

Gaussian Elimination and Row-Echelon Form

Problem 1: Use the Reduced Row-Echelon form to find all solutions of the equations

$$\begin{aligned} x + 3y + z &= 3 \\ 2x + 5y + z &= 8 \\ 3x + 8y + 2z &= 11 \end{aligned}$$

You must show all your steps and work for credit.

Problem 2: Find the general solution of

$$\begin{bmatrix} 1 & 3 & 3 & -1 \\ 3 & 3 & 6 & 2 \\ -3 & 3 & 9 & -1 \\ 3 & 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \\ 5 \end{bmatrix}.$$

Problem 3: (a) Find the row-reduced echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

(b) What are the solutions of the system $Ax = 0$? (Check!)

Problem 4: Given the equations

$$\begin{aligned} x + 2y + 3z - 3w &= 1 \\ 4x + 5y + 6z - 6w &= 1 \\ 7x + 8y + 9z - 8w &= 1 \end{aligned}$$

a) Give the Reduced Row-Echelon form of the associated augmented matrix.

b) Which are the free variables? Which are the dependent variables?

c) Give the general solution of the system of equations.

Problem 5: Given the two equations

$$\begin{aligned} x + 2y + 3z - 4w &= 2 \\ 2x + 4y + 3z + w &= 5 \end{aligned}$$

Use the method of row reduction to solve the system. Indicate which are the free variables, which are the dependent variables. What is the geometric interpretation of the solution?

Problem 6: Let a, b, c be constants, and consider the system of equations

$$\begin{aligned} 3x + 3y + z &= a \\ x + y + 2z &= b \\ 5x + 5y &= c \end{aligned}$$

Find the equation that the constants a, b, c must satisfy so that these equations are consistent.

Problem 1: In each case, give an example of a matrix which is

- not the identity matrix
- not the zero matrix,

and satisfies:

- a) A is a 2×2 diagonal matrix with an inverse.
 - b) B is a 2×2 matrix with rank 1.
 - c) C is a 2×2 symmetric matrix with no inverse.
 - d) O is a 2×2 orthogonal matrix.
- So, you must find four matrices A , B , C and O .

Matrix Determinants

Problem 1: Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

Problem 2: Use either the definition of determinant in terms of cofactors, or the method of row operations, to calculate the determinant of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Problem 3: Calculate the determinant of the matrix $B = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{bmatrix}$.

Problem 4: Find the determinant of the matrix A_3 where $A = \begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix}$.

Problem 5: Given the matrices $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$, calculate the following determinants:

- a) $|A|$, $|B|$ and $|C|$
- b) $|ABC^2|$

- c) $|7 \cdot B|$
- d) $|A^T \cdot B|$
- e) $|B \cdot C^{-1}|$
- f) $|B^T C A^{-1}|$
- g) $|A - B|$

Problem 6: a) Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix}$.

b) Use the solution to part a) to explain how many solutions the equation $A\vec{x} = \vec{b}$ has, where

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 1: a) Find the inverse (by any method) of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

b) Use the above to express the solutions of $A\vec{x} = \vec{b}$ in terms of the constants b_1 and b_2 .

Problem 2: Give the formula for the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Problem 3: Use the method of Gaussian Elimination to find the inverse for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 9 & 12 \end{bmatrix}$.

Problem 4: Use the method of Cofactors to find the inverse for $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$.

Problem 5: Find the inverse of the following matrices (and check your answers).

Do not use a calculator – you will be required to show all your work and computations.

a) $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

b) $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Problem 6: For what values of the variable λ does the matrix **D** below have an inverse? **Explain your answer!**

$$D = \begin{bmatrix} 3 - \lambda & 3 & 1 \\ 0 & 2 - \lambda & 5 \\ 0 & 0 & \lambda + 1 \end{bmatrix}$$

Problem 7: Let A be an $n \times n$ matrix. Suppose that the system of equations $AX = 0$ has a unique solution. Explain why the inverse A^{-1} has to exist.

Problem 1: Consider the subset of vectors in \mathbf{R}^2 given by

$$S = \{(x, x^2) \mid \text{where } x \text{ is any real number}\}$$

Is S a vector subspace? Justify your answer carefully.

Problem 2: Is the set $\left\{ \begin{bmatrix} x \\ x \\ x_3 \end{bmatrix} \mid \text{where } x \in \mathbb{R} \right\}$ a vector subspace of \mathbb{R}^3 ? Justify your answer.

Problem 3: Let V be the space of real-valued functions of x . Show the solution set S of the equation

$$f'(x) = xf(x)$$

is a subspace of V .

Problem 4: Let V be the space of all differentiable functions on the line. Let W be the subset of all functions f which are solutions of the differential equation $f'' + 5f = 0$. Show that the solution set W is a subspace of V .

Problem 5: Let $A_{m \times n}$ be a matrix with m rows and n columns. What are the four fundamental subspaces associated to A ? Give the definition of each of the following:

- $\text{Col}(A)$ = the column space of A .
- $\text{Row}(A)$ = the row space of A .
- $\text{Null}(A)$ = the null space of A .
- $\text{Cnull}(A)$ = conull space of A .

Problem 1: Find a basis for the subspace V of \mathbb{R}^3 spanned by the vectors

$$w_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Problem 2: In the space \mathcal{P}_3 of polynomials of degree 2 or less, are the “vectors” $\{1 + x, 1 - x, 1 + x + x^2\}$ linearly dependent or independent?

Problem 3: a) For $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 2 \\ 3 & 1 & -1 & 6 \end{bmatrix}$ find a basis for the row space and the column space.

b) Is $Ax = b$ solvable for all b ?

Problem 4: For the vectors

$$w_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}$$

Is x in the span of $\{w_1, w_2\}$? If so, write x as a linear combination of $\{w_1, w_2\}$.

Problem 5: Is $[1, 2, 3]^T$ in the span of $[4, 0, 5]^T$ and $[6, 0, 7]^T$?

Problem 6: a) Find a basis for the subspace of \mathbb{R}^4 spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

b) What is the dimension of the span of the vectors $\{v_1, v_2, v_3, v_4\}$?

Problem 7: Do the “vectors” $1 + x, 1 - x, x^2$ span the space \mathcal{P}_3 of polynomials of degree at most 2?

Problem 8: Find a basis for the subspace of 2×2 matrices $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ satisfying $a_{1,1} + a_{2,2} = 0$.

Problem 1: A is an $m \times n$ matrix. Let

- $\text{Col}(A)$ denote the column space of A
- $\text{Row}(A)$ denote the row space of A
- $\text{Null}(A)$ denote the null space of A
- $\text{Conull}(A)$ the co-null space of A

For each of the following questions, your answer should be one of the above 4 spaces. Justify your answer by stating why you think it is correct.

- a) The set of vectors perpendicular to the column space is what space?
- b) The vector equation $A\vec{x} = \vec{b}$ has a solution if \vec{b} belongs to what subspace?
- c) The set of vectors perpendicular to the row space is what space?
- d) The vector equation $A\vec{x} = \vec{b}$ has a unique solution if what space is $\{0\}$?
- e) What number do you get if you add the dimensions of all 4 spaces?

Problem 2: Give a basis for the column space, row space and null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 4 & 7 & 1 \\ 2 & 4 & 4 & 2 \\ 2 & 4 & 4 & 2 \end{bmatrix}$$

Problem 3: Find a basis for the null-space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 5 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix}$$

Problem 4: a) Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

b) Find a basis for the perpendicular space $\text{Col}(A)^\perp$

c) Find a basis for $\text{Conull}(A)$

Problem 5: Let $B = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & 2 & 4 \\ 3 & 6 & -2 & 1 \\ 10 & 4 & 1 & 0 \end{bmatrix}$

Find a basis for the four fundamental spaces of B : the column space, the row space, the null space and the co-null space (the null space of the transpose B^T).

Problem 6: Given the system of equations

$$\begin{aligned} x + y + z &= c_1 \\ x + 2y + 2z &= c_2 \\ x + 3y + 3z &= c_3 \end{aligned}$$

a) For what values of $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ does the system have a solution?

b) If there exists a solution for a given \vec{c} , how many are there?

c) Find the basis for the co-null space of the matrix associated to the system of equations above.

d) What is the relation between your answers to part a) and c)?

Problem 7: A is a 3×5 matrix and $L: \mathbb{R}^5 \rightarrow \mathbb{R}^3$ is defined by $L(\vec{v}) = A \cdot \vec{v}$. Suppose that A has rank 3.

a) What is the dimension of the kernel of L ?

b) What is the dimension of the range of L ?

Explain your answers in terms of how you would find basis of these spaces if the matrix of A were given!

Problem 8: Let $A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 6 & 2 \\ 6 & 3 & 9 & 3 \end{bmatrix}$

a) Give the **Reduced Row Echelon form** of the matrix A

b) Find a basis for the **null-space** of the matrix A

c) Find a basis for the **column space** of the matrix A

d) What is the **dimension** of the null-space $N(A)$ and the column space $C(A)$?

e) Answer **True** or **False**, and explain your answer:

The equation $A\vec{x} = \vec{b}$ has a solution for every vector $\vec{b} \in \mathbb{R}^3$.

Change of Basis and Coordinates

Problem 1: Find the coordinates of $\vec{p} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ with respect to the basis $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Problem 2: Find the new coordinates $[a, b, c]^T$ of the point $\vec{x} = [7, 5, 6]^T$ with respect to the basis for \mathbb{R}^3 given by the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Problem 3: Given the vectors in \mathbb{R}^2

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- a) Find the transition matrix S corresponding to change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{u}_1, \vec{u}_2\}$.
 b) Find the coordinate expression of $\vec{p} = 3\vec{v}_1 - \vec{v}_2$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$.

Problem 1: Let \mathcal{P}_3 be the space of polynomials of degree 2. Show that the map $L: \mathcal{P}_3 \rightarrow \mathcal{P}_3$ given by

$$L(p(x)) = p(x) - x \cdot p'(x)$$

is linear. (Here, $p'(x)$ denotes the first derivative of the polynomial $p(x)$.)

Problem 2: Find the matrix, in the standard basis for \mathbb{R}^3 , for the linear transformation

$$L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - y - z \\ x - 2y + z \\ -x + 3y + 2z \end{bmatrix}.$$

b) Find the kernel of L

Problem 3: Define the linear transformation $L: \mathcal{P}_3 \rightarrow \mathcal{P}_3$ by

$$L(p(x)) = x p''(x) - 2x p'(x) + p(x)$$

Find the matrix representing L with respect to the basis $\{1, x, x^2\}$ of \mathcal{P}_3 .

Problem 4: Find the matrix representation for the linear transformation

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - y \\ -x + 4y \end{bmatrix}.$$

with respect to the basis $\underline{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Problem 5: Let V be the space of functions with basis $\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$.

Define the linear transformation $L: V \rightarrow V$ by

$$L(f) = f'' + f' - 4f$$

a) Find the matrix representing L with respect to the given basis.

b) Find the kernel of L

Problem 6: Let a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$$T(v_1, v_2, v_3) = (3v_1 + 2v_2 + v_3, 2v_1 + v_2, v_2).$$

Give the matrix (in the *standard* basis) for T .

Problem 7: Let V be the vector space spanned by the functions $\{e^x, e^{2x}, e^{3x}\}$,

and let $L: V \rightarrow V$ be the linear transformation defined by $L(f) = f' - 2f$.

a) Find the matrix representing L with respect to the basis $\{e^x, e^{2x}, e^{3x}\}$ of V .

b) Find the kernel of L .

Problem 8: Define the linear transformation $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L(\underline{v}) = A\underline{v}$ where $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.
Find the matrix of L with respect to the new basis $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Problem 1: The linear transformation $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ with respect to the standard basis $\{e_1, e_2\}$ of \mathbb{R}^2 . Find the matrix of L with respect to the new basis

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 2: a) Find the matrix representation A with respect to the standard basis $\{e_1, e_2\}$ of \mathbb{R}^2 for the linear transformation

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - y \\ -x + 4y \end{bmatrix}.$$

b) Find the matrix representation B of L with respect to the basis $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 3: Let $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by $\mathbf{L} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4y + 6z \\ -2x - 3y \\ x + 2y + z \end{bmatrix}$.

a) Find the matrix representing L with respect to the standard basis $\{e_1, e_2, e_3\}$ of \mathbb{R}^3 .

b) Use the answer to part a) to find the matrix representing L with respect to the new basis

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Problem 4: Given the vectors in \mathbb{R}^2

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a) Find the transition matrix S corresponding to change of basis from $\{v_1, v_2\}$ to $\{u_1, u_2\}$.

b) The linear transformation $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has a matrix representation $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ with respect to the

basis $\{u_1, u_2\}$. Find the matrix representation B of L with respect to the basis $\{v_1, v_2\}$.

Problem 5: For the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

a) Find the transition matrix S corresponding to the change of basis from the standard basis $\{e_1, e_2, e_3\}$ of \mathbb{R}^3 to the new basis $\{v_1, v_2, v_3\}$.

b) Let $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$L(v_1) = v_1, \quad L(v_2) = 2 \cdot v_2, \quad L(v_3) = 3 \cdot v_3$$

Find the matrix representing L with respect to the standard basis $\{e_1, e_2, e_3\}$ of \mathbb{R}^3 .

Problem 6: a) Let $\{v_1, v_2\}$ be a basis for \mathbb{R}^2 , and let L be a linear transformation of \mathbb{R}^2 so that

$$L(c_1v_1 + c_2v_2) = (c_1 + 3c_2)v_1 + (2c_1 + 4c_2)v_2$$

Find the matrix representing L with respect to the basis $\{v_1, v_2\}$.

b) Suppose that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find the matrix representing L with respect to the standard basis of \mathbb{R}^2 .

Problem 7: Let $A = \begin{bmatrix} 7 & -2 \\ 15 & -4 \end{bmatrix}$. Define the linear map $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L(x) = Ax$

a) Find the matrix B for the linear map L with respect to the new basis $u_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

b) Suppose that p has coordinates $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ with respect to the basis $\{u_1, u_2\}$. Find $L(p)$ with respect to the basis $\{u_1, u_2\}$.

c) Suppose that $p = u_2$. Find $L^{101}(u_2) = L(L(\dots(L(L(u_2))\dots)))$.

Eigenvalues, Eigenvectors and Eigenspaces

Problem 1: Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Problem 2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 0$.

a) Find the eigenvectors for these eigenvalues.

b) Note A is symmetric. Find an *orthogonal* matrix S with $S^{-1}AS = D$ diagonal.

Problem 3: Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$.

Systems of Differential Equations

Problem 1: Given the differential equations with initial conditions

$$\begin{aligned} x' &= 3x + 4y & ; & \quad x(0) = 1 \\ y' &= -2x - 3y & ; & \quad y(0) = 2 \end{aligned}$$

Find the functions $x(t)$ and $y(t)$.

Problem 2: a) Give the general solution of the differential system

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= -2y_1 + 4y_2 \end{aligned}$$

b) Give the particular solution when $y_1(0) = 3$ and $y_2(0) = 1$.

Problem 1: For the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

- a) Find 2×2 matrices S and D such that $A = S \cdot D \cdot S^{-1}$
 b) Use your answer to part a) to calculate A^5 .

Problem 2: For $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ find the matrix e^A . (Your answer should be a 2×2 matrix.)

Problem 3: For $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, find the 2×2 matrix e^{tA} .

Problem 4: For $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

- a) Calculate A^2, A^3, A^4, A^5 .
 b) Find the eigenvalues and eigenvectors for A .
 c) Use your answer to part b) to calculate A^{10} .
 d) Use your answer to part b) to get a formula for A^n when n is a positive integer.