comes in more than one flavor due to different possibilities of orientation and choice of under or over crossing line. Two tangles are said to be *equivalent* if one can be obtained from the other by a finite sequence of elementary moves.

In illustrating the elementary moves for oriented tangles, we have not listed all the cases. In the case of the second Reidemeister move, we show only the move with reversed orientations on the lines. In the case of the move of type four, we show only one of the numerous cases. In most moves there are other cases not illustrated but obtainable from the given illustration by switching one or more crossings. We leave the full enumeration to the reader. Note that in this move four a vertical crossing is exchanged for a horizontal

crossing. This relationship allows us (below) to define the horizontal crossings in terms of the vertical crossings and the cups and caps. In the type three move we have illustrated the two main types - all arrows up and two arrows up, one down. In the discussion below we will show that the all arrows up or all arrows down move is sufficient to generate the other type three moves (in the presence of both moves of type two).

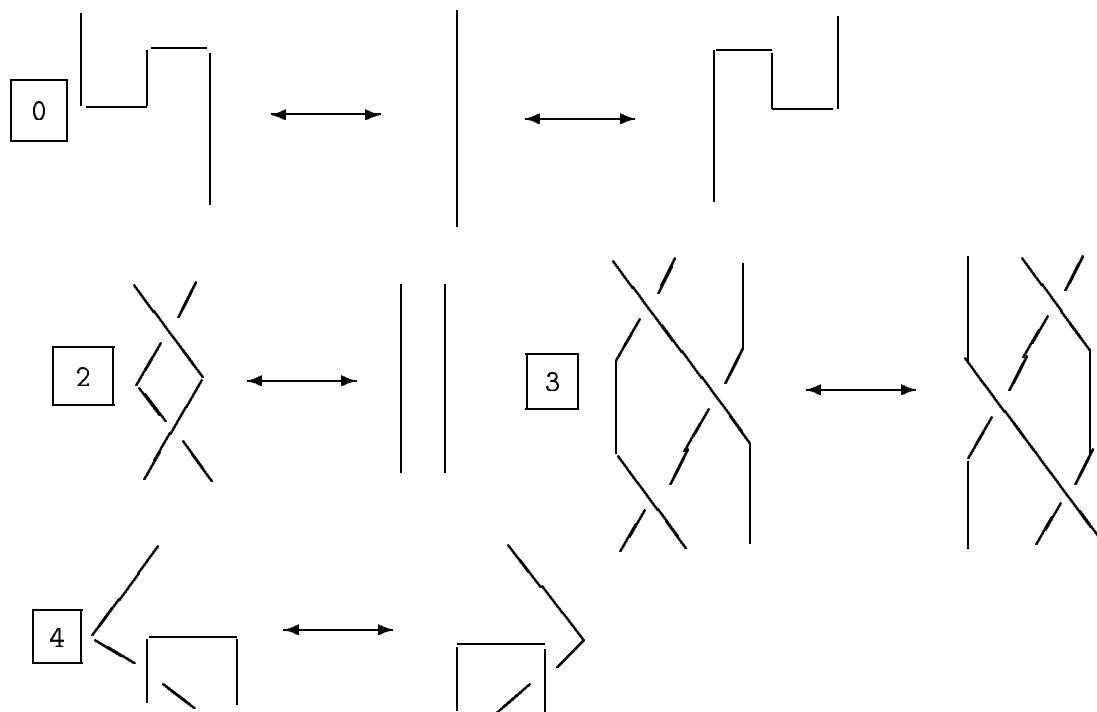


Figure 3 - Unoriented Moves on Tangles

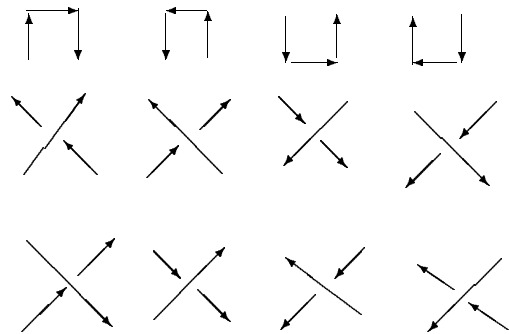


Figure 4 - Oriented Elementary Tangles

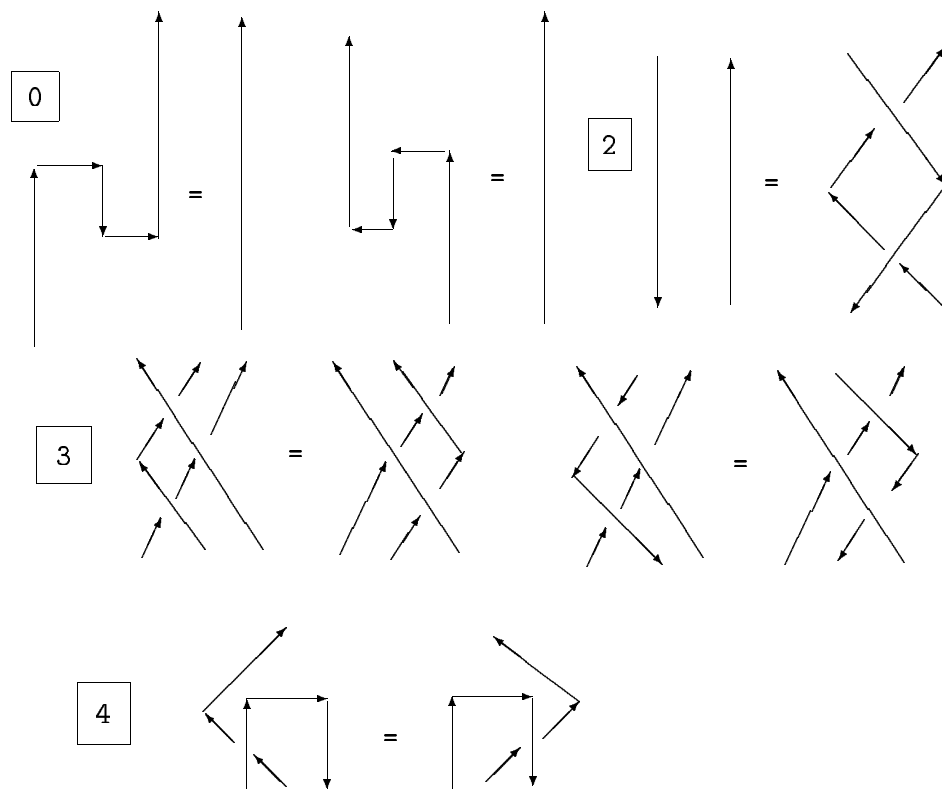


Figure 5 - Representative Moves on Oriented Tangles

In considering the oriented moves on tangles we see that there are two basic types of Reidemeister 2 move. The first (2_A) has both strings oriented

in parallel. The second (2_B) has the strings oriented in opposite directions. Similarly there are two basic types of third Reidemeister move that we denote by 3_C and 3_{NC} where the former has a cyclic triangle and the latter does not. See Figure 5. It turns out that the cyclic type three move is a consequence of the reverse oriented two move and the non-cyclic type three move. The proof of this statement is illustrated in Figure 6.

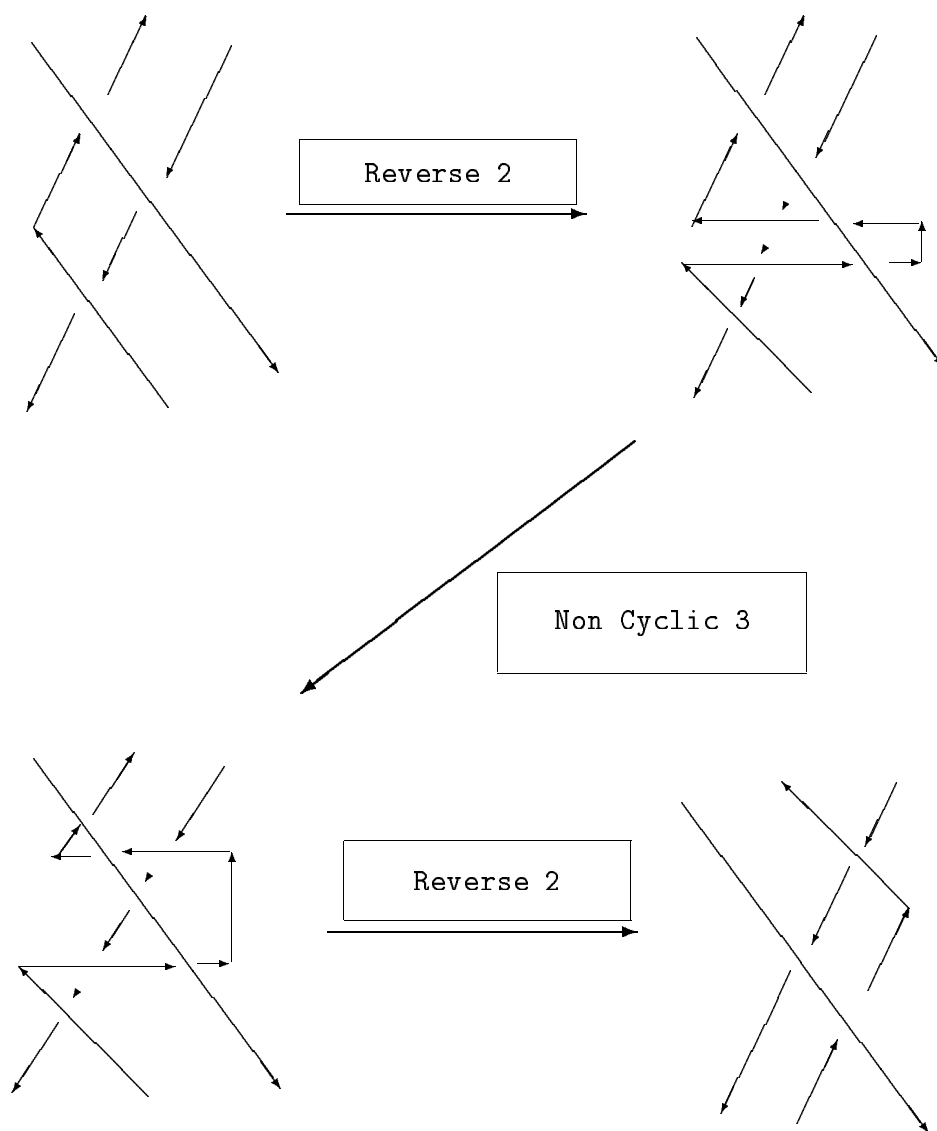


Figure 6 - Reverse Type Two Move Plus Non Cyclic Type three Implies Cyclic Type Three Move

This reduction of the number of type three moves makes working with the tangle category easier. It is still the case that one must verify invariance under both varieties of type two move for any functor on $Tang$. Note also

the switchback moves shown as type 4 moves in Figure 5. These moves give necessary relations between the crossings and the maxima and minima.

In order to make *Tang* a category, we define the *objects* of *Tang* to be ordered lists of signs (+1 or -1), including the empty list. Thus the objects of *Tang* are the lists

$$\langle \rangle, \langle \epsilon_1 \rangle, \langle \epsilon_1, \epsilon_2 \rangle, \dots, \langle \epsilon_1, \epsilon_2, \dots, \epsilon_n \rangle, \dots$$

We let ϵ denote the list $\langle \epsilon_1, \epsilon_2, \dots, \epsilon_n \rangle$ with \emptyset standing for the empty list. We let $||\epsilon||$ denote the number of entries in the list ϵ . A tangle is a morphism from ϵ to ϵ' where $||\epsilon||$ and $||\epsilon'||$ are the number of input and output strands of this tangle, respectively. Each sign corresponds to a single input or output strand of the tangle. If the sign is positive, then the strand is oriented upwards. If the sign is negative, then the strand is oriented downwards. Two morphisms are equal if they are equivalent as tangles. It is clear that this assignment makes *Tang* into a category, where composition of tangles corresponds to composition of morphisms in the category. The juxtaposition or tensor operation on tangles makes *Tang* into an associative tensor category with $\epsilon \otimes \epsilon'$ the list obtained by appending the second list of signs to the first list of signs. Note that the empty list is an identity element for the tensor product.

2.1 Flat Tangles

We now discuss the simpler category *Flat* of flat tangles. A *flat tangle* is exactly the same as an ordinary tangle diagram except that the crossings in the flat tangle have no over or under specification associated with them. Thus the generating morphisms of *Flat* are the oriented cups and caps and the various orientations of the flat crossing. All the generalized Reidemeister moves hold in *Flat* with no stipulations about under and over crossings. This means that there is much more freedom to perform moves in *Flat*. See Figure 7 for sample illustrations of some of the moves in *Flat*. Note that the crossing in *Flat* has all the formal properties of a permutation. In fact, the category *Flat* naturally contains copies of all the symmetric groups with the symmetric group on n letters occurring as the flat tangles on n upward oriented strands that do not use any cups or caps.

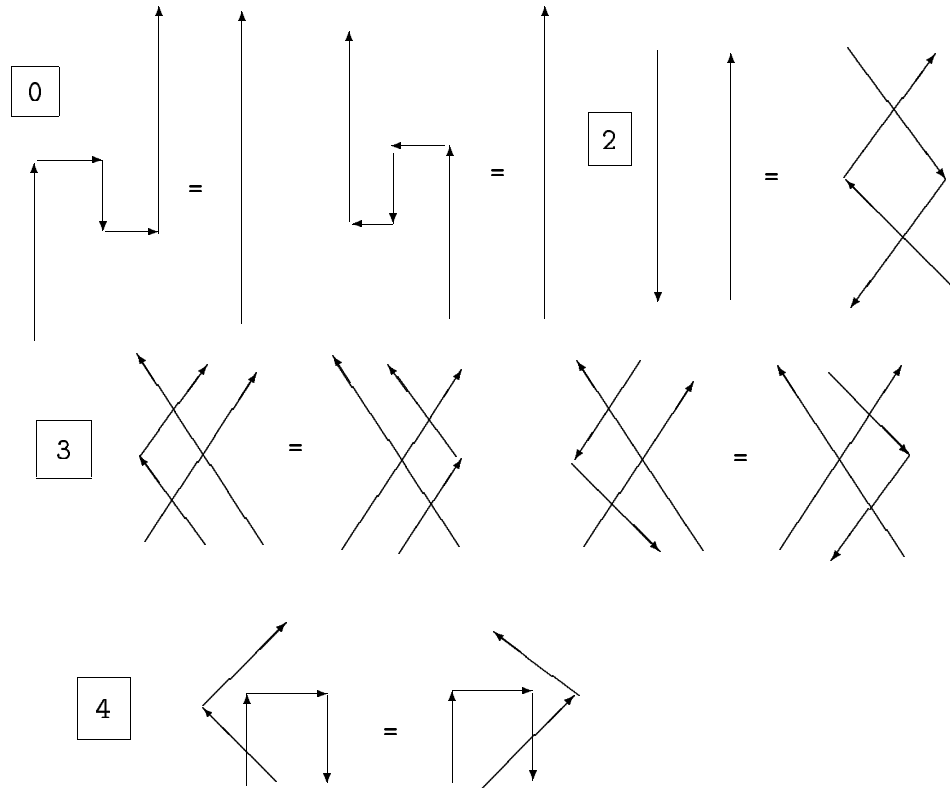


Figure 7 - Representative Oriented Moves in $Flat$

3 The Category of an Oriented Quantum Algebra

Let A be an oriented quantum algebra. In this section we build a category $Cat(A)$ associated to A by decorating morphisms of the category $Flat$ of flat tangles with “beads” from the quantum algebra. The category $Cat(A)$ has built in bead sliding rules that allow the reduction of individual strings to pure algebra. We will construct a functor, denoted $F : Tang \rightarrow Cat(A)$, from the tangle category to the category of the quantum algebra and show this functor is well-defined. In the course of this construction the reasons for our definition of oriented quantum algebra will become transparent. We will

discuss the structure of evaluations on $Cat(A)$ that give rise to invariants of links and tangles.

First note that given any algebra A there is a category $C(A)$, associated with this algebra with one object $[+]$ and a morphism a for every element a in the algebra (we use the same symbol for the element and its corresponding morphism). We simply declare that each element a of the algebra is a morphism $a : [+] \longrightarrow [+]$ and that composition of morphisms corresponds to the multiplication of elements in the algebra. Since A is an algebra, we can also add morphisms of $C(A)$ in correspondence to the addition of elements of A .

The category $Cat(A)$ will be constructed by thinking of the morphisms in $C(A)$ as labelled arrows and generalizing them to labelled flat oriented diagrams from the flat tangle category. In this way, we see at once that we must deal with down arrows as well as up arrows, and since down arrows in $Flat$ terminate in the object $[-]$, we must enlarge the list of objects to those generated by the empty object $[\]$ and the objects $[+]$ and $[-]$ from $Flat$.

Using the same object structure that we described for the tangle categories, we can regard a basic element for $C(A^{\otimes n})$ as the object

$$[n] = [+] \otimes [+] \otimes \dots \otimes [+]$$

(n factors). Morphisms in $C_n(A) = C(A^{\otimes n})$ are sums of morphisms that correspond to n -fold tensor products of elements of A . However, by the motivation above, we also need morphisms whose objects are negative signed lists. The simplest such object is $[-]$, and a morphism from $[-]$ to $[-]$ is a downward pointing arrow in $Flat$. We can decorate this arrow with an element a of the algebra A . As a morphism we denote it by $a^t : [-] \longrightarrow [-]$. Formally, the morphisms $a : [+] \longrightarrow [+]$ and a^t determine each other. Informally, a^t is just what you see if you reverse the sense of external vertical direction for the morphisms of $Cat(A)$.

We can bundle all of these categories together into one tensor category $C_\infty(A)$ with morphisms in 1 – 1 correspondence with the elements in arbitrary-fold tensor products of A and objects in 1 – 1 correspondence with finite sequences of signs.

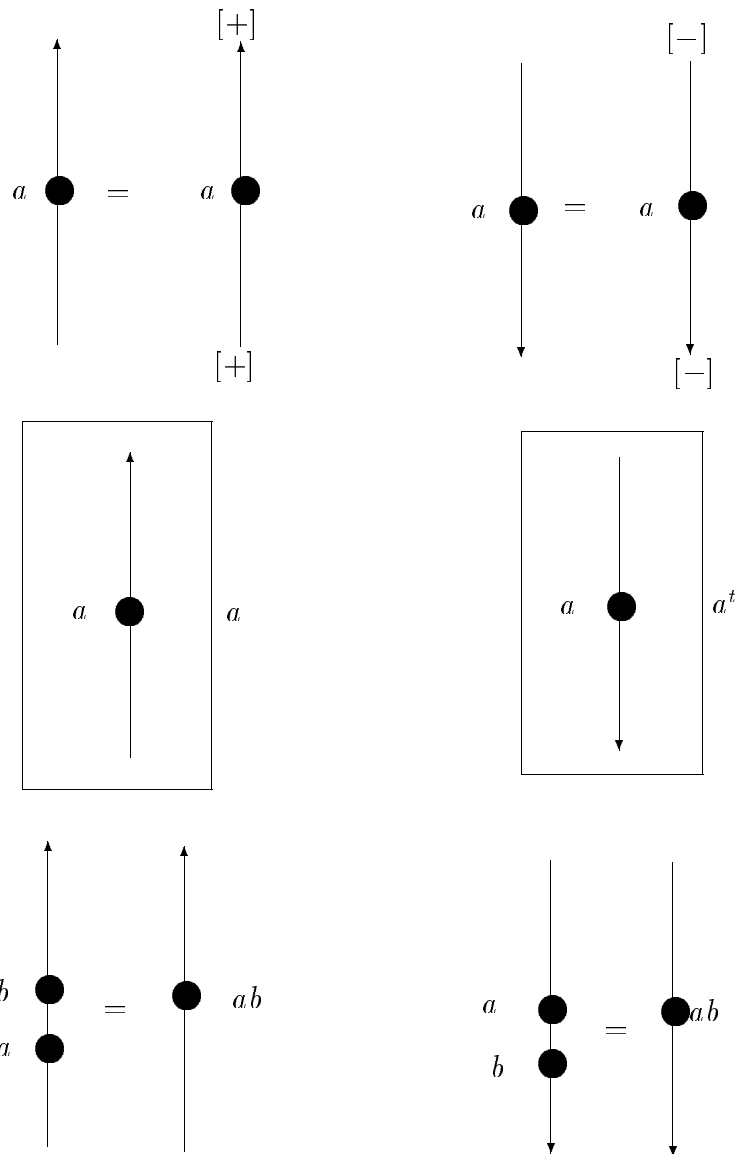


Figure 8 - Decorated Single Arrows and the Transpose of a Morphism

For a quantum algebra A , we now extend this tensor category $C_\infty(A)$ to encompass both the automorphisms U and D of the oriented quantum algebra and the structure of the flat tangle category $Flat$. We accomplish this by interpreting morphisms in $C_1(A)$ as single vertical lines (that is, as flat

(1, 1) tangles) decorated by a label that corresponds to the algebra element that gives this morphism. In our notation, this decoration is a round node (a “bead”) on the line that is usually accompanied by a text label in the plane next to the bead and disjoint from the line. Since the algebra element may be a sum of elements, we may use a sum of such labelled vertical segments. An unlabelled single segment denotes the identity morphism in $C_1(A)$. Similarly, a morphism in $C_n(A)$ is denoted by a labelling of a set of n parallel vertical segments. If the segments (from left to right) are labelled a_1, a_2, \dots, a_n , then this labelled bundle of segments is the morphism in $C_n(A)$ that corresponds to the tensor product

$$a_1 \otimes a_2 \otimes \dots \otimes a_n.$$

So far, we have indicated how to represent the category $C_\infty(A)$ via labelling of the identity n -tangles in *Flat*. We now extend this to a category we call *the category of the oriented quantum algebra A* and denote by $Cat(A)$. The objects in $Cat(A)$ are identical to the objects in *Flat*. The morphisms in $Cat(A)$ are flat tangles that have been decorated on some of their vertical segments by elements of the algebra A . (Thus all the elements of $C_\infty(A)$ are represented in $Cat(A)$ by decorations of the identity tangles.) The relations on generating morphisms for *Flat* still hold in $Cat(A)$. In addition we have extra relations for the cups and caps as illustrated in Figure 10. The basic form of this relation is:

$$(a \otimes 1) \circ Cup = (1 \otimes \tau(a)) \circ Cup,$$

$$Cap \circ (1 \otimes a) = Cap \circ (\tau'(a) \otimes 1)$$

where $\tau, \tau' : A \rightarrow A$ are automorphisms of the algebra A .

Here we have not specified the orientations on the *Cup* or the *Cap*. Up to the choice of the automorphism τ or τ' , these relations are independent of the choice of orientation, and apply to any element a of the algebra A . The topological relations for *Cup* and *Cap* that hold in *Flat* are extended to $Cat(A)$. Once we orient the cups and caps we can specify the choice of automorphism τ or τ' from two possibilities that we call U (“up”) and D (“down”). Caps with clockwise orientation and cups with counterclockwise orientation receive U . Caps with counterclockwise orientation and cups with clockwise orientation receive D . See Figures 9 and 10.

The upshot of these relations of the *Cup* and *Cap* with the automorphisms U and D of A is that U and D have diagrammatic interpretations as shown in Figure 10. One way to think about this diagrammatic interpretation is that $U(a)$ is obtained from the upward pointing diagram for the morphism a by turning this diagram upside down and running a vertical line upward that turns through a cap (on the left) and connects to the top part of the inverted a . Then a *Cup* is connected to the bottom part and continues upward to form a globally upward pointing morphism that represents $U(a)$. For $D(a)$ the same diagram is used but all the arrows are reversed.

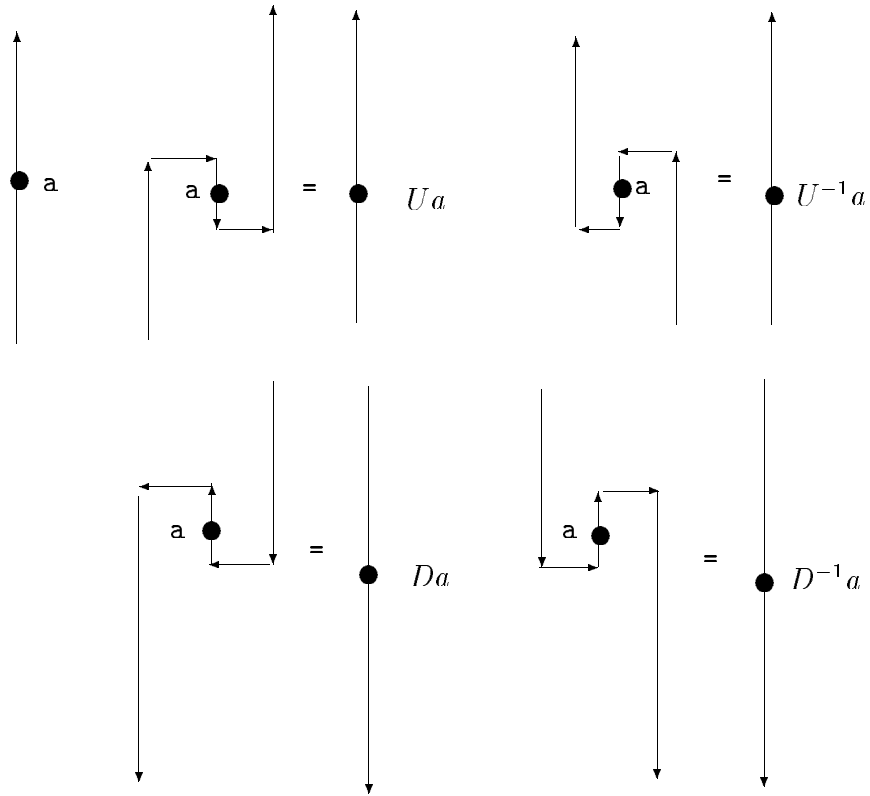


Figure 9 - Diagrams for the Automorphisms U and D

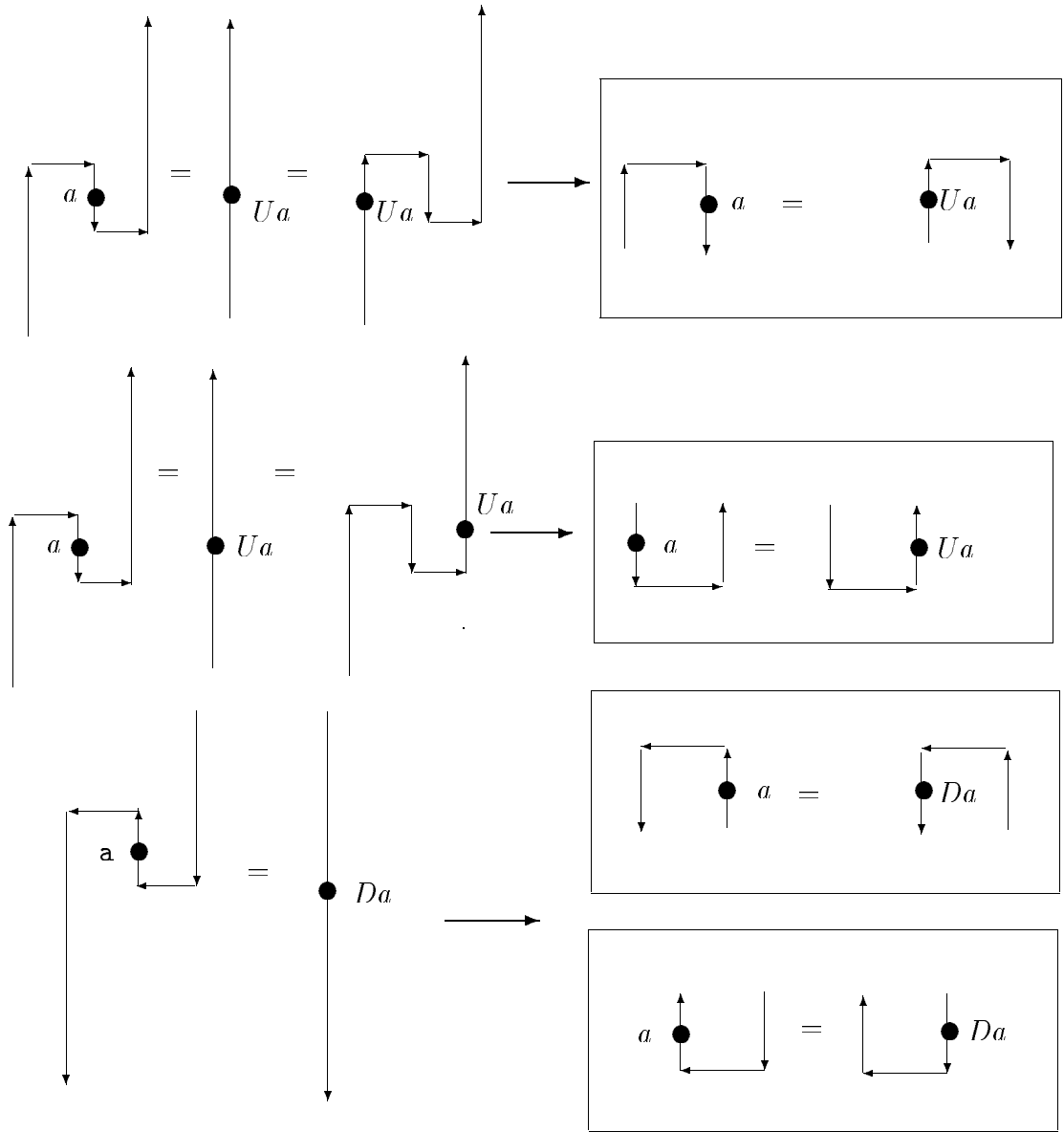


Figure 10 - Bead Sliding and the Automorphisms U and D

The reader may wonder if it is necessary to have two distinct automorphisms U and D . In fact, we shall see that there are cases where it is most convenient to have $U = D$ and other cases where it is most natural to take D to be the identity mapping while U is non-trivial. The latter occurs in representing a

quasi-triangular ribbon Hopf algebra, as we will see later in Section 6.

It is interesting to note that the morphisms UD and DU are both of the form $UD(a) = DU(a) = GaG^{-1}$ where G is the flat curl morphism illustrated in Figure 11. In the case of a ribbon Hopf algebra it is convenient to interpret G as a certain grouplike element in the algebra itself.

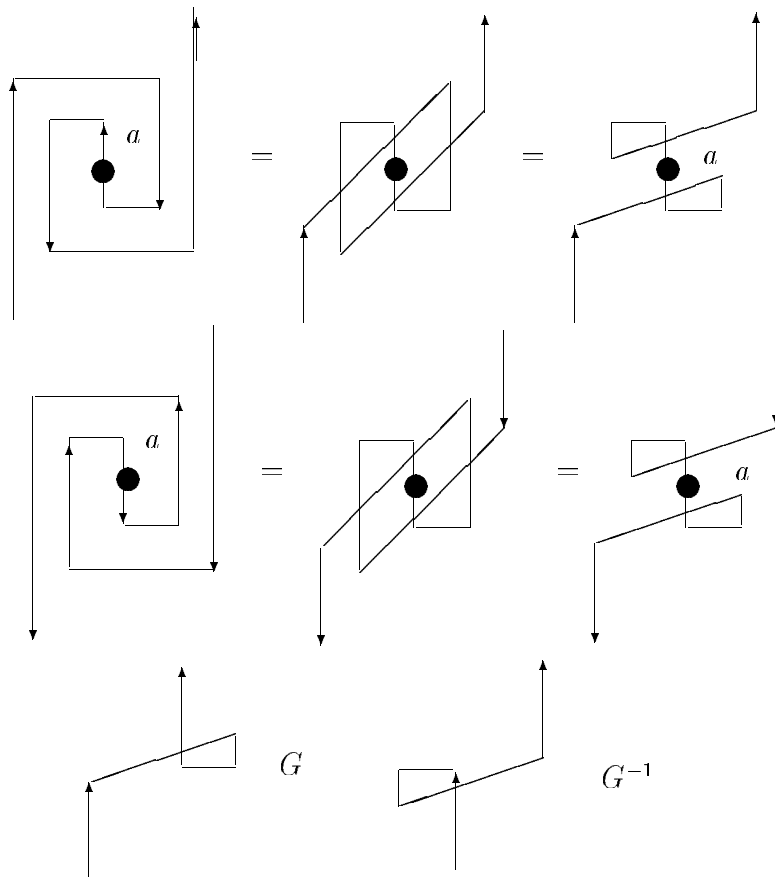


Figure 11 - $UDa = DUa = GaG^{-1}$

The functor $F : Tang \rightarrow Cat(A)$ is defined by replacing each crossing in a tangle Q by a flat crossing that is decorated with a corresponding Yang-Baxter element as shown in Figures 12 and 13. The resulting flat diagram is then a morphism in $Cat(A)$. This serves to define $F(Q)$. Figure 12 shows how F is naturally defined for vertically oriented crossings. The choice of placement of the algebra on the crossings is crucial. The reader should verify that

these choices ensure that F applied to the positive and negative crossings, respectively, are inverses of one another. Figure 13 shows how the switchback move implies the definitions of F on horizontally oriented crossings. Then in Figure 14 we point out how the two ways of performing the switchback move are compatible through our axiomatic assumptions that $(U \otimes U)\rho = \rho$ and $(D \otimes D)\rho = \rho$. This explains and justifies the first axiom for an oriented quantum algebra.

Remark. The reader should note the compatibility of these symmetry axioms about $D \otimes D$ and $U \otimes U$ with the categorical turn axiom that states (as in Figure 12) that the downward versions of the braiding operators (images under F of the crossings) are exactly the 180 degree turns of the upward versions. We leave as an exercise in bead sliding to show that the algebraic symmetry axioms plus the categorical bead-sliding axioms imply the categorical turn axiom.

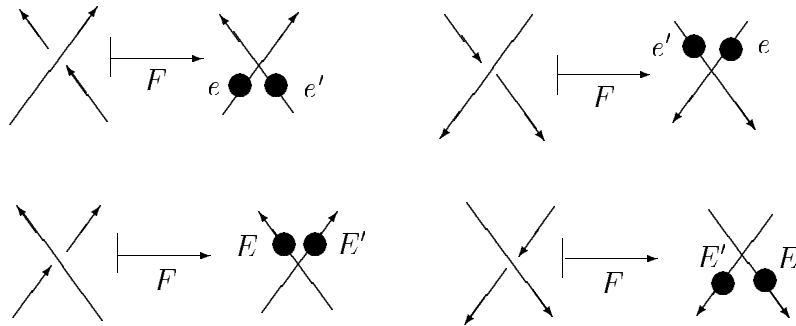


Figure 12 - The Functor F defined on vertical crossings

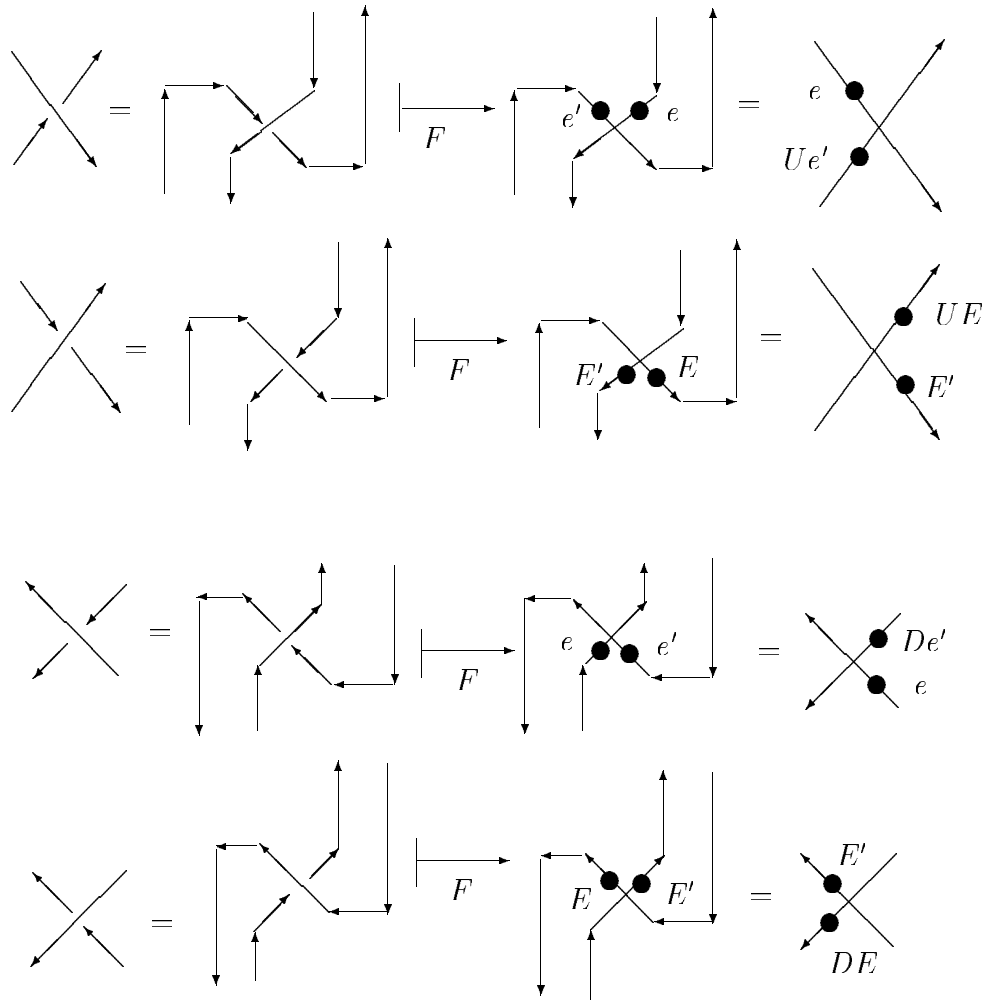


Figure 13 - Horizontal Crossings

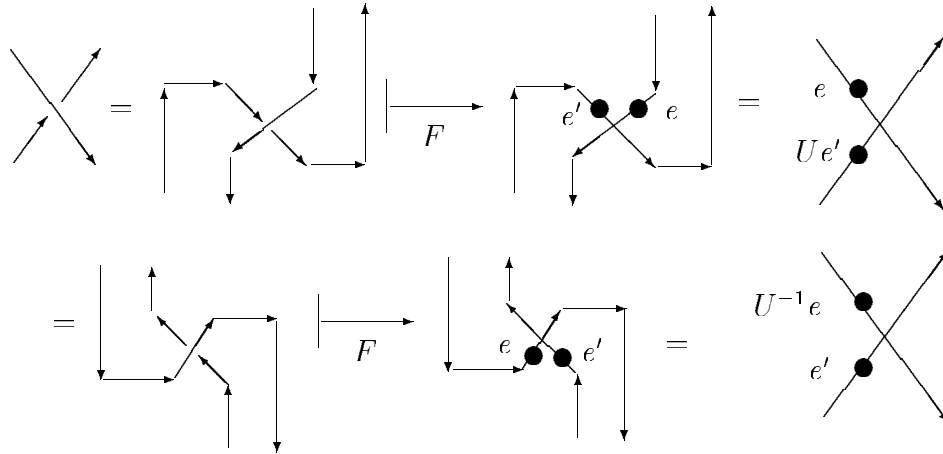


Figure 14 - The relation $(U \otimes U)\rho = \rho$

In order to see that F is well-defined it remains to verify invariance under the reverse Reidemeister two move. In Figure 15 we show that this invariance is equivalent to the equation

$$1_A \otimes 1_A = eDE \otimes E'Ue'$$

(summation convention of the pairs E, E' and e, e'). This, in turn, is equivalent to the condition that

$$[(1_A \otimes U)\rho][(D \otimes 1_{A^{op}})\rho^{-1}] = 1_{A \otimes A^{op}}.$$

This proves that the functor F is well-defined and it gives us the motivation for the second axiom for an oriented quantum algebra. Note that in $A \otimes A^{op}$ the formula for multiplication is

$$(a \otimes b)(c \otimes d) = ac \otimes db.$$

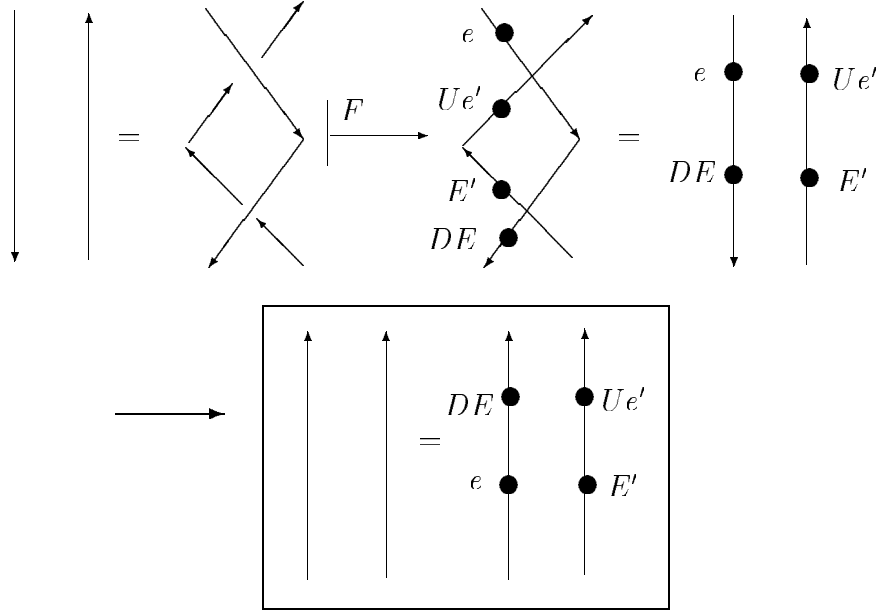


Figure 15 - The Reverse Type Two Move

3.1 Bead Sliding

Morphisms in the category $Cat(A)$ are precisely the morphisms in $Flat$ decorated by elements of the algebra A . The rules of interaction of cups and caps with algebra elements amount to this: If an algebra element a decorates the right side of a cap then it can be moved to the left side of that cap at the expense of relabeling it as $U(a)$ if the left side of the cap is a rising orientation, and $D(a)$ if the left side of the cap is a falling orientation. If an algebra element a decorates the left side of a cup then it can be moved to the right side of that cup at the expense of relabeling it as $U(a)$ if the right side of the cup is a rising orientation, and $D(a)$ if the right side of the cup is a falling orientation.

This amounts to a counterclockwise turn or *bead slide* taking a around the cap and changing it into $U(a)$ or $D(a)$. This rule of transformation is uniform. Clockwise turns correspond to applications of the automorphisms U^{-1} and D^{-1} , while counterclockwise turns correspond to applications of

U and D . The upshot of these transformations is that, given a morphism in $Cat(A)$, we can move all the decorations on a given component to any single vertical segment of that component. In particular, this means that in the case of a $1 - 1$ tangle, we can move all the algebra to the top of the tangle. What remains below the algebra is an immersed curve that can be regularly homotoped to a string of curls (as illustrated in Figure 11). Each curl is a special morphism in this category, not necessarily corresponding to an element of the algebra. It is convenient to label the curls G and G^{-1} as shown in Figure 11. Then we get an algebraic expression for every morphism in the form $G^n w$ where w is an expression in the algebra A . Note that in general w is a summation of products since the decoration of each crossing consists in a sum in $A \otimes A$. The exponent n is the Whitney degree of the plane curve that underlies the morphism.

For a $1 - 1$ tangle T we let $Inv(T)$ denote the expression $G^n w$. Up to equivalence in the oriented algebra A (possibly augmented by an element corresponding to G) the expression $Inv(T)$ is an invariant of the regular isotopy class of the tangle. The following proposition shows what happens when we reverse the orientation of the tangle T .

Proposition. Let T be a $1 - 1$ tangle of one component (a knot with ends). Let $r(T)$ denote the result of reversing the orientation of T . If $Inv(T) = G^n w$ then $Inv(r(T)) = G^{-n} r(w)$ where $r(w)$ is obtained by reversing the order of the products in w .

Proof. First note that we can arrange the diagram for the tangle so that all the crossings are vertical. It then follows from our conventions that reversing the orientation of the diagram does not affect the decoration of the crossings with algebra. See Figure 12 for an illustration. The cup and cap operators are obtained for the reverse orientation by interchanging U and D . Note that if (for the sake of argument) the pair of signifiers for the ρ^{-1} at a crossing are E and E' then, when the beads are slid to the top of the tangle, E and E' will differ by a composition of the operators U and D that is obtained from going around a closed loop (that loop being the difference in the journeys taking E and E' to the top of the tangle). Such a composition of operators is necessarily a power of $t = UD = DU$. We know that $(U \otimes U)\rho = \rho$ and that $(D \otimes D)\rho^{-1} = \rho^{-1}$. Note that $U \otimes U$ is an automorphism of $A \otimes A$. Hence $\rho^{-1} = ((U \otimes U)\rho)^{-1} = (U \otimes U)\rho^{-1}$. Similarly, $(D \otimes D)\rho = \rho$. The

upshot of this remark is that in the algebra expression $Inv(T)$ if E and E' receive identical operators as compositions of U and D , then these operators can be removed from both of them. Since, referring specifically to E and E' in the discussion above, we have that E and E' will differ by a power of $t = DU = UD$, it follows that the entire algebra expression $Inv(T)$ involves only the operator t (after removing common operators from the pairs E, E' and e, e'). The Proposition follows easily from these observations. //

Corollary. The knot invariant derived via $1 - 1$ tangles from an oriented quantum algebra is equivalent to an invariant derived from a standard oriented quantum algebra obtained by replacing U by $t = UD = DU$ and replacing D by the identity automorphism.

Proof. It is not hard to see that if (A, ρ, D, U) is an oriented quantum algebra, then $(A, \rho, 1, DU)$ is also a quantum algebra. The method of the proof of the proposition shows that these two algebras yield the same invariants. //

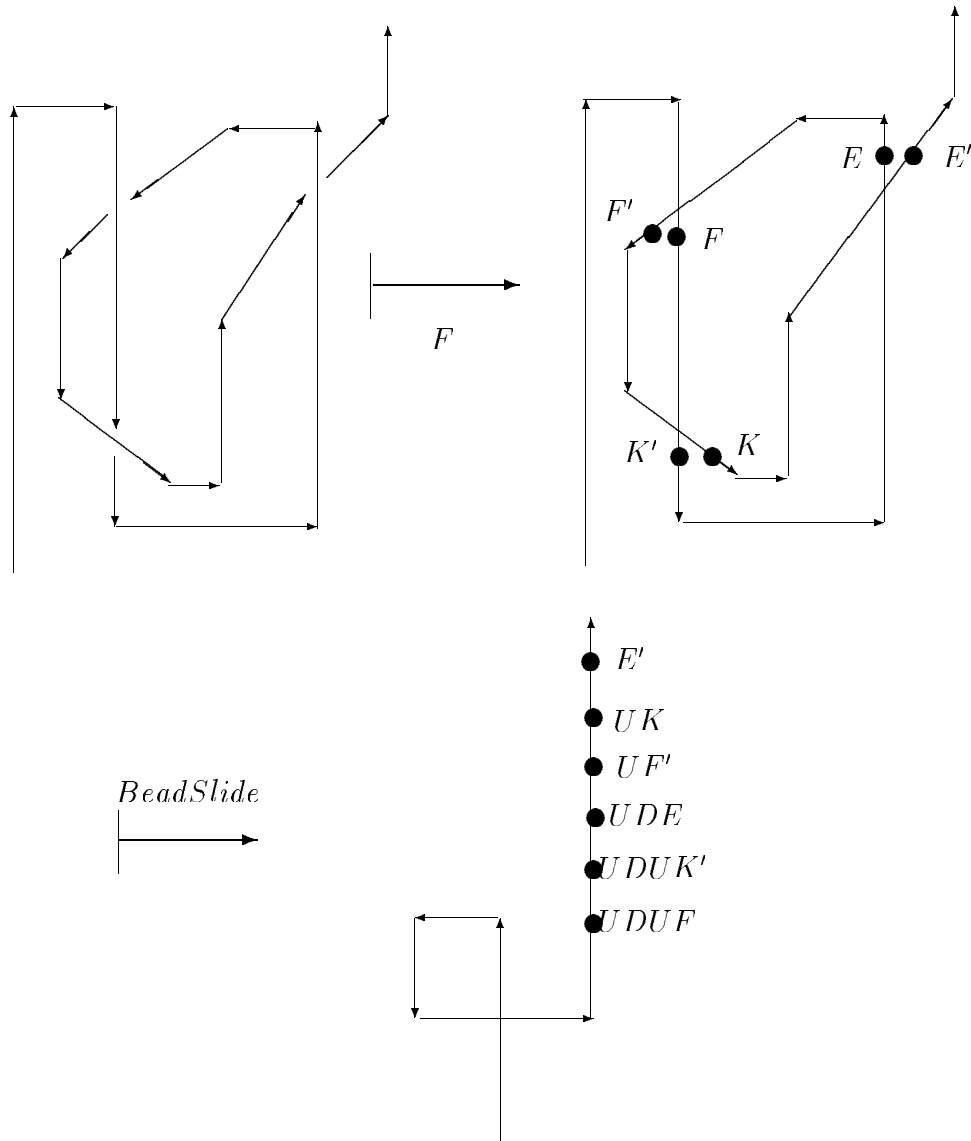


Figure 16 - Bead Sliding Trefoil

Remark. In Figure 16 we have

$$\begin{aligned}
 w(T) &= G^{-1}(UDUF)(UDUK')(UDE)(UF')(UK)(E') \\
 &= G^{-1}(DUF)(DUK')(UDE)F'KE' = G^{-1}(tF)(tK')(tE)F'KE'.
 \end{aligned}$$

Here we removed some operators as a consequence of the identity $(U \otimes U)\rho = \rho$. This example is a concise illustration of the content of the Proposition. This proposition suggests the conjecture that the 1 – 1 tangle invariants can detect the difference between some knots and their reversals.

4 Matrix Models

This section will show how matrix representations of a quantum algebra (or a matrix quantum algebra) give rise to invariants that can be construed directly as state summations via the vertex weights from cup, cap and crossing matrices. Thus we get a double description in terms of bead sliding and in terms of the state sums. We will *also* discuss how our theory of quantum algebras corresponds to the usual way of augmenting a solution to the Yang-Baxter equation to produce a quantum link invariant. This is a case of taking an oriented quantum algebra associated with a solution to the Yang-Baxter equation. We shall restrict the discussion to balanced oriented quantum algebras where $U = D = T$. We modify our discussion for the general case and for other specific cases of quantum algebras in [14].

In a matrix model for a knot or link invariant we are given a diagram for the knot or link K that is arranged with respect to a vertical direction so that there are a finite number of transverse directions to the vertical that have critical points (maxima, minima or crossings) and these critical points are separated. Thus K is given as a morphism in the tangle category. We then traverse the diagram for K , marking one point on each arc in the diagram from critical point to critical point. This divides the diagram (by deleting the marked points) into a collection of generators of the tangle category (cups, caps and crossings). See Figure 17.

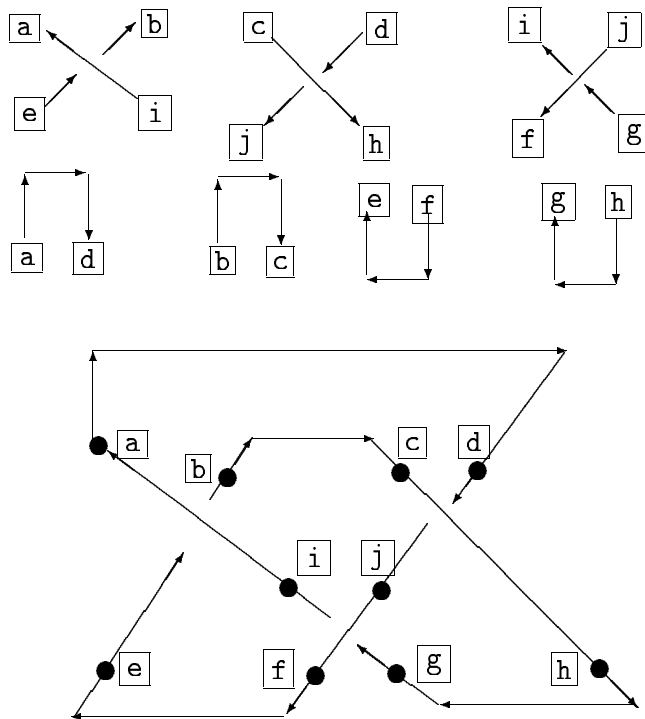


Figure 17 – A knot K with marked points

We further assume that matrices with entries in an appropriate commutative ring k have been assigned to each of the different orientation types of cups, caps and crossings. With such an assignment of matrices, we can create an evaluation $Z(K)$ of a given marked diagram by the following algorithm: Let I denote the index set for the individual indices on the cup, cap and crossing matrices. (Cups and caps have two indices while crossings have four indices.) A *coloring* of a marked diagram is an arbitrary assignment of indices to the marked points on the diagram. A colored diagram then has indices assigned to each of its component cup, cap and crossing matrices. We define $Z(K)$ to be the sum over all colorings of the products of these matrix entries. Thus in the diagram shown in Figure 17 we have that

$$Z(K) = \sum_{a, \dots, h} [\overline{M}_{ad}^>] [\overline{M}_{bc}^>] [\overline{S}_{ei}^{ab}] [S_{fg}^{ij}] [S_{jh}^{cd}] [M_{<}^{ef}] [M_{<}^{gh}].$$

Here $[\overline{M}^>]$ stands for a right-oriented (clockwise) cap, $[\overline{M}^<]$ stands for a left-oriented (counter-clockwise) cap, $[M_{>}]$ stands for a counter-clockwise cup,

$[M_{<}]$ stands for a clockwise cup, \overline{R} is the positive upward pointing crossing matrix and \overline{S} is its inverse. A bar below the R or S connotes a downward-pointing crossing and a bar to the left or to the right connotes a crossing that points to the left or to the right respectively. Thus we have the following list of matrices that are directly associated with the link diagram:

$$\overline{M}^>, \overline{M}^<, \underline{M}_{>}, \underline{M}_{<}, \overline{R}, \underline{R}, |R, R|, \overline{S}, \underline{S}, |S, S|.$$

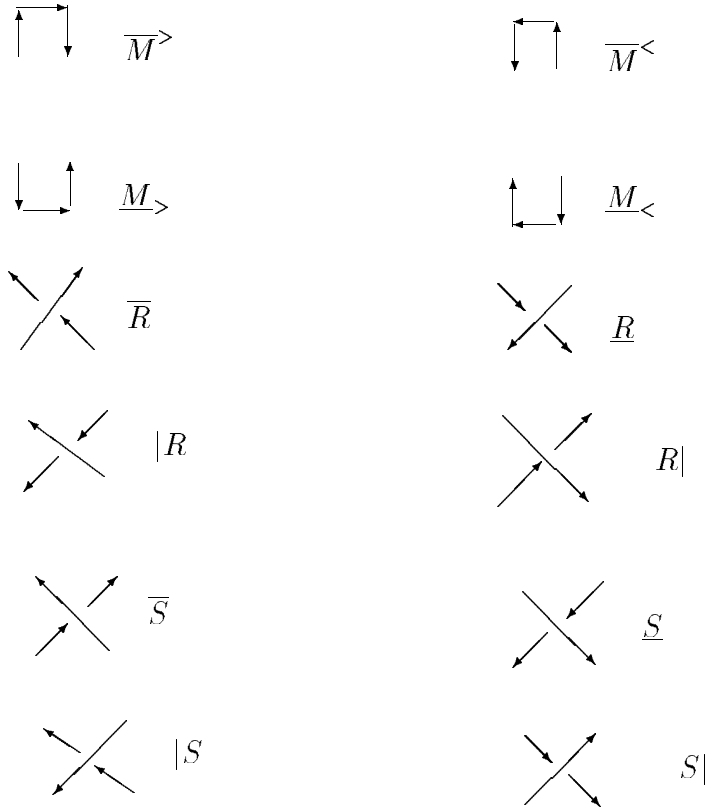


Figure 18 - Matrix Notations

In the models all these oriented “matrices” (as above and in Figure 18) will be defined by a smaller collection of ordinary matrices with the standard multiplication convention. We shall call these oriented matrices of Figure 18 the *diagram matrices* since they can be read directly from a decorated Morse

link diagram in the process of translating from topology to algebra. We shall call the smaller collection of standard matrices the *background matrices* for the matrix model. There will be a single background matrix M (written with lower indices M_{ab}) and its inverse M^{-1} so that $\sum_i M_{ai} M_{ib}^{-1} = \delta_{ab}$. The background matrices R and S written with indices in the form R_{cd}^{ab} and S_{cd}^{ab} , and $\sum_{ij} R_{cd}^{ij} S_{ij}^{ab} = \delta_c^a \delta_d^b$. Thus, if we think of ab as a single index and write $R_{cd}^{ab} = R_{ab,cd}$ and $S_{cd}^{ab} = S_{ab,cd}$ then R and S multiply in the standard matrix convention where the left lower index is the input index and the right lower index is the output index. In this convention we can write $RS = SR = I$ where I denotes the identity matrix of the appropriate dimension.

The rewrite definitions for the diagram matrices in terms of the background matrices are as follows:

$$\overline{M}_{ab}^> = M_{ab}$$

$$\overline{M}_{ab}^< = M_{ba}^{-1}$$

$$\underline{M}_{>}^{ab} = M_{ab}^{-1}$$

$$\underline{M}_{<}^{ab} = M_{ba}$$

$$\overline{R}_{cd}^{ab} = R_{cd}^{ab}$$

$$\underline{R}_{cd}^{ab} = R_{ba}^{dc}$$

$$\overline{S}_{cd}^{ab} = S_{cd}^{ab}$$

$$\underline{S}_{cd}^{ab} = S_{ba}^{dc}$$

$$|R_{cd}^{ab} = \overline{M}_{ci}^< \overline{R}_{dj}^{ia} \underline{M}_{<}^{jb} = \underline{M}_{<}^{ai} \underline{R}_{ic}^{bj} \overline{M}_{jd}^<$$

$$R|_{cd}^{ab} = \overline{M}_{ci}^> \underline{R}_{dj}^{ia} \underline{M}_{>}^{jb} = \underline{M}_{>}^{ai} \overline{R}_{ic}^{bj} \overline{M}_{jd}^>$$

$$|S_{cd}^{ab} = \overline{M}_{ci}^< \overline{S}_{dj}^{ia} \underline{M}_{<}^{jb} = \underline{M}_{<}^{ai} \underline{S}_{ic}^{bj} \overline{M}_{jd}^<$$

$$S_{cd}^{ab} = \overline{M}_{ci}^> \underline{S}_{dj}^{ia} \underline{M}_{>}^{jb} = \underline{M}_{>}^{ai} \overline{S}_{ic}^{bj} \overline{M}_{jd}^>$$

The last four equations expressing the horizontal versions of the braiding matrices in terms of the vertical ones and the cup and cap matrices follow from the switchback moves as illustrated in Figures 5 and 13. The conditions for invariance under regular isotopy in the tangle category (Figure 5) translate into conditions on these matrices. For example, the type three move translates to the Yang-Baxter equation in braiding form. Assuming that the matrices satisfy these conditions, it follows that $Z(K)$ is a regular isotopy invariant of knots and links.

In fact the same method of assignment gives a functor from the tangle category to the tensor category associated with the basic representation module associated with these matrices or equivalently to the category of an oriented quantum algebra $M_n(k)$ associated with the $n \times n$ matrices over the ring k where n is the size of the index set. In working with $n \times n$ matrices A it is convenient for diagram purposes to write $A_i^j = A_{ij}$ where the second half of this equation denotes the standard convention for matrix indices (ij stands for row- i and column- j) so that

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}.$$

In writing A_i^j we indicate that for the standard upward orientation with respect to the vertical, the input index for the matrix is at the bottom and the output index for the matrix is at the top. With these conventions, Figures 9 and 10 (interpreted for matrices) show that the automorphism $T : M_n(k) \rightarrow M_n(k)$ is given by the equation

$$T(A) = MAM^{-1}.$$

To see this note that

$$T(A)_{ij} = T(A)_i^j = \overline{M}_{ik}^> A_k^l \underline{M}_{>}^{lj} = M_{ik} A_{kl} M_{lj}^{-1} = (MAM^{-1})_{ij}$$

when we write A for the matrix corresponding to an algebra element a .

In this picture the generators of the algebra are the elementary matrices E_a^b (with entry 1 in the a -th row, b -th column place and zero elsewhere). Here we think of the lower index on the elementary matrix as the index corresponding to the entrance to the arrow for the corresponding morphism in $Cat(M_n(k))$ and the upper index corresponds to the exit from the arrow. In this convention we have $E_a^b E_c^d = \delta_c^b E_a^d$ where δ_c^b is the Kronecker delta (equal to one when $b = c$ and 0 otherwise). These conventions are important for specific calculations of these quantum algebras associated to the matrix models.

In order to complete the relationship between matrix models and our description of oriented quantum algebras we note that given $\rho \in M_n(k) \otimes M_n(k)$, we obtain braiding matrices R and S by permuting the upper and lower indices of ρ and ρ^{-1} respectively. That is

$$R_{cd}^{ab} = \rho_{cd}^{ba}$$

and

$$S_{cd}^{ab} = (\rho^{-1})_{dc}^{ab}.$$

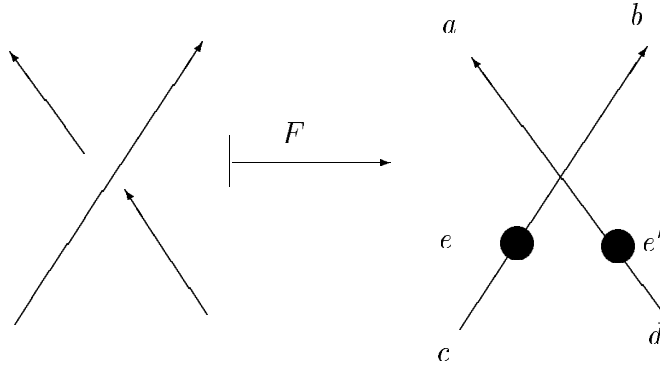
These assignments follow directly from the definition of the functor from the tangle category to the category of the algebra $M_n(k)$. Figure 19 illustrates the corresponding diagrams. In these diagrams we denote the matrix representation of $\rho = e \otimes e'$ by

$$\rho_{cd}^{ab} = e_c^a e_d^b.$$

The functor places the signifiers e and e' of ρ on the lines of a flat crossing and the resulting diagrammatic matrix is given by

$$R_{cd}^{ab} = e_d^{ja} e_c^b = e_c^b e_d^{ja} = \rho_{cd}^{ba}$$

Note that this identity is dependent on the fact that the matrix elements that correspond to the signifiers of ρ are commuting scalars in the base ring k .



$$R_{cd}^{ab} = e_d^{ta} e_c^b = \rho_{cd}^{ba}$$

Figure 19 - Braiding Matrix R and Algebraic Matrix ρ

4.1 Matrices and Bead Sliding

Note that when we use a matrix model based on a representation of a (balanced) oriented quantum algebra we require not only a matrix representation of the algebra, but also a matrix representation of the basic automorphism T of A . This data entails the matrices e_b^a and $e_b^{\prime a}$ that define the matrix representation of ρ , the matrices E_b^a and $E_b^{\prime a}$ that define the matrix representation of ρ^{-1} and the background matrices M and M^{-1} that define the representation of T via $T(v) = MvM^{-1}$. Note that we apply T functionally on the left. It has been our convention to multiply algebra in the order of its appearance on the oriented lines of the diagram from bottom to top. With our index conventions for matrices this corresponds to the left-right order of matrix multiplication. This means that if we regard the category $Cat(M_n(k))$ as containing morphisms that represent individual matrices, then the composition of such morphisms by attaching directed arrows head-to-tail corresponds to the multiplication of these matrices.

Remark. It is also natural to represent A to $End(V)$, endomorphisms of a given vector space V . The reader should note, however, that in this language the composition of morphisms of vector spaces proceeds in the opposite order from the matrix multiplication that we have preferred. In order to rectify this,

one must speak of representations of the opposite algebra A^{op} to $End(V)$. This point of view is useful in other contexts, but will not be pursued here.

Theorem. Let A be a (balanced) oriented quantum algebra. Let $Rep : A \longrightarrow M_n(k)$ be a representation of A to the ring of $n \times n$ matrices over the ground ring k . Suppose there is an invertible matrix M in $M_n(k)$ such that the automorphism $\tau : M_n(k) \longrightarrow M_n(k)$ given by $\tau(x) = MxM^{-1}$ gives $Rep(A)$ the structure of a (balanced) oriented quantum algebra such that $Rep(T(a)) = \tau(Rep(a))$ for all a in A . Then the matrices $Rep(\rho)$, $Rep(\rho^{-1})$ and M give data for a matrix model oriented link invariant of regular isotopy.

Proof. The proof of this Theorem is contained in the discussion that precedes it. //

Some discussion of this Theorem is in order. First of all, note that the evaluation of a matrix model can be read directly from the pattern of matrices on the diagram. One simply writes down the list of the diagram matrix entries corresponding to the cups, caps and crossings and then sums the product of the elements in this list over all possible values of the repeated indices. The resulting evaluation is a function of the non-repeated indices.

On the other hand, we can view the matrices that come from the representation of an oriented quantum algebra as having the specific forms such as

$$R_{cd}^{ab} = e_d'^a e_c^b.$$

By putting the expression of the invariant in terms of the signifier matrices $Rep(e)$, $Rep(e')$, $Rep(E)$, $Rep(E')$, M and M^{-1} we bring the expression of the invariant close to the abstract expression in $Cat(A)$. In particular, there is a corresponding matrix expression for any diagram that is obtained from the original abstract diagram by bead sliding (just replace the signifiers by their corresponding matrices and write down the resulting sum of products of terms involving them and the cups and caps). Thus every diagram obtained by sliding beads on the original abstract diagram has a matrix evaluation. It is easy to see that these evaluations are invariant under bead sliding. (We leave the proof to the reader.) The consequence is that we can slide the beads first and then evaluate the matrix model. (This pattern was first observed in

[5].) In particular if K is a closed loop diagram and we slide all the algebra to one segment where it takes the form w and regularly homotop the flat diagram to the form G^n then the matrix model yields the invariant

$$INV(K) = \text{trace}((M^2)^n \text{Rep}(w))$$

as the value of the matrix model. Here trace denotes standard matrix trace and the term M^2 is the matrix evaluation of G (Bead sliding shows that $GxG^{-1} = UD(x)$ for all matrices x . The formula $G = M^2$ then follows from $U(x) = D(x) = MxM^{-1}$.)

Remark on General Matrix Models. Everything that we have said in this section generalizes to matrix models with two background matrices corresponding to the two automorphisms U and D in the general case of oriented quantum algebras. We have restricted ourselves to the balanced case only for ease of exposition. The general case follows the same lines. Since, in the general case there are two automorphisms, there will be two background matrices M and M' with $U(x) = MxM^{-1}$ and $D(x) = M'xM'^{-1}$ for all matrices x . Specifically we will have

$$\overline{M}^> = M, \quad \underline{M}_> = M^{-1}$$

and

$$\overline{M}^< = M'^{-1}, \quad \underline{M}_< = M'.$$

Otherwise, the model for a general oriented quantum algebra behaves in all respects like the model for a balanced algebra.

4.2 A Matrix Balanced Oriented Quantum Algebra

In this subsection we give a specific example of a matrix oriented balanced quantum algebra. This algebra can be used to produce a sequence of models of specializations of the *HOMFLYPT* polynomial [6] [3], as we shall see in the next section.

Recall from the previous section the elementary matrices E_a^b and the rule of multiplication $E_a^b E_c^d = \delta_c^b E_a^d$.

We let

$$z = q - q^{-1}.$$

In the equations below there is an implicit summation over the repeated indices for an index set of the form $\{1, 2, 3, \dots, N\}$ for a natural number N . Logical conditions on the indices are expressed within the formulas. The logical symbol

$$\otimes^>$$

refers to indices at the same level in the expression to which it belongs. For example,

$$E_b^a \otimes^> E_a^b = \Sigma_{a>b} E_b^a \otimes E_a^b.$$

We begin by defining ρ , ρ^{-1} and the automorphism T .

$$\rho = z E_b^a \otimes^> E_a^b + q E_a^a \otimes E_a^a + E_a^a \otimes^{\neq} E_b^b$$

$$\rho^{-1} = -z E_b^a \otimes^> E_a^b + q^{-1} E_a^a \otimes E_a^a + E_a^a \otimes^{\neq} E_b^b$$

$$T(E_b^a) = q^{a-b} E_b^a$$

Then

$$(1 \otimes T)\rho = z q^{a-b} E_b^a \otimes^> E_a^b + q E_a^a \otimes E_a^a + E_a^a \otimes^{\neq} E_b^b$$

$$(T \otimes 1)\rho^{-1} = -z q^{b-a} E_b^a \otimes^> E_a^b + q^{-1} E_a^a \otimes E_a^a + E_a^a \otimes^{\neq} E_b^b.$$

Here is the calculation:

$$\begin{aligned} [(1 \otimes T)\rho][(T \otimes 1)\rho^{-1}] &= -\Sigma_{a>b, a'>b'} z^2 q^{a-b} q^{b'-a'} E_b^a E_{b'}^{a'} \otimes E_a^{b'} E_a^b \\ &+ \Sigma_{a>b} z q^{a-b} q^{-1} E_b^a E_{a'}^{a'} \otimes E_{a'}^{a'} E_a^b + \Sigma_{a>b, a' \neq b'} z q^{a-b} E_b^a E_{a'}^{a'} \otimes E_{b'}^{b'} E_a^b \\ &- \Sigma_{a'>b'} z q q^{b'-a'} E_a^a E_{b'}^{a'} \otimes E_{a'}^{b'} E_a^a + \Sigma_{a, a'} E_a^a E_{a'}^{a'} \otimes E_{a'}^{a'} E_a^a \\ &+ \Sigma_{a' \neq b'} q E_a^a E_{a'}^{a'} \otimes E_{b'}^{b'} E_a^a - \Sigma_{a \neq b, a' > b'} z q^{b'-a'} E_a^a E_{b'}^{a'} \otimes E_{a'}^{b'} E_b^b \\ &+ \Sigma_{a \neq b} q^{-1} E_a^a E_{a'}^{a'} \otimes E_{a'}^{a'} E_b^b + \Sigma_{a \neq b, a' \neq b'} E_a^a E_{a'}^{a'} \otimes E_{b'}^{b'} E_b^b \end{aligned}$$

$$\begin{aligned}
&= -\sum_{a'>a>b'} z^2 q^{a-b'} q^{a-a'} F_{b'}^{a'} \otimes F_{a'}^{b'} + \sum_{a'>b'} z q^{a'-b'-1} F_{b'}^{a'} \otimes F_{a'}^{b'} \\
&\quad - \sum_{a'>b'} z q^{b'-a'+1} F_{b'}^{a'} \otimes F_{a'}^{b'} + \sum_a F_a^a \otimes F_a^a + \sum_{a \neq b} F_a^a \otimes F_b^b \\
&= \sum_{a'>b'} z [-z \sum_{a'>a>b'} q^{a-b'} q^{a-a'} + q^{a'-b'-1} - q^{b'-a'+1}] F_{b'}^{a'} \otimes F_{a'}^{b'} \\
&\quad + \sum_a F_a^a \otimes F_a^a + \sum_{a \neq b} F_a^a \otimes F_b^b
\end{aligned}$$

But

$$[-z \sum_{a'>a>b'} q^{a-b'} q^{a-a'} + q^{a'-b'-1} - q^{b'-a'+1}] = q^{-a'-b'} [-z \sum_{a'>a>b'} q^{2a} + q^{2a'-1} - q^{2b'+1}]$$

and

$$-z \sum_{a'>a>b'} q^{2a} = (q^{-1} - q)(q^{2b'+2} + \dots + q^{2a'-2}) = q^{2a'-1} - q^{2b'+1}$$

Thus

$$[(1 \otimes T)\rho][(T \otimes 1)\rho^{-1}] = \sum_a F_a^a \otimes F_a^a + \sum_{a \neq b} F_a^a \otimes F_b^b = 1_{A \otimes A^{\text{op}}}.$$

This verifies that the algebra generated by elementary matrices in conjunction with ρ , ρ^{-1} and T form a balanced oriented quantum algebra. In order to associate a link invariant to this algebra we can construct a matrix model by taking

$$M_{ij} = q^i \delta_{ij}.$$

We leave it as an exercise for the reader to verify that the resulting invariant is an unnormalized version of the *HOMFLYPT* polynomial. This exercise is clarified by the remarks on state models in the next section.

Remark. Note that for such a matrix, $M_{ij} = q^i \delta_{ij}$, it follows from the previous section that

$$T(F_b^a) = q^{a-b} F_b^a.$$

For a general R -matrix R_{cd}^{ab} we will have that

$$\rho_{cd}^{ab} = R_{cd}^{ba}$$

and, letting $\bar{\rho}$ denote ρ^{-1} ,

$$\bar{\rho}_{cd}^{ab} = \bar{R}_{dc}^{ab}$$

whence

$$\rho = \sum R_{cd}^{ba} E_c^a \otimes E_d^b$$

and

$$\bar{\rho} = \sum \bar{R}_{dc}^{ab} E_c^a \otimes E_d^b.$$

It follows that the equation $\rho = (T \otimes T)\rho$ implies that the non-zero entries R_{cd}^{ab} of the R -matrix satisfy the equation $a+b = c+d$, and that the remaining conditions in $A \otimes A^{op}$ for the balanced oriented quantum algebra

$$[(1_A \otimes T)\rho][(T \otimes 1_{A^{op}})\rho^{-1}] = 1_{A \otimes A^{op}}$$

and

$$[(T \otimes 1_{A^{op}})\rho^{-1}][(1_A \otimes T)\rho] = 1_{A \otimes A^{op}}$$

impose the conditions (using Einstein summation convention)

$$\bar{R}_{jc}^{ib} R_{id}^{ja} q^{i-c} q^{j-d} = \delta_c^a \delta_d^b$$

and

$$R_{cj}^{bi} \bar{R}_{di}^{aj} q^{b-j} q^{a-i} = \delta_c^a \delta_d^b.$$

These are the same as the conditions of [24] and [7] for producing a quantum link invariant. In this way the matrix models and the oriented quantum algebra dovetail neatly.

5 State Sums

There are many versions of the general notion of a state summation model for a link invariant. Given a link diagram K , a *combinatorial structure associated with K* is a graph with decorations (possibly algebraic) that is obtained from K by some well-defined process of labelling and replacement. The exact details of what a combinatorial structure can be are left open, as there are many possibilities. In general we do not assume, as in a matrix model, that the link diagram has been arranged with respect to a vertical direction. Nevertheless, matrix models are examples of state summations. One well known way to obtain combinatorial structures associated to a link diagram L is to replace each crossing of L with either a *smoothing* of that crossing or a *flattening* of that crossing. In smoothing a crossing we replace the connections at the crossing so that it fits in the plane without any arcs crossing over or under one another. In flattening a crossing, the crossing is replaced by a four-valent vertex in the plane. See Figure 20 for illustrations of smoothing and flattening.

By a *combinatorial state sum* I mean that for each link diagram K there will be associated a set of combinatorial structures $\mathbf{S}(K)$, called the *states* of K , and a functional $\langle K|S \rangle$ that associates to each state of K and state S an element $\langle K|S \rangle$ in a commutative ring k . Then we define the *state sum for K* to be the summation

$$Z_K = \sum_{S \in \mathbf{S}(K)} \langle K|S \rangle \langle S \rangle .$$

where $\langle S \rangle$ is a specifically given state evaluation in k . It is intended that Z_K be a regular isotopy invariant of the link K .

In this section we shall consider state sums of the following special form. We specify a given index set $I = \{1, 2, 3, \dots, n\}$. In the state sum each crossing of the link diagram can be replaced by either oriented parallel arcs (an oriented smoothing) with a sign of either equality or inequality between them (this constitutes three possibilities, since the inequality can be oriented in two ways between the two lines), or by crossed arcs (a flattening) with a sign of inequality between them. In Figure 20 we have illustrated these local replacements in the form of symbolic summations (we use the diagram and its evaluation interchangeably)

$$\begin{aligned}
K_+ &= AsK_= + BsK_< + CsK_> + DfK_\neq \\
K_- &= A'sK_= + B'sK_< + C'sK_> + D'fK_\neq
\end{aligned}$$

where K_+ and K_- denote the link with a specific crossing that is either positive or negative, $sK_=$ denotes the result of smoothing with the vertical left line equal to the vertical right line (where vertical denotes the local directionality of the crossing), $sK_<$ denotes the result of smoothing with the left vertical line less than the right vertical line, $sK_>$ denotes the result of smoothing with the left vertical line greater than the right vertical line, fK_\neq denotes the result of flattening with the left top line unequal to the left top right line (note that the lines cross over one another in a flattening). See Figure 20. These equations express the state summation symbolically. We expand in this way on each crossing until a formal sum of products is reached. The diagrams that are implicated in these products have a medley of stipulations of equality and inequality inscribed upon them. Those that are impossible (e.g. a line is asked to be less than itself) are given value zero. A diagram D with a possible assignments of equality and inequality is evaluated as $\langle D \rangle$ as in the general state sum description of the previous paragraph. This recursive description of the state summation is identical to the description

$$Z_K = \sum_{S \in \mathbf{S}(K)} \langle K|S \rangle \langle S \rangle$$

where the states S are the diagrams D obtained by smoothing and flattening, and the evaluations $\langle K|S \rangle$ denote the product of labels A, A', \dots, D, D' that are implicit in the recursive expansion. That is, one should think of that state diagram D as decorated not only with signs of equality and inequality, but also with the alphabetic labels that are implicit in the recursive expansion. Then $\langle K|S \rangle$ is the product of the labels that is the coefficient of the corresponding diagram in the recursive expansion.

We now turn to the specific definition of $\langle D \rangle$ for a state diagram D . This diagram is labelled with equalities and inequalities that make it possible to create actual labellings of its curves (one numerical label per curve) from the index set $I = \{1, 2, 3, \dots, n\}$. Let Λ denote an actual labelling of D and $D(\Lambda)$ that particular labelling of D . Let

$$\langle D \rangle = \sum_{\Lambda} \langle D(\Lambda) \rangle = \sum_{\Lambda} q^{\sum_{i \in \Lambda} i \text{Rot}_D(i)}$$

where Λ runs over all admissible labellings of D from the index set I . Here q is a commuting algebraic variable, and $\text{Rot}_D(i)$ denotes the Whitney degree in the plane of the component of D that is labelled by the index i . (We think of Λ as a list of indices (with multiplicities) such that each index is attached to a specific component of D .)

This specification completely defines the state summation. That is, the state summation is now well-defined on link diagrams. It is not in general an invariant of regular isotopy for links. We will now show that this specification corresponds to a specific matrix model and that a special case gives specializations of the *HOMFLYPT* polynomial.

The *HOMFLYPT* specialization has vertex weights given by the expansion

$$\begin{aligned} K_+ &= qsK_+ + (q - q^{-1})sK_< + fK_{\neq} \\ K_- &= q^{-1}sK_+ + (q^{-1} - q)sK_> + fK_{\neq}. \end{aligned}$$

Note that

$$K_+ - K_- = (q - q^{-1})[sK_+ + sK_< + sK_>] = (q - q^{-1})sK$$

where sK denotes the diagram obtained by smoothing the crossing indicated by the context, and fK denotes the diagram obtained by flattening the crossing. This is the source of the skein relation for the *HOMFLYPT* polynomial.

The matrix model arises by interpreting the expansion formulas for K_+ and K_- as definitions for the braiding matrices in the matrix model. Translating these braiding matrices to the algebraic form (by composing with the appropriate permutation) yields the ρ and ρ^{-1} of the previous section. We omit these details. The upshot is that the present state sum can be used to prove that the matrix model in the last section does give the *HOMFLYPT* polynomial specializations.

In Figure 21 we show how to interpret the smoothed and flattened local states as matrices, using the convention that $\delta[a, b, c, d]$ is equal to 1 when

$a = b = c = d$ and is equal to 0 otherwise, $\delta[a, b]$ is equal to 1 when $a = b$ and is equal to 0 otherwise and $[P]$ is equal to 1 when the proposition P is true and is equal to 0 otherwise. With these interpretations for the braiding matrices, it is not hard to see that this state model is identical to the link invariant derived from the oriented quantum algebra of the previous section. This completes our discussion of these relationships among state models, matrix models and oriented quantum algebras.

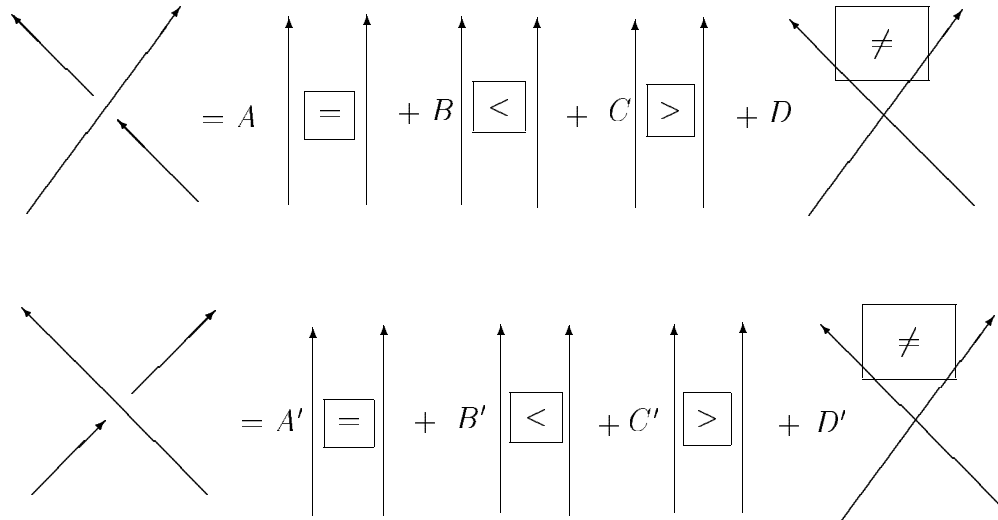


Figure 20 - Local States

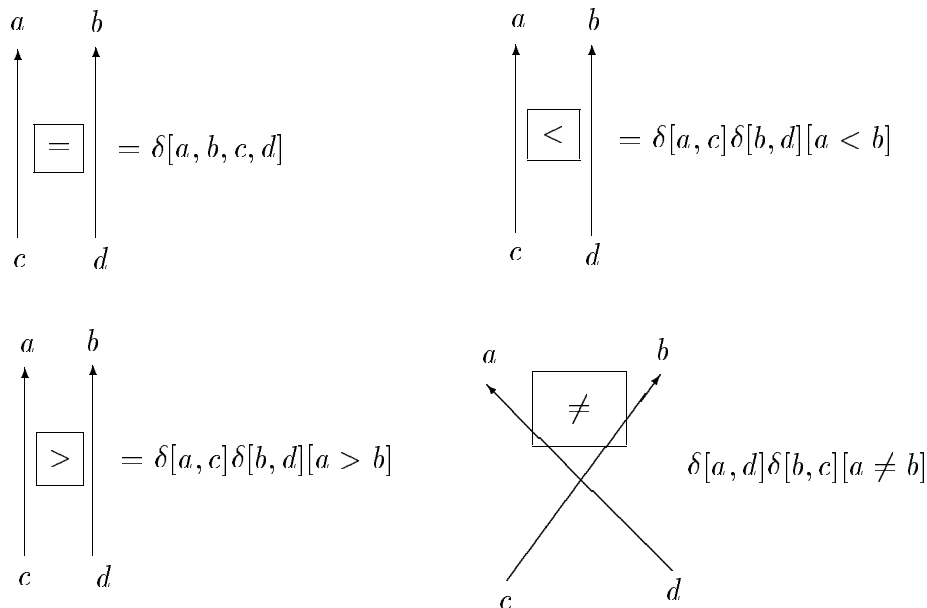


Figure 21 - Local Matrices

6 Quasitriangular Ribbon Hopf Algebras

In this section we show how to recover the results of Reshetikhin and Turaev [22, 21] that associate an oriented link invariant to each representation of a quasitriangular ribbon Hopf algebra. In our language, each such representation gives rise to a $D = 1$ (standard) oriented quantum algebra and a corresponding matrix model.

Any quasitriangular Hopf algebra is an example of an (unoriented) quantum algebra. In a quantum algebra with antipode s we have

$$(s \otimes s)\rho = \rho$$

and

$$(s \otimes 1)\rho = (1 \otimes s^{-1})\rho = \rho^{-1}.$$

We represent

$$\rho = \Sigma e \otimes e' = \Sigma f \otimes f'$$

and taking the convention that the Σ is implicit, we write

$$\rho = e \otimes e' = f \otimes f'.$$

Then

$$(se)f \otimes e'f' = 1 \otimes 1.$$

Now apply $s \otimes 1$ to both sides and get

$$(sf)s^2(e) \otimes e'f' = 1 \otimes 1.$$

Now transfer the s^2 using $(s \otimes s)\rho = \rho$ in the form $(s \otimes 1)\rho = (1 \otimes s^{-1})\rho$ and get

$$(sf)e \otimes s^{-2}(e')f' = 1 \otimes 1.$$

Thus

$$(sf \otimes f')(e \otimes s^{-2}e') = 1_A \otimes 1_{A^{op}}.$$

Hence

$$[(s \otimes 1)\rho][(1 \otimes s^{-2})\rho] = 1_A \otimes 1_{A^{op}}$$

$$[\rho^{-1}][(1 \otimes s^{-2})\rho] = 1_A \otimes 1_{A^{op}}.$$

Similarly,

$$[(1 \otimes s^{-2})\rho][\rho^{-1}] = 1_A \otimes 1_{A^{op}}.$$

This shows that any quantum algebra A is a standard oriented quantum algebra with $U = t = s^{-2}$ and $D = 1$. In particular, This applies to quasi-triangular ribbon Hopf algebras. We now obtain in this way a new proof of the results of Reshetikhin and Turaev on the existence of link invariants from representations of such algebras.

In order to define a link invariant from a representation, we will define cups and caps as morphisms of vector spaces so that they represent the

automorphisms $t = U$ and $1 = D$. Once these cups and caps are defined, the evaluation of tangles is accomplished by regarding the image of each tangle under the functor F as a morphism of vector spaces. In particular, a closed link diagram is a linear map from k to k and the value of its invariant is the value of this map on the unit 1 in k . This method of evaluation is necessary for the direct comparison of our invariant with the Reshetikhin Turaev invariant. It is also compatible with the other methods in this paper (matrices and bead sliding).

In a quasitriangular ribbon Hopf algebra A the square of the antipode is represented by a grouplike element G [1] so that $s^2(a) = G^{-1}aG$ and

$$t(a) = s^{-2}(a) = GaG^{-1}$$

for all a in A . The existence of this grouplike element allows us, given a finite dimensional representation of A , to represent the cups and the caps so that they correctly represent the automorphisms t and the identity. The definitions are given below.

Remark. A quantum algebra with anti-automorphism s whose square s^2 is represented by conjugation by an element G is called (by us) a *twist quantum algebra* [11]. The results of this section can be stated in full generality for twist quantum algebras.

Since we are in a representation of the algebra A , we can assume that each element of A corresponds to an endomorphism of a vector space V . Let β run over a basis \mathcal{B} for V . Let β^* run over the dual basis for V^* . Thus $\beta^*(\beta') = \delta(\beta, \beta')$.

Note that for any v in V that

$$v = \sum_{\beta} \beta^*(v)\beta.$$

We define v^* for any v in V by the equation $v^*(\beta) = \beta^*(v)$ where β is in the basis \mathcal{B} .

The transpose of an element a in A will be denoted by a^t . Then a^t corresponds to an element in the endomorphisms of the dual space V^* . Note also

that if a is an endomorphism of V , then a^t is an endomorphism of V^* defined by the formula

$$a^t f(v) = f(a(v))$$

for any f in V^* . By our usual conventions the transpose of the morphism for an element a is diagrammed by drawing a down line that is still labelled a .

First we define *CapRight* and *CupRight*. It is understood that these formulas occur in the representation with $x \otimes y^*$ representing an element of $V \otimes V^*$. Each summation runs over a basis for V .

$$CapRight(x \otimes y^*) = y^*(G^{-1}x)$$

for any x and y in V .

$$CupRight(1) = \Sigma_{\beta}(\beta^* \otimes G\beta).$$

where β runs over a basis for V .

Then, letting a be any endomorphism of V and x an element of V , we let $t(a)$ be the endomorphism of V obtained from sandwiching a in the middle of the *CapRight* and *CupRight* as in Figure 10. We have

$$t(a)(x) = \Sigma_{\beta} CapRight(x \otimes a^t \beta^*) G\beta = \Sigma_{\beta} a^t \beta^*(G^{-1}x) G\beta = \Sigma_{\beta} \beta^*(aG^{-1}x) G\beta$$

Now

$$aG^{-1}x = \Sigma_{\beta} \beta^*(aG^{-1}x) \beta$$

thus

$$t(a)x = GaG^{-1}x.$$

hence

$$t(a) = GaG^{-1}.$$

CapLeft and *CupLeft* are defined by evaluation and coevaluation without the use of G so that $D = 1$.

These constructions show that any representation of a quasitriangular ribbon Hopf algebra A or an oriented twist quantum algebra [11] gives rise

to an invariant of regular isotopy of knots and links. The specification of the cups and caps gives a matrix model for this invariant in any basis for the representation space. In fact, the specification of cups and caps that we have used matches (up to reversals of arrow conventions) the cups and caps of Reshetikhin and Turaev in [22]. The choice of cups and caps determines the automorphisms of the corresponding oriented quantum algebra. The oriented quantum algebra underlying the Reshetikhin Turaev invariants for a representation of a ribbon Hopf algebra is identical to the algebra that we have described in this section. From this it follows that these methods reproduce the Reshetikhin Turaev invariants.

Remark. The categorically minded reader should note that in the case of linear representations of oriented quantum algebras and in the case of the matrix models, we can interpret the automorphisms U and D in terms of cups and caps by the adjoint relation of tensor product and module endomorphisms. For example, if $RightCup$ is represented by a mapping

$$Cup^> : V \otimes V^* \longrightarrow k$$

then this mapping is adjoint to an endomorphism

$$U : V \longrightarrow V.$$

With appropriate conditions on $Cup^>$ this endomorphism will be an automorphism, and is one of the generating automorphisms for the (represented) quantum algebra. We could have imported this duality into the original definitions of our categories, and in this way, made the relationships between cups, caps and automorphisms implicit from the outset.

References

- [1] V. G. Drinfel'd, Quantum Groups, *Proceedings of the International Congress of Mathematicians*, Berkeley, California, USA (1987), 798–820.
- [2] Mark Hennings, Invariants of links and 3-manifolds obtained from Hopf algebras, *J. London Math. Soc. (2)* **54** (1996), 594–624.

- [3] V. F. R. Jones, On knot invariants related to some statistical mechanics models, *Pacific J. Math.*, Vol. 137, No. 2 (1989), 311 - 334.
- [4] Louis H. Kauffman, State Models and the Jones Polynomial, *Topology* **26**, (1987), 395-407.
- [5] Louis H. Kauffman, Gauss Codes, quantum groups and ribbon Hopf algebras, *Reviews in Mathematical Physics* **5** (1993), 735-773. (Reprinted in [6], 551–596.)
- [6] Louis H. Kauffman, “Knots and Physics,” World Scientific, Singapore/New Jersey/London/Hong Kong, 1991, 1994.
- [7] Louis H. Kauffman, Knots and Diagrams, in “Lectures at Knots ’96”, edited by Shin’ichi Suzuki (1996), p. 123-194, World Scientific Pub. Co.
- [8] Louis H. Kauffman, Knot theory and statistical mechanics, *Intl. J. of Mod. Phys. B*, Vol. 11, Nos. 1 and 2 (1997), 39-49.
- [9] Louis H. Kauffman and David E. Radford, A Separation Result for Quantum Coalgebras with an Application to Pointed Quantum Coalgebras of Small Dimension, *J. of Algebra* **225** (1999), 161-200.
- [10] Louis H. Kauffman and David E. Radford, Invariants of 3-manifolds derived from finite-dimensional Hopf algebras, *J. Knot Theory Ramifications* **4** (1995), 131–162.
- [11] Louis H. Kauffman and David E. Radford, On invariants of knots and links which arise from quantum coalgebras, *in preparation*.
- [12] Louis H. Kauffman and David E. Radford, Oriented quantum algebras and invariants of knots and links. *submitted*.
- [13] Louis H. Kauffman and David E. Radford, Oriented quantum coalgebras and oriented quantum coalgebras, *in preparation*.
- [14] Louis H. Kauffman and David E. Radford, Oriented knot and link invariants in the context of quantum algebras and coalgebras, *in preparation*.
- [15] Louis H. Kauffman and David E. Radford, Quantum algebra structures on $n \times n$ matrices, *J. Algebra* **213** (1999), 405–436.

- [16] Louis H. Kauffman and David E. Radford, Quantum algebras, quantum coalgebras, invariants of 1–1 tangles and knots, *Comm. in Algebra* (to appear). 62 \LaTeX pages
- [17] Louis H. Kauffman, David Radford and Stephen Sawin, Centrality and the KRH invariant. *J. Knot Theory Ramifications* **7** (1998), 571–624.
- [18] D. E. Radford, Minimal quasitriangular Hopf algebras. *J. Algebra* **157** (1993), 285–315.
- [19] David E. Radford, On parameterized family of twist quantum coalgebras and the bracket polynomial, *J. of Algebra* **225** (2000), 93-123.
- [20] K. Reidemeister, *Knotentheorie*, Chelsea Pub. N.Y. (1948).
- [21] N. Yu. Reshetikhin, Quasitriangle Hopf algebras and invariants of tangles. *Leningrad Math. J.*, Vol. 1, No. 2 (1990), 491-513.
- [22] N. Yu. Reshetikhin and V. G. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Commun. Math. Phys.*, Vol. 127, (1990), 1-26.
- [23] M. Rosso, Groupes quantiques et modèles à vertex de V. Jones en théorie des noeuds, *C. R. Acad. Sci. Paris*, t. 307, Série I (1988), p. 207-210.
- [24] V. G. Turaev, The Yang-Baxter equation and invariants of links, *Invent. Math.*, Vol. 92, (1988), p. 527-553.