Problem 1: Use the Reduced Row-Echelon form to find all solutions of the equations

\[
\begin{align*}
    x + 3y + z &= 3 \\
    2x + 5y + z &= 8 \\
    3x + 8y + 2z &= 11
\end{align*}
\]

You must show all your steps and work for credit.

Problem 2: Find the general solution of

\[
\begin{bmatrix}
    1 & 3 & 3 \\
    2 & 6 & 9 \\
    -1 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix} =
\begin{bmatrix}
    1 \\
    5 \\
    5
\end{bmatrix}
\]

Problem 3: (a) Find the row-reduced echelon form of

\[
\begin{bmatrix}
    1 & 2 & 3 \\
    4 & 5 & 6 \\
    7 & 8 & 9
\end{bmatrix}
\]

(b) What are the solutions of the system? Indicate which are the free variables, which are the dependent variables.

Problem 4: Given the two equations

\[
\begin{align*}
    x + 2y + 3z - 3w &= 1 \\
    4x + 5y + 6z - 6w &= 1 \\
    7x + 8y + 9z - 8w &= 1
\end{align*}
\]

(a) Give the Reduced Row-Echelon form of the associated augmented matrix.

(b) Which are the free variables? Which are the dependent variables?

(c) Give the general solution of the system of equations.

Problem 5: Given the two equations

\[
\begin{align*}
    x + 2y + 3z - 4w &= 2 \\
    2x + 4y + 3z + w &= 5
\end{align*}
\]

Use the method of row reduction to solve the system. Indicate which are the free variables, which are the dependent variables, and the geometric interpretation of the solution.

Problem 6: Let \( a, b, c \) be constants, and consider the system of equations

\[
\begin{align*}
    3x + 3y + z &= a \\
    x + y + 2z &= b \\
    5x + 5y &= c
\end{align*}
\]

Find the equation that the constants \( a, b, c \) must satisfy so that these equations are consistent.

---

**Gaussian Elimination and Row-Echelon Form**
Matrix Algebra and Manipulating Matrices

Problem 1: In each case, give an example of a matrix which is
• not the identity matrix
• not the zero matrix,
and satisfies:

\( A \) is a \( 2 \times 2 \) diagonal matrix
\( B \) is a \( 2 \times 2 \) orthogonal matrix
\( C \) is a \( 2 \times 2 \) symmetric matrix with no inverse.
\( D \) is a \( 2 \times 2 \) matrix with rank 1.
\( E \) is a \( 2 \times 2 \) diagonal matrix with an inverse.

\( A \) is a \( 2 \times 2 \) diagonal matrix which is

not the zero matrix
not the identity matrix

Problem 1: In each case, gives an example of a matrix which is

Matrix Determinants

Problem 2: Find the determinant of the matrix

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 4 \\
1 & 3 & 8
\end{bmatrix}
\]

Problem 3: Calculate the determinant of the matrix

\[
\begin{bmatrix}
3 & 1 & 0 & 1 \\
6 & 1 & 0 & 1 \\
6 & 1 & 0 & 0 \\
2 & 3 & 1 & 0
\end{bmatrix}
\]

Problem 4: Find the determinant of the matrix

\[
\begin{bmatrix}
5 & 1 \\
6 & 2
\end{bmatrix}
\]

Problem 5: Given the matrices \( A \), \( B \), \( C \), and \( D \):

\[
\begin{bmatrix}
4 & 5 \\
6 & 8
\end{bmatrix} = C,
\begin{bmatrix}
5 & 0 \\
6 & 7
\end{bmatrix} = D,
\begin{bmatrix}
1 & 3 & 2 \\
2 & 3 & 1
\end{bmatrix} = A
\]

Problem 1: Calculate the following determinants:

\[ |A|, |B|, |C| \]
Problem 6:  

(a) Find the determinant of the matrix 
\[
\begin{bmatrix}
111 \\
124 \\
137 \\
\end{bmatrix}
\]

(b) Use the solution to part (a) to explain how many solutions the equation \( Ax = b \) has, where \( b = Ax \).

(c) Find the determinant of the matrix 
\[
\begin{pmatrix}
2 & 3 & 1 \\
4 & 2 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]

(d) Find the determinant of the matrix 
\[
\begin{pmatrix}
|A^T - v| \ (g) \\
|1 - v^T A| \ (i) \\
|1 - v \cdot v| \ (e) \\
|\bar{B} \cdot v| \ (d) \\
|\bar{B} \cdot \bar{v}| \ (c) \\
\end{pmatrix}
\]
Matrix Inverses

Problem 1:
(a) Find the inverse (by any method) of
\[
\begin{bmatrix}
1 & 2 \\
3 & 5
\end{bmatrix}
\]
(b) Use the above to express the solutions of \( AX = \vec{b} \) in terms of the constants \( b_1 \) and \( b_2 \).

Problem 2:
Give the formula for the inverse of
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

Problem 3:
Use the method of Gaussian Elimination to find the inverse for
\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 9 & 12
\end{bmatrix}
\]

Problem 4:
Use the method of Cofactors to find the inverse for
\[
\begin{bmatrix}
-1 & 2 & 1 \\
2 & -1 & 2 \\
1 & 2 & -1
\end{bmatrix}
\]

Problem 5:
(a) Find the inverse of the following matrices: (and check your answers.)
(b) Do not use a calculator – you will be required to show all your work and computations.

Problem 6:
For what values of the variable \( \lambda \) does the matrix \( D \) below have an inverse?

\[
\begin{bmatrix}
3 & -1 & 0 \\
-\lambda & 3 & 1 \\
0 & 2 & -\lambda
\end{bmatrix}
\]

Problem 7:
Let \( A \) be an \( n \times n \) matrix. Suppose that the system of equations \( AX = \vec{0} \) has a unique solution. Explain why the inverse \( A^{-1} \) must exist.
Vector Spaces and Subspaces

Problem 1: Consider the subset of vectors in \( \mathbb{R}^2 \) given by
\[
S = \{ (x, x^2) \mid x \text{ is any real number} \}
\]
Is \( S \) a vectors subspace? Justify your answer carefully.

Problem 2: Let \( A \) be a square matrix with \( n \) rows and \( n \) columns. What are the four fundamental subspaces associated to \( A \)?

Problem 3: Let \( V \) be the space of all real-valued functions of \( x \). Show the solution set of the equation
\[
f'(x) = xf(x)
\]
is a subspace of \( V \).

Problem 4: Let \( V \) be the space of all real-valued functions of \( x \). Let \( W \) be the subset of all functions \( f \) which are solutions of the differential equation \( f'' + 5f = 0 \). Show that the solution set \( W \) is a subspace of \( V \).

Problem 5: Let \( A \) be an \( m \times n \) matrix with \( m \) rows and \( n \) columns. What are the four fundamental subspaces associated to \( A \)? Give the definition of each of the following:

- \( \text{Col}(A) \): the column space of \( A \)
- \( \text{Row}(A) \): the row space of \( A \)
- \( \text{Null}(A) \): the null space of \( A \)
- \( \text{Conull}(A) \): the conull space of \( A \)

\[
(x)f = (x)f
\]
Problem 1: Find a basis for the subspace of \( \mathbb{R}^3 \) spanned by the vectors \( \vec{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \), \( \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \), \( \vec{u}_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \).

Problem 2: In the space \( \mathcal{P}_3 \) of polynomials of degree 2 or less, are the "vectors" \( \{1+x, 1-x, 1+x+x^2\} \) linearly dependent or independent?

Problem 3:

(a) For \( A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \) find a basis for the row space and the column space.

(b) Is \( A \vec{x} = \vec{b} \) solvable for all \( \vec{b} \)?

Problem 4: For the vectors \( \vec{w}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \), \( \vec{w}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \), and \( \vec{x} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix} \) is \( \vec{x} \) in the span of \( \{\vec{w}_1, \vec{w}_2\} \)? If so, write \( \vec{x} \) as a linear combination of \( \{\vec{w}_1, \vec{w}_2\} \).

Problem 5: Is \( [1, 2, 3]^T \) in the span of \( [4, 0, 5]^T \) and \( [6, 0, 7]^T \)?

Problem 6:

(a) Find a basis for the subspace of \( \mathbb{R}^4 \) spanned by the vectors \( \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix} \), \( \vec{v}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix} \), \( \vec{v}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix} \).

(b) What is the dimension of the span of the vectors \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \)?

Problem 7: Do the "vectors" \( \{x+1, x-1, x^2+1, x+1\} \) span the space of polynomials of degree at most 2?

Problem 8: Find a basis for the subspace of \( 2 \times 2 \) matrices that span the space of polynomials of degree \( \leq 1 \) in the variables \( x \).

Problem 9: In the space of polynomials of degree at most 2, are the vectors \( \{x^2, x, 1\} \) linearly dependent or independent?
Problem 1: Let $A$ be an $m \times n$ matrix. Let

- $\text{Col}(A)$ denote the column space of $A$.
- $\text{Row}(A)$ denote the row space of $A$.
- $\text{Null}(A)$ denote the null space of $A$.
- $\text{Conull}(A)$ denote the co-null space of $A$.

For each of the following questions, your answer should be one of the above 4 spaces. Justify your answer by stating why you think it is correct.

(a) The set of vectors perpendicular to the column space of $A$ is what space?

(b) The vector equation $A\vec{x} = \vec{b}$ has a unique solution if what space is $\{0\}$?

(c) The set of vectors perpendicular to the null space of $A$ is what space?

(d) The vector equation $A\vec{x} = \vec{b}$ has a solution if what subspace of $A$ is contained in $\vec{b}$?

Problem 2: Give a basis for the column space, row space and null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 4 & 0 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 2 & 1 & 1 \end{bmatrix} = B$$

Problem 3: Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix} = B$$

Problem 4: a) Find a basis for the column space of $A$.

b) Find a basis for the perpendicular space $\text{Col}(A)^\perp$.

c) Find a basis for $\text{Conull}(A)$.

Problem 5: Let $B$ be the matrix

$$B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix}$$

Find a basis for the four fundamental spaces of $B$: the column space, the row space, the null space and the co-null space ($\text{Null}(B^T)$).
Problem 6:
Given the system of equations

\[ \begin{align*}
    x + y + z &= c_1 \\
    x + 2y + 2z &= c_2 \\
    x + 3y + 3z &= c_3
\end{align*} \]

(a) For what values of \( \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \) does the system have a solution?

(b) If there exists a solution for a given \( \vec{c} \), how many are there?

(c) Find the basis for the null-space of the matrix associated to the system of equations above.

(d) What is the dimension of the null-space \( N(A) \) and the column space \( C(A) \) of the matrix \( A \)?

(e) Find a basis for the null-space of the matrix \( A \).

\[ \begin{bmatrix} 6 & 3 & 9 \\ 4 & 6 & 2 \\ 0 & 3 & 2 \end{bmatrix} \]

Problem 7:
\( A \) is a 3 \( \times \) 5 matrix and \( L: \mathbb{R}^5 \rightarrow \mathbb{R}^3 \) is defined by \( L(\vec{v}) = A \cdot \vec{v} \). Suppose that \( A \) has rank 3.

(a) What is the dimension of the kernel of \( L \)?

(b) What is the dimension of the range of \( L \)?

(c) Explain your answers in terms of how you would find bases of these spaces if the matrix of \( A \) were given.

\[ \begin{align*}
    \text{Problem 8:} \ Let \ 
    \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 6 & 2 \\ 6 & 3 & 9 & 3 \end{bmatrix} \]
\]

(a) Give the Reduced Row Echelon form of the matrix \( A \).

(b) Find a basis for the null-space of the matrix \( A \).

(c) Find a basis for the column space of the matrix \( A \).

(d) What is the dimension of the null-space \( N(A) \) and the column space \( C(A) \)?

(e) Answer True or False, and explain your answer:

\[ \begin{align*}
    \text{The equation } A\vec{x} = \vec{b} \text{ has a solution for every vector } \vec{b} \in \mathbb{R}^3. \]
\]
Change of Basis and Coordinates

Problem 1: Find the coordinates of \( \vec{p} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) with respect to the basis \( \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

Problem 2: Find the new coordinates \([a, b, c]_T\) of the point \( \vec{x} = [7, 5, 6]_T \) with respect to the basis for \( \mathbb{R}^3 \) given by the vectors \( \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \).

Problem 3: Given the vectors in \( \mathbb{R}^2 \)

(a) Find the transition matrix \( S \) corresponding to change of basis from \( \{\vec{v}_1, \vec{v}_2\} \) to \( \{\vec{u}_1, \vec{u}_2\} \).

(b) Find the coordinate expression of \( \vec{p} = 3\vec{v}_1 - \vec{v}_2 \) with respect to the basis \( \{\vec{u}_1, \vec{u}_2\} \).
Problem 1: Let $P_3$ be the space of polynomials of degree 2. Show that the map $L: P_3 \to P_3$ given by

$$L(p(x)) = p(x) - x \cdot p'(x)$$

is linear. (Here, $p'(x)$ denotes the first derivative of the polynomial $p(x)$.)

Problem 2: Find the matrix, in the standard basis for $\mathbb{R}^3$, for the linear transformation $L: \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 2x - y - z \\ x - 2y + z - x + 3y + 2z \end{bmatrix}$.

Problem 3: Define the linear transformation $L: P_3 \to P_3$ by

$$L(p(x)) = xp''(x) - 2xp'(x) + p(x)$$

Find the matrix representing $L$ with respect to the basis $\{1, x, x^2\}$ of $P_3$.

Problem 4: Find the matrix representation for the linear transformation $L: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 4x - y - x + 4y \end{bmatrix}$ with respect to the basis $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Problem 5: Let $V$ be the space of functions with basis $\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$. Define the linear transformation $L: V \to V$ by

$$L(f) = f'' + f' - 4f$$

(a) Find the matrix representing $L$ with respect to the given basis.

(b) Find the kernel of $L$.

Problem 6: Let a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(v_1, v_2, v_3) = (3v_1 + 2v_2 + v_3, 2v_1 + v_2, v_2)$$

Give the matrix (in the standard basis) for $T$.

Problem 7: Let $V$ be the space spanned by the functions $\{e^x, e^{2x}, e^{3x}\}$, and let $L: V \to V$ be the linear transformation defined by $L(f) = f' - 2f$. Find the matrix of $L$ with respect to the basis $\{e^x, e^{2x}, e^{3x}\}$ of $V$.

(a) Find the matrix representing $L$ with respect to the basis $\{e^x, e^{2x}, e^{3x}\}$ of $V$.

(b) Find the kernel of $L$.

Problem 8: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$(x, y) \mapsto A \begin{bmatrix} x \\ y \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$. Find the matrix of $T$ with respect to the new basis $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
Problem 1: The linear transformation $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ with respect to the standard basis $\{ \vec{e}_1, \vec{e}_2 \}$ of $\mathbb{R}^2$. Find the matrix of $L$ with respect to the new basis $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Problem 2:

a) Find the matrix representation $A$ with respect to the standard basis $\{ \vec{e}_1, \vec{e}_2 \}$ of $\mathbb{R}^2$ for the linear transformation $L : (x, y) \mapsto (4x - y - x + 4y)$.

b) Find the matrix representation $B$ of $L$ with respect to the basis $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 3:

Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by $L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4y + 6z - 2x - 3y \\ x \\ x + 2y + z \end{bmatrix}$.

a) Find the matrix representing $L$ with respect to the standard basis $\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ of $\mathbb{R}^3$.

b) Use the answer to part a) to find the matrix representing $L$ with respect to the new basis $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Problem 4:

a) Given the vectors in $\mathbb{R}^2$ $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find the transition matrix $S$ corresponding to change of basis from $\{ \vec{v}_1, \vec{v}_2 \}$ to $\{ \vec{u}_1, \vec{u}_2 \}$.

b) The linear transformation $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has a matrix representation $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ with respect to the basis $\{ \vec{u}_1, \vec{u}_2 \}$. Find the matrix representation $B$ of $L$ with respect to the basis $\{ \vec{v}_1, \vec{v}_2 \}$.

Problem 5:

For the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, find the transition matrix $S$ corresponding to change of basis from the standard basis $\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ of $\mathbb{R}^3$ to the new basis $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$.

b) Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z + \frac{1}{2}y + x \\ \frac{1}{2}y - xz + z + y \\ 2y + 2y \end{bmatrix}$. Find the matrix representing $L$ with respect to the standard basis $\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ of $\mathbb{R}^3$.
Problem 6:

(a) Let \( \{ \vec{v}_1, \vec{v}_2 \} \) be a basis for \( \mathbb{R}^2 \), and let \( L \) be a linear transformation of \( \mathbb{R}^2 \) so that

\[ L(c_1 \vec{v}_1 + c_2 \vec{v}_2) = (c_1 + 3c_2) \vec{v}_1 + (2c_1 + 4c_2) \vec{v}_2. \]

Find the matrix representing \( L \) with respect to the basis \( \{ \vec{v}_1, \vec{v}_2 \} \).

(b) Suppose that \( \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Find the matrix representing \( L \) with respect to the standard basis of \( \mathbb{R}^2 \).

Problem 7:

Let \( A = \begin{bmatrix} 7 & -2 \\ 15 & -4 \end{bmatrix} \). Define the linear map \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( L(\vec{x}) = A\vec{x} \).

(a) Find the matrix \( B \) for the linear map \( L \) with respect to the new basis \( \vec{u}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \).

(b) Suppose that \( \vec{p} \) has coordinates \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \) with respect to the basis \( \{ \vec{u}_1, \vec{u}_2 \} \). Find \( L(\vec{p}) \) with respect to the basis \( \{ \vec{u}_1, \vec{u}_2 \} \).

(c) Suppose that \( \vec{p} = \vec{u}_2 \). Find \( L^{100}(\vec{u}_2) = L(\cdots(L(\cdots(L(\vec{u}_2))))) \) (\( n \) times).

\( \cdot \) \( \cdot \) \( \cdot \)
Problem 1: Find the eigenvalues and corresponding eigenvectors for
\[
\begin{bmatrix}
1 & 2 & 0 \\
3 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

Problem 2: Let \( A \) be symmetric. Find an orthogonal matrix \( S \) with \( S^{-1}AS \) diagonal.

(a) Find the eigenvalues for these eigenvalues.

(b) Find the eigenvectors and corresponding eigenvectors for \( A \).

Problem 3: Given the differential equations with initial conditions
\[
\begin{align*}
x' &= 3x + 4y; \\
y' &= -2x - 3y;
\end{align*}
\]

(a) Find the general solution of the differential system.

(b) Given the particular solution when \( y(0) = 3 \) and \( y(0) = 1 \).

Problem 2: a) Find the eigenvectors for these eigenvalues.

b) Note \( A \) is symmetric. Find an orthonormal matrix \( S \) with \( S^{-1}AS \) diagonal.

Problem 1: Given the differential equations with initial conditions

\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Eigenvalues, Eigenvectors and Eigenspaces
Problem 1:

For the matrix

\[
A = \begin{bmatrix}
1 & -1 \\
0 & 2
\end{bmatrix}
\]

(a) Find 2 \times 2 matrices \(S\) and \(D\) such that \(A = S \cdot D \cdot S^{-1}\).

(b) Use your answer to part (a) to calculate \(A^{10}\).

(c) Find the eigenvalues and eigenvectors for \(A\).

Problem 2:

For \(A = \begin{bmatrix}
1 & -1 \\
0 & 2
\end{bmatrix}\), find the matrix \(e^{A}\). (Your answer should be a 2 \times 2 matrix.)

Problem 3:

For \(A = \begin{bmatrix}
1 & 2 \\
-1 & 4
\end{bmatrix}\), find the 2 \times 2 matrix \(e^{tA}\).

Problem 4:

For \(A = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}\),

(a) Calculate \(A^2, A^3, A^4, A^5\).

(b) Find the eigenvalues and eigenvectors for \(A\).

(c) Use your answer to part (b) to calculate \(A^{10}\).

(d) Use your answer to part (b) to get a formula for \(A^n\) when \(n\) is a positive integer.