New Journal of Physics
An Institute of Physics and Deutsche Physikalische Gesellschaft Journal

Quantum entanglement and topological entanglement

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New Journal of Physics **4** (2002) 73.1–73.18 (http://www.njp.org/)
Received 30 May 2002
Published 16 October 2002

Abstract. This paper discusses relationships between topological entanglement and quantum entanglement. Specifically, we propose that it is more fundamental to view topological entanglements such as braids as *entanglement operators* and to associate with them unitary operators that are capable of creating quantum entanglement.

1. Introduction

This paper discusses relationships between topological entanglement and quantum entanglement. The paper is an expanded version of [9]. Specifically, we propose that it is more fundamental to view topological entanglements such as braids as *entanglement operators* and to associate with them unitary operators that perform quantum entanglement. Then one can compare the way the unitary operator corresponding to an elementary braid has (or has not) the capacity to entangle quantum states. Along with this, one can examine the capacity of the same operator to detect linking. The detection of linking involves working with closed braids or with link diagrams. In both cases, the algorithms for computing link invariants are very interesting to examine in the light of quantum computing. These algorithms can usually be decomposed into one part that is a straight composition of unitary operators, and hence can be seen as a sequence of quantum computer instructions, and another part that can be seen either as preparation/detection, or as a quantum network with cycles in the underlying graph.

The paper is organized as follows. Section 2 discusses the basic analogy between topological entanglement and quantum entanglement. Section 3 proposes the viewpoint of braiding operators

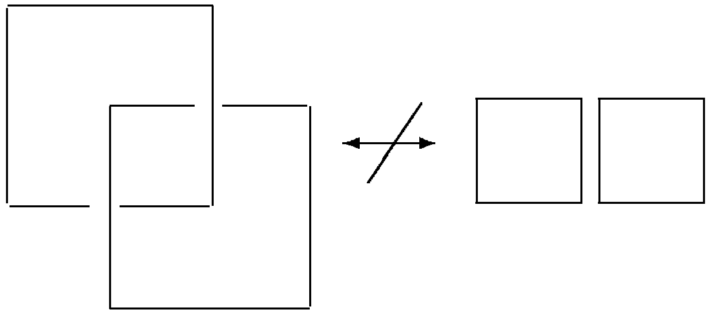

Figure 1. The Hopf link.

and gives a specific example of a unitary braiding operator, showing that it does entangle quantum states. Section 3 ends with a list of problems. Section 4 discusses the link invariants associated with the braiding operator R introduced in the previous section. Section 5 is a discussion of the structure of entanglement in relation to measurement. Section 6 is an introduction to the virtual braid group, an extension of the classical braid group by the symmetric group. We contend that unitary representations of the virtual braid group provide a good context and language for quantum computing. Section 7 is a discussion of ideas and concepts that have arisen in the course of this research. An appendix describes a unitary representation of the three-strand braid group and its relationship with the Jones polynomial. This representation is presented for contrast since it can be used to detect highly non-trivial topological states, but it does not involve any quantum entanglement.

2. The temptation of tangled states

It is quite tempting to make an analogy between topological entanglement in the form of linked loops in three-dimensional space and the entanglement of quantum states. A topological entanglement is a non-local structural feature of a topological system. A quantum entanglement is a non-local structural feature of a quantum system. Take the case of the Hopf link of linking number one (see figure 1). In this figure we show a simple link of two components and state its inequivalence to the disjoint union of two unlinked loops. The analogy that one wishes to draw is with a state of the form

$$\psi = (|01\rangle - |10\rangle)/\sqrt{2}$$

which is *quantum entangled*. That is, this state is not of the form $\psi_1 \otimes \psi_2 \in H \otimes H$ where H is a complex vector space of dimension two. Cutting a component of the link removes its topological entanglement. Observing the state removes its quantum entanglement in this case.

An example of Aravind [1] makes the possibility of such a connection even more tantalizing. Aravind compares the Borromean rings (see figure 2) and the GHZ state

$$|\psi\rangle = (|\beta_1\rangle|\beta_2\rangle|\beta_3\rangle - |\alpha_1\rangle|\alpha_2\rangle|\alpha_3\rangle)/\sqrt{2}.$$

The Borromean rings are a three-component link with the property that the triplet of components is indeed topologically linked, but the removal of any single component leaves a pair of unlinked rings. Thus, the Borromean rings are of independent intellectual interest as

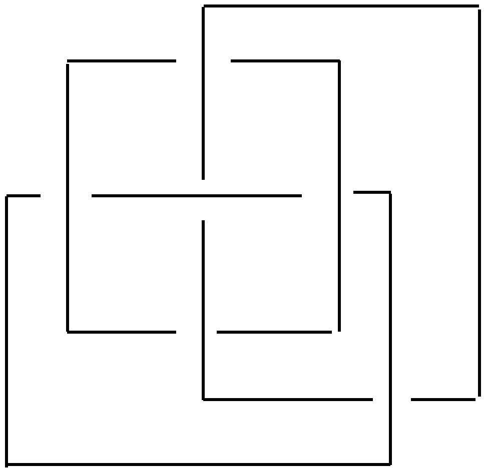

Figure 2. Borromean rings.

an example of a tripartite relation that is not expressed in terms of binary relations. The GHZ state can be viewed as an entangled superposition of three particles with (say) all their spins in the z direction. If we measure one particle of the three-particle quantum system, then the state becomes disentangled (that is, it becomes a tensor product). Thus, the GHZ state appears to be a quantum analogue to the Borromean rings!

However, Aravind points out that this analogy is basis dependent, for if one changes basis, rewriting to

$$|\psi\rangle = (|\beta_{1x}\rangle/\sqrt{2})(|\beta_2\rangle|\beta_3\rangle - |\alpha_2\rangle|\alpha_3\rangle)/\sqrt{2} + (|\alpha_{1x}\rangle/\sqrt{2})(|\beta_2\rangle|\beta_3\rangle + |\alpha_2\rangle|\alpha_3\rangle)/\sqrt{2},$$

where $|\beta_{1x}\rangle$ and $|\alpha_{1x}\rangle$ denote the spin-up and spin-down states of particle 1 in the x direction, then one sees that a measurement of the spin of particle 1 in the x direction will yield an entangled state of the other two particles. Thus, in this basis, the state $|\psi\rangle$ behaves like a triplet of loops such that each pair of loops is linked! Seeing the state as analogous to a specific link depends upon the choice of basis. From a physical standpoint, seeing the state as analogous to a link depends upon the choice of an observable.

These examples show that the analogy between topological linking and quantum entanglement is surely complex. One might expect a collection of links to exemplify the entanglement properties of a single quantum state. It is attractive to consider the question: *What patterns of linking are inherent in a given quantum state?* This is essentially a problem in linear algebra and should be investigated further. We will not pursue it in this paper.

On top of this, there is quite a bit of ingenuity required to produce links with given properties. For example, in figure 3 we illustrate a Brunnian link of four components. This link has the same property as the Borromean rings but for four components rather than three. Remove any component and the link falls apart. The obvious generalization of the GHZ state with this property just involves adding one more tensor product in the two-term formula. This raises a question about the relationship of topological complexity and algebraic complexity of the corresponding quantum state. The other difficulties with this analogy are that topological properties of linked loops are not related to quantum mechanics in any clear way. Nevertheless, it is clear that this is an analogy worth pursuing.

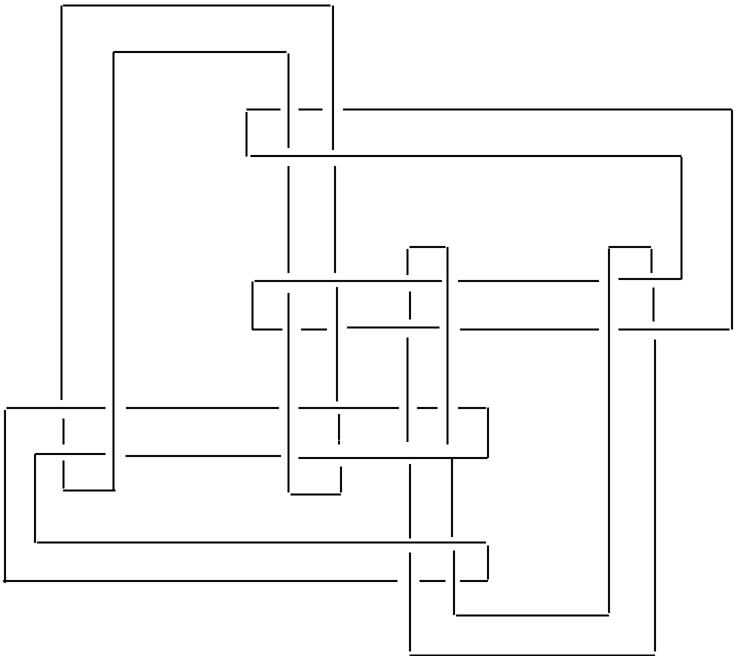

Figure 3. A Brunnian link.

3. Entanglement operators

Braids and the Artin braid group form a first instance in topology where a space (or topological configuration) is also seen as an *operator* on spaces and configurations. It is a shift that transmutes the elements of a topological category to morphisms in an associated category. While we shall concentrate on braids as an exemplar of this shift, it is worth noting that such a shift is the basis of quantum topology and topological quantum field theory, where spaces are viewed (through appropriate functors) as morphisms in a category analogous to a category of Feynman diagrams. This pivot from spaces to morphisms and back is the fundamental concept behind topological quantum field theory.

Braids are patterns of entangled strings. A braid has the form of a collection of strings extending from one set of points to another, with a constant number of points in each cross section. Braids start in one row of points and end in another. As a result, one can multiply two braids to form a third braid by attaching the end points of the first braid to the initial points of the second braid. Up to topological equivalence, this multiplication gives rise to a group, the Artin braid group B_n on n strands.

Each braid is, in itself, a pattern of entanglement. Each braid is an operator that operates on other patterns of entanglement (braids) to produce new entanglements (braids again).

We wish to explore the analogy between topological entanglement and quantum entanglement. From the point of view of braids this means *the association of a unitary operator with a braid that respects the topological structure of the braid and allows exploration of the entanglement properties of the operator*. In other words, we propose to study the analogy between topological entanglement and quantum entanglement by looking at *unitary representations of the Artin braid group*. It is not the purpose of this paper to give an exhaustive account of

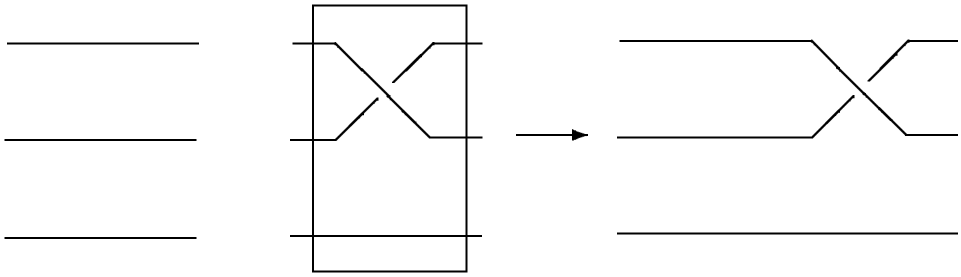

Figure 4. A braiding operator.

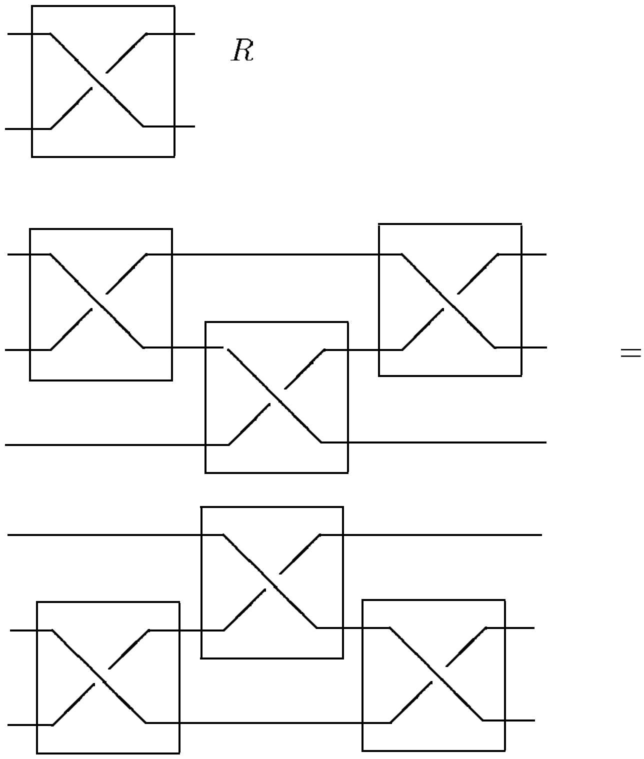

Figure 5. The Yang–Baxter equation.

such representations. Rather, we shall concentrate on one particularly simple representation and analyse the relationships between topological and quantum entanglement that are implicit in this representation. The main point for the exploration of the analogy is that, from the point of view of a braid group representation, each braid is seen as an operator rather than a state (see figure 4).

We will consider representations of the braid group constructed in the following manner. To an elementary two-strand braid there is associated an operator

$$R : V \otimes V \longrightarrow V \otimes V.$$

Here V is a complex vector space, and for our purposes, V will be two-dimensional so that V can hold a single qubit of information. One should think of the two input and two output lines from the braid as representing this map of tensor products. Thus the left endpoints of R as shown in figures 4–6 represent the tensor product $V \otimes V$ that forms the domain of R and the right

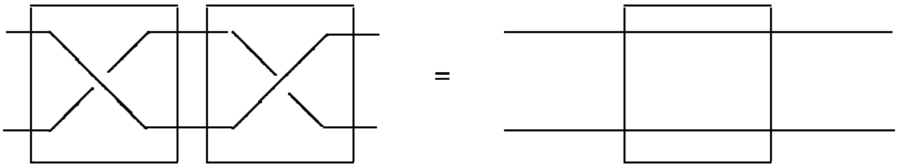

Figure 6. Inverses.

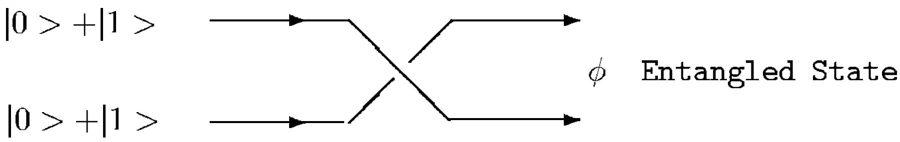

Figure 7. Braiding operator entangling a state.

endpoints of the diagram for R represent $V \otimes V$ as the range of the mapping. In the diagrams with three lines shown in figure 5, we have mappings from $V \otimes V \otimes V$ to itself. The identity shown in figure 5 is called the Yang–Baxter equation, and it reads algebraically as follows, where I denotes the identity transformation on V:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R).$$

This equation expresses the fundamental topological relation in the Artin braid group, and is the main requirement for producing a representation of the braid group by this method. We also need an inverse to R and this will be associated with the reversed elementary braid on two strands as shown in figure 6. One then defines a representation τ of the Artin braid group to automorphisms of $V^{\otimes n}$ by the equation

$$\tau(\sigma_k) = I \otimes \cdots \otimes I \otimes R \otimes I \cdots \otimes I,$$

where the R occupies the k and $k+1$ places in this tensor product. If R satisfies the Yang–Baxter equation and is invertible, then this formula describes a representation of the braid group. If R is unitary, then this construction provides a unitary representation of the braid group.

Here is the specific R matrix that we shall examine. The point of this case study is that R, being unitary, can be considered as a quantum gate *and* since R is the key ingredient in a unitary representation of the braid group, it can be considered as a operator that performs topological entanglement. We shall see that it can also perform quantum entanglement in its action on quantum states:

$$R = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & b \end{bmatrix}.$$

Here a, b, c, d can be any scalars on the unit circle in the complex plane. *Then R is a unitary matrix and it is a solution to the Yang–Baxter equation.* It is an interesting and illuminating exercise to verify that R is a solution to the Yang–Baxter equation. We will omit this verification here, but urge the reader to perform it. In fact, the following more general construction gives a large class of unitary R matrices: let $M = (M_{ij})$ denote an $n \times n$ matrix with entries in the unit circle in the complex plane. Let R be defined by the equation

$$R^{ij}_{kl} = \delta^i_l \delta^j_k M_{ij}.$$

It is easy to see that R is a unitary solution to the Yang–Baxter equation. Our explicit example is the special case of R where the matrix M is 2×2. It turns out, just as we shall show here for the special case, that R detects no more than linking numbers for braids, knots and links. This is interesting, but it would be even more interesting to see other unitary R matrices that have subtler topological properties. The reader may enjoy comparing this situation with the unitary representation of the Artin braid group discussed in [8].

One can use that representation to calculate the Jones polynomial for three-strand braids. There is still a problem about designing a quantum computer to find the Jones polynomial, but this braid group representation does encode subtle topology. At the same time the representation in [8] cannot entangle quantum states. Thus the question of the precise relationship between topological entanglement and quantum entanglement certainly awaits the arrival of more examples of unitary representations of the braid group. We are indebted to David Meyer for asking sharp questions in this domain [14]. Now let P be the swap permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and let $\tau = RP$ so that

$$\tau = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & b \end{bmatrix}.$$

Then from the point of view of quantum gates, we have the phase gate τ and the swap gate P with $\tau = RP$. From the point of view of braiding and algebra, we have that R is a solution to the braided version of the Yang–Baxter equation, τ is a solution to the algebraists version of the Yang–Baxter equation, and P is to be regarded as an algebraic permutation *or* as a representation of a virtual or flat crossing. We discuss the virtual braid group [3, 4, 5, 7] in section 5, but for here suffice it to say that it is an extension of the classical braid group by the symmetric group and so contains braiding generators and also generators of order two. Now the point is that by looking at unitary representations of the virtual braid group, we can (as with the matrices above) pick up both phase and swap gates, and hence the basic ingredients for quantum computation. This means that the virtual braid group provides a useful topological language for quantum computing. This deserves further exploration.

The matrix R can also be used to make an invariant of knots and links that is sensitive to linking numbers. We will discuss this point in section 4.

But now, consider the action of the unitary transformation R on quantum states. We have

(1) $R|00\rangle = a|00\rangle$
(2) $R|01\rangle = c|10\rangle$
(3) $R|10\rangle = d|01\rangle$
(4) $R|11\rangle = b|11\rangle$.

Here is an elementary proof that the operator R can entangle quantum states. Note how this comes about through its being a composition of a phase and a swap gate. This decomposition is available in the virtual braid group.

Lemma. *If R is chosen so that $ab \neq cd$, then the state $R(\psi \otimes \psi)$, with $\psi = |0\rangle + |1\rangle$, is entangled.*

Proof. By definition,

$$\phi = R(\psi \otimes \psi) = R((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)) = a|00\rangle + c|10\rangle + d|01\rangle + b|11\rangle.$$

If this state ϕ is unentangled, then there are constants X, Y, X', Y' such that

$$\phi = (X|0\rangle + Y|1\rangle) \otimes (X'|0\rangle + Y'|1\rangle).$$

This implies that

(1) $a = XX'$

(2) $c = X'Y$

(3) $d = XY'$

(4) $b = YY'$.

It follows from these equations that $ab = cd$. Thus, when $ab \neq cd$ we can conclude that the state ϕ is entangled as a quantum state. □

Remark. Note that if $\alpha = a|0\rangle + b|1\rangle$ and $\beta = c|0\rangle + d|1\rangle$ then $\alpha \otimes \beta = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$. Thus a state $\gamma = X|00\rangle + Y|01\rangle + Z|10\rangle + W|11\rangle$ is entangled if $XW \neq YZ$.

3.1. Questions

This phenomenon leads to more questions than we have answers.

(1) How does one classify quantum entanglements in terms of braids (and corresponding braiding operators) that can produce them?

(2) Can all quantum entangled states be lifted to braidings?

(3) How do protocols for quantum computing look from this braided point of view?

(4) What is the relationship between the analogy between quantum states and entangled loops when viewed through the lens of the braiding operators?

(5) Does the association of unitary braiding operators shed light on quantum computing algorithms for knot invariants and statistical mechanics models? Here one can think of the computation of a knot invariant as separated into a braiding computation that is indeed a quantum computation, plus an evaluation related to the preparation and detection of a state (see [6, 8]).

(6) How does one classify all unitary solutions to the Yang–Baxter equation?

4. Link invariants from R

The unitary R matrix that we have considered in this paper gives rise to a non-trivial invariant of links. In this section we shall discuss the invariant associated with the specialization of R with $c = d$ so that

$$R = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & b \end{bmatrix}.$$

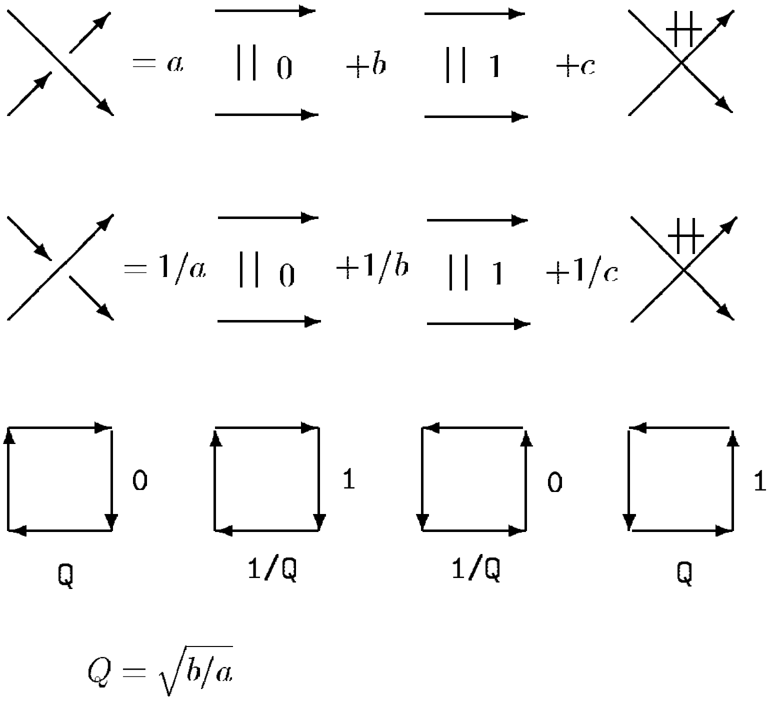

Figure 8. Formulae for the state summation.

Later we will specialize further so that $a = b$. We omit the details here, and just give the formula for this invariant in the form of a state summation. The invariant has the form

$$Z_K = a^{-w(K)}(\sqrt{a/b})^{\mathrm{rot}(K)}\langle K\rangle,$$

where $w(K)$ is the sum of the crossing signs of the oriented link K and $\mathrm{rot}(K)$ is the rotation number (or Whitney degree) of the planar diagram for K (see figure 8). The bracket $\langle K\rangle$ is the un-normalized state sum for the invariant. This state sum is defined through the equations shown in figure 8.

In this figure, the first crossing is positive, the second negative. The first two diagrammatic equations correspond to terms in the matrices R and R^{-1} respectively. Note that the glyphs in these equations are labelled with 0 or 1. The first two terms correspond to the action of R on $|00\rangle$ and on $|11\rangle$ respectively. The third term refers to the fact that R acts on $|01\rangle$ and $|10\rangle$ in the same way (by multiplying by c). However, these equations are interpreted for the state summation as instructions for forming local states on the link diagram. A global state on the link diagram is a choice of replacement for each crossing in the diagram so that it is either replaced by parallel arcs (as in the first two terms of each equation) or by crossed arcs (as in the third term of each equation). The local assignments of 0 and 1 on the arcs must fit together compatibly in a global state. Thus in a global state one can think of the 0 and 1 as qubits 'circulating' around simple closed curves in the plane. Each such state of circulation is measured in terms of the qubit type and the sense of rotation. These are the evaluations of cycles indicated below the two main equations for the state sum. Each cycle is assigned either Q or $1/Q$ where $Q = \sqrt{b/a}$. The state sum is the summation of evaluations of all of the possible states of qubit circulation where each state is evaluated by the product of weights

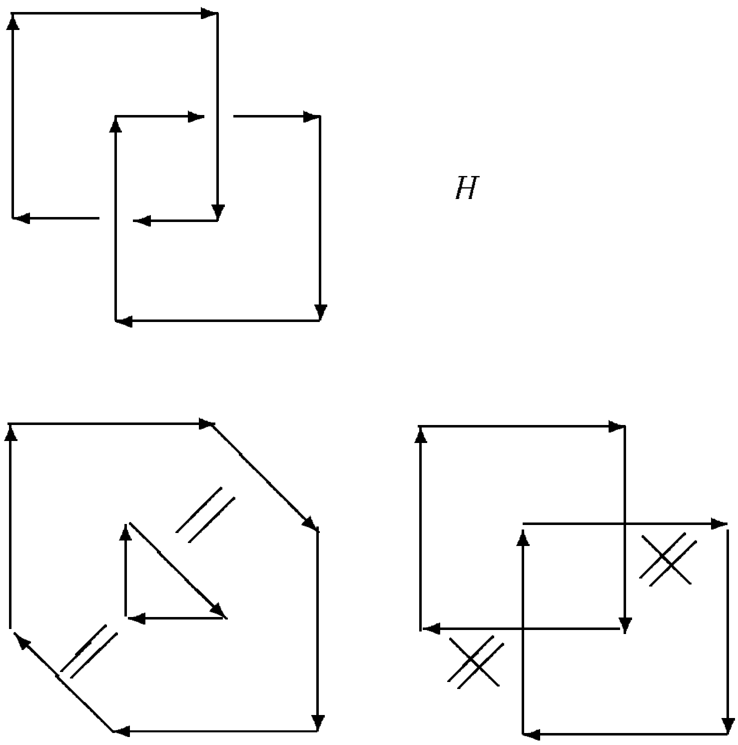

Figure 9. States for the Hopf link H.

a, b, c (and their inverses) coming from the expansion equations, multiplied by the product of the evaluations Q or $1/Q$ of the simple closed curves in the state. This completes a summary of the algorithm.

There are many ways to construe a state summation such as this. One can arrange the knot or link with respect to a given direction in the plane, and see the calculation as a vacuum–vacuum amplitude in a toy quantum field theory [6]. One can look directly at it as a generalized statistical mechanics state summation as we described it above. One can write the link as a closed braid and regard a major part of the calculation as a composition of unitary braiding operators. In this picture, a good piece of the algorithm can be construed as quantum. We believe that algorithms of this type, inherent in the study of so-called quantum link invariants, should be investigated more deeply from the point of view of quantum computing. In particular, the point of view of the algorithm as a sum over states of circulating qubits can be formalized, and will be the subject of another paper.

An example of a computation of this invariant is in order. In figure 9 we show the admissible states for a Hopf link (a simple link of two circles) where both circles have the same rotation sense in the plane. We then see that if H denotes the Hopf link, then $\langle H \rangle = a^2Q^2 + b^2Q^{-2} + 2c^2$ whence

$$Z_H = Q^{-2}\langle H \rangle = a^2 + b^2Q^{-4} + 2c^2Q^{-2}.$$

From this it is easy to see that the invariant Z detects the linkedness of the Hopf link. In fact, Z cannot detect linkedness of links with linking number equal to zero. For example, Z cannot detect the linkedness of the Whitehead link shown in figure 10.

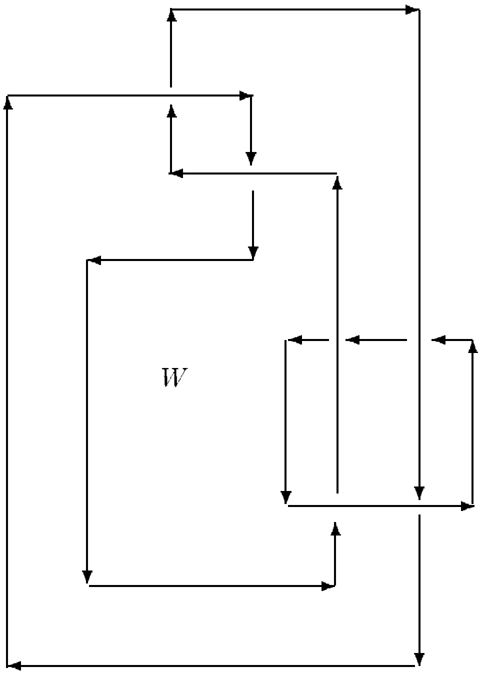

Figure 10. The Whitehead link.

4.1. A further specialization of Z_K

If we let $a = b$ in the definition of Z_K, then the state summation becomes particularly simple with $Q = 1$. It is then easy to see that for a two-component link Z_K is given by the formula

$$Z_K = 2(1 + (c^2/a^2)^{lk(K)})$$

where $lk(K)$ denotes the linking number of the two components of K. Thus we see that *for this specialization of the R matrix the operator R entangles quantum states exactly when it can detect linking numbers in the topological context.*

Here is another description of the state sum: instead of smoothing or flattening the crossings of the diagram, label each component of the diagram with either 0 or 1. Take vertex weights of a or c (in this special case, and the corresponding matrix entries in the general case) for each local labelling of a positive crossing as shown in figure 11. For a negative crossing the corresponding labels are $1/a$ and $1/c$ (which are the complex conjugates of a and c respectively, when a and c are unit complex numbers). Let each state (labelling of the diagram by zeros and ones) contribute the product of its vertex weights. Let $\Sigma(K)$ denote the sum over all the states of the products of the vertex weights. Then one can verify that $Z(K) = a^{-w(K)}\Sigma(K)$ where $w(K)$ is the sum of the crossing signs of the diagram K.

For example, view figure 12. Here we show the zero–one states for the Hopf link H. The 00 and 11 states each contributes a^2, while the 01 and 10 states contribute c^2. Hence $\Sigma(H) = 2(a^2 + c^2)$ and $a^{-w(H)}\Sigma(H) = 2(1 + (c^2/a^2)^1) = 2(1 + (c^2/a^2)^{lk(H)})$, as expected.

The calculation of the invariant in this form is actually an analysis of quantum networks with cycles in the underlying graph. In this form of calculation we are concerned with those states of the network that correspond to labellings by qubits that are compatible with the entire network structure. A precise definition of this concept will be given in a sequel to this paper.

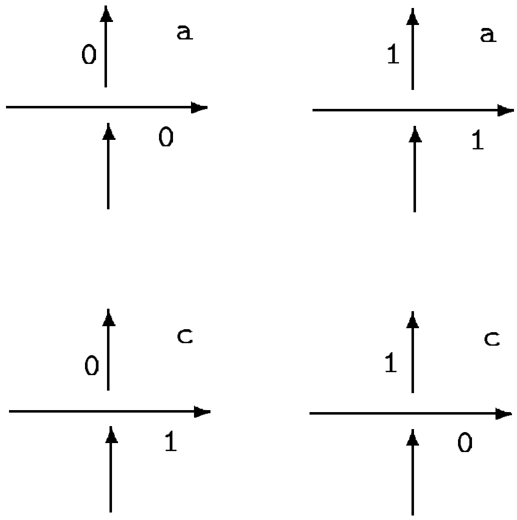

Figure 11. Positive crossing weights.

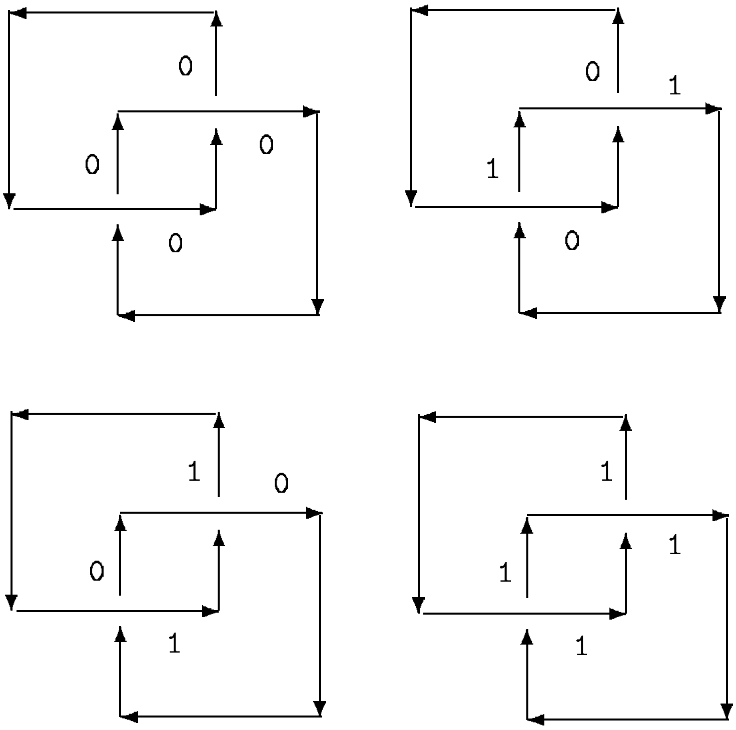

Figure 12. Zero–one states for the Hopf link.

Here one considers only those quantum states that are compatible with the interconnectedness of the network as a whole.

The example of the Hopf link shows how subtle properties of topological entanglement are detected through the use of the operator R in circularly interconnected quantum networks. It remains to do a deeper analysis that can really begin to disentangle the roles of quantum entanglement and circularity in such calculations.

5. A remark about EPR

It is remarkable that the simple algebraic situation of an element in a tensor product that is not itself a tensor product of elements of the factors corresponds to subtle nonlocality in physics. It helps to place this algebraic structure in the context of a gedanken experiment to see where the physics comes in. Consider

$$S = |0\rangle|1\rangle + |1\rangle|0\rangle.$$

In an EPR thought experiment, we think of two 'parts'of this state that are separated in space. We want a notation for these parts and suggest the following:

$$L = \{|0\rangle\}|1\rangle + \{|1\rangle\}|0\rangle, \qquad R = |0\rangle\{|1\rangle\} + |1\rangle\{|0\rangle\}.$$

In the left state L, an observer can only observe the left-hand factor. In the right state R, an observer can only observe the right-hand factor. These 'states' L and R together comprise the EPR state S, but they are accessible individually just as are the two photons in the usual thought experiment. One can transport L and R individually and we shall write

$$S = L * R$$

to denote that they are the 'parts' (but not tensor factors) of S.

The curious thing about this formalism is that it includes a little bit of macroscopic physics implicitly, and so it makes it a bit more apparent what EPR were concerned about. After all, lots of things that we can do to L or R do not affect S. For example, transporting L from one place to another, as in the original experiment where the photons separate. On the other hand, if Alice has L and Bob has R and Alice performs a local unitary transformation on 'her' tensor factor, this applies to both L and R since the transformation is actually being applied to the state S. This is also a 'spooky action at a distance' whose consequence does not appear until a measurement is made.

6. Virtual braids

This section expands the remarks about how the inclusion of a swap operator in the braid group leads to a significant generalization of that structure to the virtual braid group.

The *virtual braid group* is an extension of the classical braid group by the symmetric group. If V_n denotes the n-strand virtual braid group, then V_n is generated by braid generators $\sigma_1, \ldots, \sigma_{n-1}$ and virtual generators $c_1, \ldots, c_n$ where each virtual generator c_i has the form of the braid generator σ_i with the crossing replaced by a virtual crossing. Among themselves, the braid generators satisfy the usual braiding relations. Among themselves, the virtual generators are a presentation for the symmetric group S_n. The relations that relate virtual generators and braiding generators are as follows:

$$\sigma_i^{\pm} c_{i+1} c_i = c_{i+1} c_i \sigma_{i+1}^{\pm}, \qquad c_i c_{i+1} \sigma_i^{\pm} = \sigma_{i+1}^{\pm} c_i c_{i+1}, \qquad c_i \sigma_{i+1}^{\pm} c_i = c_{i+1} \sigma_i^{\pm} c_{i+1}.$$

It is easy to see from this description of the virtual braid groups that all the braiding generators can be expressed in terms of the first braiding generator σ_1 (and its inverse) and the virtual generators. One can also see that Alexander's theorem generalizes to virtuals: every virtual knot is equivalent to a virtual braid [4]. In [7] a Markov theorem is proven for virtual braids.

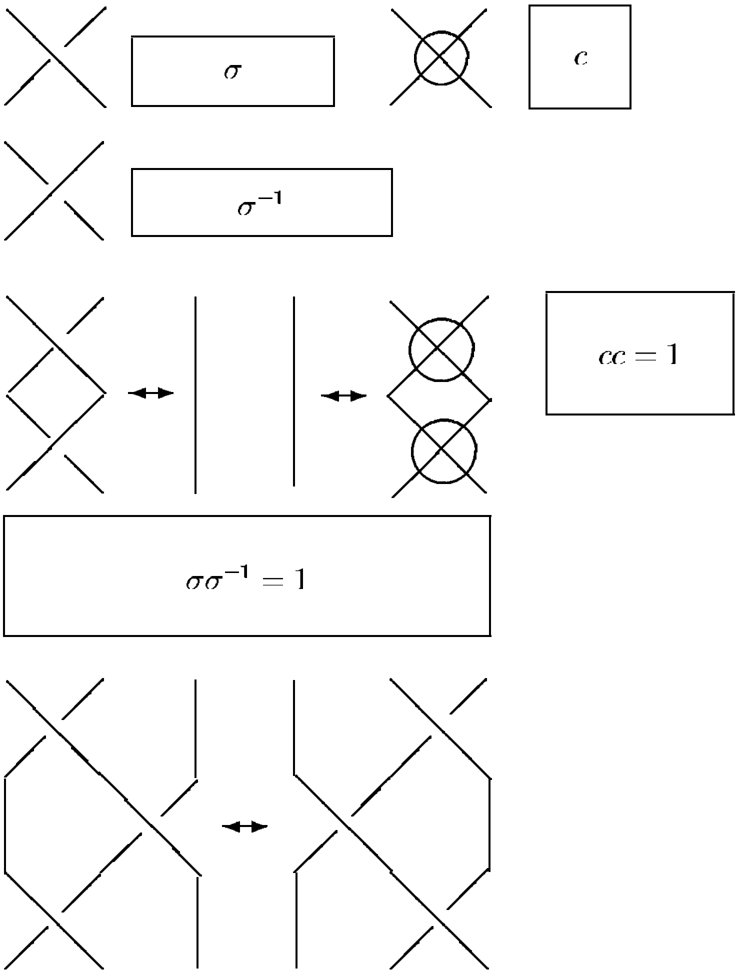

Figure 13. Braid generators and virtual braid generators.

From the point of view of quantum computing, it is natural to add the virtual braiding operators to the Artin braid group. *Each virtual braiding operator can be interpreted as a swap gate*. With the virtual operators in place, we can compose them with the R matrices to obtain phase gates and other apparatus as described in section 3. We then have the virtual braid group as a natural topologically based group structure that can be used as an underlying language for building patterns of quantum computation.

7. Discussion

We are now in a position to state the main problem posed by this paper. We have been exploring the analogy between topological entanglement and quantum entanglement. It has been suggested that there may be a direct connection between these two phenomena. But on closer examination, it appears that rather than a direct connection, there is a series of analogous features that are best explored by going back and forth across the boundary between topology and quantum computing. In particular, we have seen that the unitary operator R can indeed produce entangled

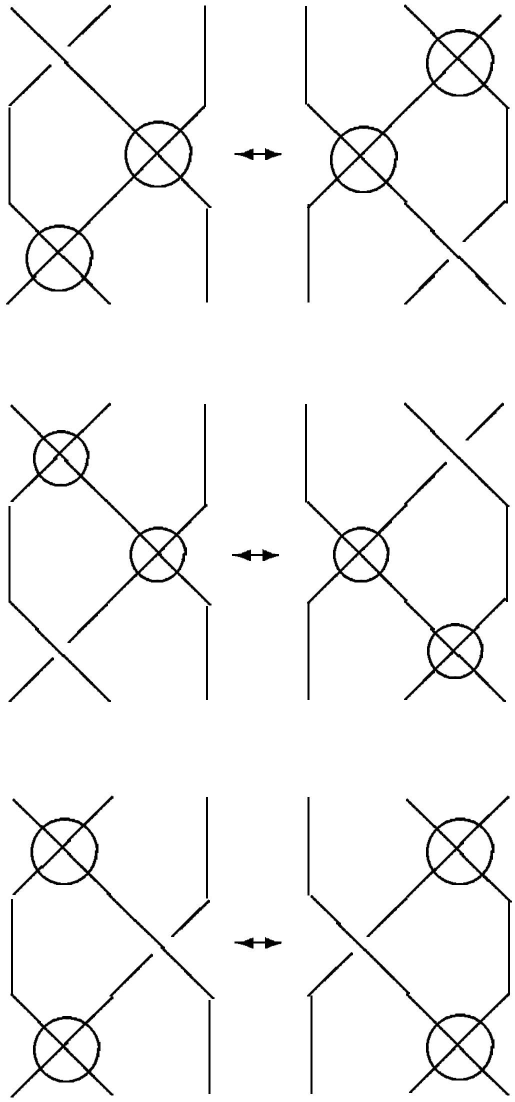

Figure 14. Relations in the virtual braid group.

quantum states from unentangled quantum states. The operator R is the basic ingredient for forming a representation of the Artin braid group. As such, it is intimately connected with topological entanglement. In fact, the operator R is also the basic ingredient in constructing the link invariant Z_K that we have studied in section 4. The construction of this link invariant is motivated by quantum statistical mechanics and its structure bears further investigation from the point of view of quantum computing. The theme that emerges is powerfully related to the circularity of the links. It is through mutual circularity that the topological linking occurs. And it is through this circularity and the measurement of circulating states of qubits that one computes the state summation model. A deep relation of quantum states and topological states will be seen through the study of the quantum states of circularly interconnected networks structurally related to three-dimensional space. These networks are both topological and quantum mechanical, and a common structure will emerge. This is the project for further papers in our series.

In the meantime, the language of the braid group and virtual braid group provides an arena for representing quantum operators that can be interpreted topologically. This framework provides a means for topology and quantum computing to converse with one another.

The papers [10, 11] and [12, 13] provide background to the considerations of the present paper. In particular, they provide a general framework for studying quantum entanglement that may be useful in investigating the role of infinitesimal braiding operators and other aspects of the representation theory of the Artin braid group.

The reader may wish to compare the points of view in this paper with the paper [2]. There the author considers the possibility of anyonic computing and follows out the possible consequences in terms of representations of the Artin braid group. We are in substantial agreement with his point of view *and* we contend that braiding is fundamental to quantum computation whether or not it is based in anyonic physics.

Acknowledgments

Most of this effort was sponsored by the Defense Advanced Research Projects Agency (DARPA) and Air Force Research Laboratory, Air Force Material Command, USAF, under agreement F30602-01-2-05022. Some of this effort was also sponsored by the National Institute for Standards and Technology (NIST). The US Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright annotations thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Defense Advanced Research Projects Agency, the Air Force Research Laboratory, or the US Government (Copyright 2002). It gives the first author great pleasure to thank Fernando Souza for interesting conversations in the course of preparing this paper.

Appendix. A unitary representation of the three-strand braid group and the corresponding quantum computer

Many questions are raised by the formulation of quantum computation associated with a given link diagram. In this appendix we give an example of a unitary representation of the three-strand braid group. We can use this representation to compute the Jones polynomial for closures of 3-braids, and therefore this representation provides a test case for the corresponding quantum computation. We now analyse this case by first making explicit how the bracket polynomial is computed from this representation. See [8] for more details about this construction.

The representation depends on two matrices U_1 and U_2 with

$$U_1 = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad U_2 = \begin{bmatrix} \delta^{-1} & \sqrt{1-\delta^{-2}} \\ \sqrt{1-\delta^{-2}} & \delta - \delta^{-1} \end{bmatrix}.$$

The representation is given on the two braid generators by

$$\lambda(\sigma_1) = AI + A^{-1}U_1 \qquad \lambda(\sigma_2) = AI + A^{-1}U_2$$

for any A with $\delta = -A^2 - A^{-2}$, and with $A = \mathrm{e}^{\mathrm{i}\theta}$, then $\delta = -2\cos(2\theta)$. We get the specific range of angles $|\theta| \leq \pi/6$ and $|\theta - \pi| \leq \pi/6$ that give unitary representations of the three-strand braid group.

Note that $\mathrm{tr}(U_1) = \mathrm{tr}(U_2) = \delta$ while $\mathrm{tr}(U_1U_2) = \mathrm{tr}(U_2U_1) = 1$. If b is any braid, let $I(b)$ denote the sum of the exponents in the braid word that expresses b. For b a three-strand braid, it follows that

$$\lambda(b) = A^{I(b)}I + \Pi(b)$$

where I is the 2×2 identity matrix and $\Pi(b)$ is a sum of products in the Temperley–Lieb algebra [8] involving U_1 and U_2. Since the Temperley–Lieb algebra in this dimension is generated by I, U_1, U_2, U_1U_2 and U_2U_1, it follows that

$$\langle \bar{b} \rangle = A^{I(b)}\delta^2 + \mathrm{tr}(\Pi(b))$$

where $\bar{b}$ denotes the standard braid closure of b, and the sharp brackets denote the bracket polynomial [8] as described in previous sections. From this we see at once that

$$\langle \bar{b} \rangle = \mathrm{tr}(\lambda(b)) + A^{I(b)}(\delta^2 - 2).$$

It follows from this calculation that the question of computing the bracket polynomial for the closure of the three-strand braid b is mathematically equivalent to the problem of computing the trace of the matrix $\lambda(b)$. To what extent can our quantum computer determine the trace of this matrix?

The matrix in question is a product of unitary matrices, the quantum gates that we have associated with the braids σ_1 and σ_2. The entries of the matrix $\lambda(b)$ are the results of preparation and detection for the two-dimensional basis of qubits for our machine:

$$\langle i|\lambda(b)|j\rangle.$$

Given that the computer is prepared in $|j\rangle$, the probability of observing it in state $|i\rangle$ is equal to $|\langle i|\lambda(b)|j\rangle|^2$. Thus we can, by running the quantum computation repeatedly, estimate the absolute squares of the entries of the matrix $\lambda(b)$. This will not yield the complex phase information that is needed for either the trace of the matrix or the absolute value of that trace. Thus our quantum computer can compute information relating to the braiding process, but it cannot approximate the full value of the bracket polynomial.

Note that our quantum computer does indeed have the capability to detect three strand braiding, since for a braid b the matrix $\lambda(b)$ can have non-trivial off-diagonal elements. The absolute squares of these elements are approximated by successive runs of the quantum computer. In this quantum computer, braiding corresponds to quantum states and is detectable by that token. The bracket polynomial itself depends upon subtler phase relationships and is not detectable by this quantum computer.

What is remarkable here is that this unitary representation of the three-strand braid group can do so much topology, *and that it does so without using any quantum entanglement*. The representation described above is defined on a two-dimensional complex vector space and hence acts upon a single qubit. No entanglement occurs in the quantum process, and yet topological linking and knotting is detected. Clearly, the relative roles of entanglement in quantum computing and in topology need a deeper examination.

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