THE PALLETT GRAPH OF A FOX COLORING

By

TAKEJI NAKAMURA, YASUTAKA NAKANISHI, AND SHIN SATOH
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Abstract. We introduce the notion of a graph associated with a Fox p-coloring of a knot, and show that any non-trivial p-coloring requires at least \(\lfloor \log_2 p \rfloor + 2\) colors. This lower bound is best possible in the sense that there is a p-colorable virtual knot which attains the bound.

1. Introduction

A \(p\)-coloring of a diagram \(D\) of a knot \(K\), introduced by Fox [1] in 1961, is a map from the set of the arcs of \(D\) to \(\mathbb{Z}/p\mathbb{Z}\),

\[ \gamma : \{\text{the arcs of } D\} \to \mathbb{Z}/p\mathbb{Z}, \]

such that at each crossing the sums of the images (called the colors) of the undercrossing arcs is equal to twice the color of the over-crossing arc. We say that a \(p\)-coloring \(\gamma\) is trivial if it is a constant map.

Harary and Kauffman [2] study the number of distinct colors appeared in a non-trivially \(p\)-colored knot diagram \((D, \gamma)\). Let \(N(D, \gamma) = \#\text{Im}(\gamma) > 1\) be the cardinality of the image of \(\gamma\). For a \(p\)-colorable knot \(K\) in \(\mathbb{R}^3\), we denote by \(C_p(K)\) the minimal number of \(N(D, \gamma)\) for all the non-trivially \(p\)-colored diagrams \((D, \gamma)\) of \(K\). We remark that the notation \(C_p(K)\) is used in the original paper [2], and also written as \(\text{mincol}_p(K)\) in some papers.

There are several studies on this number found in [4, 5, 6, 7]. In particular, it is known in [6, 7] that

- \(C_3(K) = 3\) for any 3-colorable knot \(K\),
- \(C_5(K) = 4\) for any 5-colorable knot \(K\),
- \(C_7(K) = 4\) for any 7-colorable knot \(K\), and
- \(C_{11}(K) \geq 5\) for any 11-colorable knot \(K\).

The first aim of this paper is to generalize these results as follows:

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THEOREM 1.1. Let $p$ be an odd prime. Any $p$-colorable knot $K$ satisfies

$$C_p(K) \geq \lfloor \log_2 p \rfloor + 2,$$

where $\lfloor x \rfloor$ is the maximal integer less than or equal to $x$.

All of this can be done as well for virtual knots, with virtual crossings imposing no conditions on the colors: An arc of a virtual knot diagram is a curve that begins and ends at under-crossings, possibly passing through several virtual crossings, and the coloring conditions are derived from real crossings only [3].

For a $p$-colorable virtual knot $K$, we denote by $C_v^p(K)$ the minimal number of non-trivially $p$-colored diagrams $(D, \gamma)$ of $K$ in virtual knot category. The second aim of this paper is to prove that the inequality is best possible for virtual knots as follows.

THEOREM 1.2. Let $p$ be an odd prime. There is a $p$-colorable virtual knot $K$ with

$$C_v^p(K) = \lfloor \log_2 p \rfloor + 2.$$

This paper is organized as follows: In Section 2, we introduce a graph associated with a $p$-coloring which we call the pallet graph. We prove Theorem 1.1 by calculating the determinant of a matrix associated with the pallet graph. In Section 3, we prove Theorem 1.2 by constructing a tree with $\lfloor \log_2 p \rfloor + 2$ vertices for each $p$ which is the pallet graph of some $p$-colored virtual knot diagram.

2. Determinant of a matrix

We will start this section with a calculation of a matrix. Let $\mathcal{M}_n$ be the set of $n \times n$ matrices with integer entries such that

- each row contains at most two 1’s and at most one $-2$, and
- all the entries other than 1 and $-2$ are 0.

We denote by $\det(X)$ the determinant of $X$.

LEMMA 2.1. Any matrix $X$ in $\mathcal{M}_n$ satisfies $|\det(X)| \leq 2^n$.

Proof. We prove the lemma by induction on $n$. For $n = 1$, we have $X = (0), (1)$, or $(-2)$ and the inequality holds. For $n > 1$, we divide the proof into three cases.

(i) If $X$ has a row which contains no $-2$, then the cofactor expansion along the row induces

$$|\det(X)| \leq 1 \cdot 2^{n-1} + 1 \cdot 2^{n-1} = 2^n.$$
(ii) If $X$ has a row which contains no 1 but one $-2$, then the cofactor expansion along the row induces

$$|\det(X)| \leq 2 \cdot 2^{n-1} = 2^n.$$

(iii) Consider the case other than (i) and (ii); that is, every row contains exactly

- one 1 and one $-2$, or
- two 1’s and one $-2$.

Let $\vec{v}_j$ be the $j$th column of $X$. We may assume that the $(1,1)$-entry of $X$ is $-2$. Consider the matrix

$$Y = \left( -\sum_{j=1}^n \vec{v}_j, \vec{v}_2, \ldots, \vec{v}_n \right).$$

Then we see that $Y \in M_n$ and the first row of $Y$ satisfies the case (i). Therefore, we have $|\det(X)| = |\det(Y)| \leq 2^n$. \hfill \qed

**Definition 2.2.** Let $(D, \gamma)$ be a non-trivially $p$-colored diagram. The pallet graph $G$ of $(D, \gamma)$ is a simple graph such that

- the vertices of $G$ correspond to the colors on the arcs of $(D, \gamma)$, that is, the elements of the image $\text{Im}(\gamma)$, and

- two different vertices $c$ and $c'$ of $G$ are connected by an edge labeled $c'' = (c + c')/2$ if and only if there is a crossing of $(D, \gamma)$ whose lower arcs admit the colors $c$ and $c'$ and the upper admits $c''$.

We take a maximal tree of the pallet graph $G$. Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of $T$, and $c_1, c_2, \ldots, c_n$ the vertices of $T$, where $n = N(D, \gamma)$. We define the $(n-1) \times n$ matrix $A = (a_{ij})$ with integer entries such that

- $a_{ij} = 1$ if the edge $e_i$ is incident to the vertex $c_j$,
- $a_{ij} = -2$ if the edge $e_i$ is labeled by $c_j$, and
- $a_{ij} = 0$ otherwise.

**Lemma 2.3.** Let $A$ be the $(n-1) \times n$ matrix as above, and $A_j$ the $(n-1) \times (n-1)$ submatrix obtained from $A$ by deleting the $j$th column (1 ≤ $j$ ≤ $n$).

(i) $\det(A_j)$ is divisible by $p$.
(ii) $\det(A_j)$ is odd.

**Proof.** (i) The simultaneous equation $A\vec{x} = \vec{0}$ over the field $\mathbb{Z}/p\mathbb{Z}$ has two independent solutions $\vec{x} = ^t(1,1,\ldots,1), ^t(c_1,c_2,\ldots,c_n)$. Since the rank of $A$ is at most $n-2$, we have $\det(A_j) \equiv 0 \pmod{p}$. 

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(ii) The matrix $A$ over $\mathbb{Z}_2$ is coincident with the incident matrix of $T$. For each $1 \leq i \leq n - 1$, let $c_{\sigma(i)}$ be the vertex between two endpoints of the edge $e_i$ which is farther than the other away from the vertex $c_j$. Since $T$ is a tree, we see that

$$\det(A_j) \equiv a_{1\sigma(1)}a_{2\sigma(2)} \ldots a_{n-1,\sigma(n-1)} \equiv 1 \pmod{2}. \quad \square$$

**Proof of Theorem 1.1.** Let $(D, \gamma)$ be a non-trivially $p$-colored diagram of a knot $K$, and $A$ the $(n-1) \times n$ matrix constructed as above, where $n = N(D, \gamma)$. By Lemmas 2.1 and 2.3, it holds that $p \leq |\det(A_j)| < 2^{n-1}$, that is, $n > \log_2 p + 1$. \quad \square

**Remark 2.4.** By definition, the proof of Theorem 1.1 can be also applied for a virtual knot; any $p$-colorable virtual virtual knot $K$ satisfies

$$C^n_p(K) \geq \lfloor \log_2 p \rfloor + 2.$$

### 3. Construction of a Graph

Recall that a pallet graph $G$ over $\mathbb{Z}/p\mathbb{Z}$ satisfies the following properties:

**(P1)** $G$ is a connected simple graph with two or more vertices.

**(P2)** If two different vertices $c$ and $c' \in \mathbb{Z}/p\mathbb{Z}$ are connected by an edge, then the label $c'' = (c + c')/2$ of the edge also appears as a vertex of $G$.

We remark that $G$ has at least $\lfloor \log_2 p \rfloor + 2$ vertices by Theorem 1.1.

**Lemma 3.1.** There is a graph $G$ with exactly $\lfloor \log_2 p \rfloor + 2$ vertices satisfying (P1) and (P2).

**Proof.** Put $k = \lfloor \log_2 p \rfloor$; that is, $k$ is the integer satisfying $2^k < p < 2^{k+1}$. There are integers $m_1, m_2, \ldots, m_s$ uniquely satisfying

$$2^{k+1} - p = 2^{m_s} + \cdots + 2^{m_1} + 1$$

with $1 \leq m_1 < m_2 < \cdots < m_s < k$. Since $m_{j+1} \geq m_j + 1$ and $m_1 \geq 1$, it holds that $m_j \geq j$ for each $j$. Similarly, since $m_{j-1} \leq m_j - 1$ and $m_s \leq k - 1$, it holds that $m_j \leq k - 1 - (s - j)$ for each $j$. Therefore, we obtain

$$0 \leq m_j - j \leq k - s - 1.$$
We take $1 + (k - s + 1) + s = k + 2$ elements $a, b(0), b(1), \ldots, b(k - s), c(1), c(2), \ldots, c(s)$ in $\mathbb{Z}/p\mathbb{Z}$ such that

$$\begin{cases} 
  a = 0, \\
  b(i) = 2^i & \text{for } i = 0, 1, \ldots, k - s, \text{ and} \\
  c(j) = 2^{k-j+1} - (2^{m_j-j} + \cdots + 2^{m_j-j}) & \text{for } j = 1, 2, \ldots, s.
\end{cases}$$

We connect the vertices corresponding to these numbers to obtain a graph $G$ as follows:

(i) $b(0)$ is connected to $a$ by an edge labeled

$$\frac{a + b(0)}{2} = \frac{p + 1}{2} = 2^k - (2^{m_s-1} + \cdots + 2^{m_1-1}) = c(1).$$

(ii) For each $1 \leq i \leq k - s$, $b(i)$ is connected to $a$ by an edge labeled

$$\frac{a + b(i)}{2} = 2^{i-1} = b(i - 1).$$

(iii) For each $1 \leq j \leq s - 1$, $c(j)$ is connected to $b(m_j - j)$ by an edge labeled

$$\frac{b(m_j - j) + c(j)}{2} = 2^{k-j} - (2^{m_j-j} + \cdots + 2^{m_j-j-1}) = c(j + 1).$$

(iv) $c(s)$ is connected to $b(m_s - s)$ by an edge labeled

$$\frac{b(m_s - s) + c(s)}{2} = 2^{k-s} = b(k - s).$$

Since the graph $G$ is connected, we have the conclusion. $\square$

Figure 1 shows an example of the graph constructed in Lemma 3.1 for $p = 601$. 

![Figure 1](image-url)
**Lemma 3.2.** Let $G$ be a graph satisfying the properties (P1) and (P2). Then there is a non-trivially $p$-colored virtual knot diagram whose pallet graph is $G$.

**Proof.** It is sufficient to construct a Gauss diagram instead of a virtual knot diagram (cf. [3]). We take a closed path of $G$ which passes all the edges of $G$. Let $c_1, c_2, \ldots, c_n$ be the vertices of $G$, and $c_{k(1)}, c_{k(2)}, \ldots, c_{k(m)}$ the sequence of vertices on the path in this order.

To construct a Gauss diagram, we divide a circle into $m$ arcs by $m$ points $P_1, P_2, \ldots, P_m = P_0$, and assign the color $c_{k(i)}$ to each arc between $P_{i-1}$ and $P_i$ ($i = 1, 2, \ldots, m$). We take $m$ points $Q_1, Q_2, \ldots, Q_m$ on the circle such that $Q_i$ is in the interior of an arc labeled $(c_{k(i)} + c_{k(i+1)})/2$, where $c_{k(m+1)} = c_{k(1)}$.

We consider a Gauss diagram equipped with the oriented chords $\overrightarrow{Q_iP_i}$ ($i = 1, 2, \ldots, m$) and any signs on them. The Gauss diagram presents a non-trivially $p$-colored diagram such that $P_i$ and $Q_i$ correspond to lower and upper crossings, respectively. Then we see that $G$ is the pallet graph of the $p$-colored diagram. 

**Proof of Theorem 1.2.** By Lemmas 3.1 and 3.2, there is a non-trivially $p$-colored virtual knot diagram $(D, \gamma)$ such that its pallet graph $G$ has exactly $\lfloor \log_2 p \rfloor + 2$ vertices. The virtual knot $K$ presented by $D$ satisfies $C_p(K) \leq N(D, \gamma) = \lfloor \log_2 p \rfloor + 2$. The opposite inequality follows by Theorem 1.1.

**Remark 3.3.** (i) Several statements proved in this paper hold even for any odd composite $p$.

(ii) It is an open question whether any $p$-colorable knot $K$ satisfies

$$C_p(K) = \lfloor \log_2 p \rfloor + 2.$$ 

The equality holds for $p = 3, 5, 7$ (cf. [6, 7]).

(iii) Let $c(K)$ denote the crossing number of $K$. Since $c(K) \geq C_p(K)$, any $p$-colorable knot $K$ satisfies

$$c(K) \geq \lfloor \log_2 p \rfloor + 2$$

by Theorem 1.1. It is an open question whether the equality does not hold for other than the trefoil knot ($p = 3$) and the figure-eight knot ($p = 5$).

**References**


Takuji Nakamura
Department of Engineering Science,
Osaka Electro-Communication University,
Hatsu-cho 18-8, Neyagawa, 572-8530, Japan
E-mail: n-takuji@isc.osakac.ac.jp

Yasutaka Nakanishi
Department of Mathematics, Kobe University,
Rokkodai-cho 1-1, Nada-ku, Kobe 657-8501, Japan
E-mail: nakanisi@math.kobe-u.ac.jp

Shin Satoh
Department of Mathematics, Kobe University,
Rokkodai-cho 1-1, Nada-ku, Kobe 657-8501, Japan
E-mail: shin@math.kobe-u.ac.jp