REFLEXIVITY, EIGENFORM AND FOUNDATIONS OF PHYSICS

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Abstract: This essay is a discussion of the concept of reflexivity and its relationships with self-reference, re-entry, eigenform and the foundations of physics.

I. Introduction

Reflexive is a term that refers to the presence of a relationship between an entity and itself. One can be aware of one's own thoughts. An organism produces itself through its own action and its own productions. A market or a system of finance is composed of actions and individuals, and the actions of those individuals influence the market just as the global information from the market influences the actions of the individuals. Here it is the self-relations of the market through its own structure and the structure of its individuals that moves its evolution forward. Nowhere is there a way to effectively cut an individual participant from the market and make him into an objective observer. His action in the market is concomitant to his being reflexively linked with that market. Just so for theorists of the market for their theories, if communicated become part of the action and decision-making of the market. Social systems partake of this same reflexivity, and so does apparently objective science and mathematics. In order to see the reflexivity of the practice of physical science or mathematics, one must leave the idea of an objective domain of investigation in brackets and see the enterprise as a large conversation among a group of investigators. Then, at once, the process is seen to be a reflexive interaction among the members of this group. Mathematical results, like all technical inventions, have a certain stability over time that gives them an air of permanence, but the process that produces these novelties is every bit as fraught with circularity and mutual influence as any other conversation or social interaction.

How then, shall we describe a reflexive domain? It is the purpose of this paper to give a very abstract definition that nevertheless captures, what I believe to be the main conceptual feature of reflexivity. We then immediately prove that eigenforms, fixed points of transformations, are present for all transformations of the reflexive domain. This will encourage us and it will give us pause.

The existence of eigenforms will encourage us, for we have previously studied them with the notion that "objects are tokens for eigenbehaviour". Eigenforms are the natural emergence of those tokens by way of recursion. So to find the eigenforms dictated by a larger concept is pleasing. But we shall also need to pause.

For the existence of fixed points for arbitrary transformations will show us that the domain we have postulated is indeed very wide.

It is not an objectively existing domain. It is a clearing in which structures can arise and new structures can arise. A reflexive domain is not an already-existing structure. Not at all. To be what it claims to be, a reflexive domain must be a combination of existing structure and an invitation to create new structure and new concepts. The new will become platforms from which further flights of creativity can be made. Thus in the course of examining the concept of reflexivity we will find that the essence of the matter is an opening into creativity, and that will become the actual theme of this paper.

We are particularly interested in the way these concepts of reflexivity affect fundamentals of topology and fundamentals of physics. The last parts of this essay are a reformulation of elementary mathematics of matrices, complex numbers and exponentials in terms of process, reflexivity and eigenform.

We then show how quantum mechanics and discrete physics acquire a new point of view in the light of these interpretations. The reader may wish to skip directly to Section XII to see how this part of the argument proceeds.

Our essay begins with explication of the notion of eigenform as pioneered by Heinz von Foerster in his papers [4, 5,6,7] and explored in papers of the author [11, 12]. In [5] The familiar objects of our existence can be seen to be nothing more than tokens for the behaviors of the organism, creating apparently stable forms. Such an attitude toward objects makes it impossible to discriminate between the object as an element of a world and the object as a token or symbol that is simultaneously a process.

The notion of an eigenform is inextricably linked with second order cybernetics. One starts on the road to such a concept as soon as one begins to consider a pattern of patterns, the form of form or the cybernetics of cybernetics. Such concepts appear to loop around upon themselves, and at the same time they lead outward to new points of view. Such circularities suggest a possibility of transcending the boundaries of a system within. When the circular concept is called into being, the boundaries turn inside out.

An object , in itself , is a symbolic entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions. We ourselves are such objects, we as human beings are "signs for ourselves", a concept originally due to the American philosopher C. S. Peirce [10]. Eigenforms are mathematical companions to Peirce's work.

In an observing system, what is observed is not distinct from the system itself, nor can one make a separation between the observer and the observed. The observer and the observed stand together in a coalescence of perception. From the stance of the observing system all objects are non-local, depending upon the presence of the system as a whole. It is within that paradigm that these models begin to live, act and enter into conversation with us.

After this journey into objects and eigenforms, we take a wider stance and consider the structure of spaces and domains that partake of the reflexivity of object and process. We make a definition of a *reflexive domain* (compare [1] and [18]). Our definition populates a space (domain) with entities that could be construed as objects, and we assume that each object acts as a transformation on the space. Essentially this means that given entities A and B, then there is a new entity C that is the result of A and B acting together in the order AB (so that one can say that "A acts on B" for AB and one can say "B acts on A" for BA). This means that the reflexive space is endowed with a non-commutative and non-associative algebraic structure. The reflexive space is expandable in the sense that whenever we define a process, using entities that have already been constructed or defined, then that process can take a name, becoming a new entity/transformation of a space that is expanded to include itself.. Reflexive spaces are open to evolution in time, as new processes are invented and new forms emerge from their interaction.

Remarkably, reflexive spaces always have eigenforms for every element/transformation/entity in the space! The proof is simple but requires discussion.

Given F in a reflexive domain, define G by Gx = F(xx).
Then GG = F(GG) and so GG is an eigenform for F.

Just as promised, in a reflexive domain, every entity has an eigenform. From this standpoint, one should start with the concept of reflexivity and see that from it emerge eigenforms. Are we satisfied with this approach? We are not satisfied. For in order to start with reflexivity, we need to posit objects and processes. As we have already argued in this essay, objects are tokens for eigenbehaviours. And a correct or natural beginning is process where objects are seen as tokens of processes.

By now the reader begins to see that the story we have to tell is a circular one. We give a way to understand this circularity with Section X where we discuss creativity in recursive process and the emergence of novelty.

The reader will see that we have woven a tale the goes back and forth between recursion and idealized eigenforms. This means that we are sometimes considering abstractions such as reflexive domains and their algebraic properties and we are sometimes looking at the particulars of recursions directly related to automata or to specific complex numbers. Here follows a précis of the paper from the point of view of both the algebras and the physics.

This paper explores the analogies of fixed points, observations and observables, eigenvectors and recursive processes in relation to foundations of physics. In particular we shall re-open the books on the complex numbers and view them in terms of recursion and reflexivity, finding new and natural ways to think about their roles in physical theory (Section XIII).

To give a hint, think of the oscillatory process generated by $\mathbf{R(x) = -1/x}$. The fixed point is $\mathbf{i}$ with $\mathbf{i^2 = -1}$, but the processes generated over the real numbers must be directly related to the idealized i. We shall let **I{+1,-1}** stand for an undisclosed alternation or ambiguity between **+1** and **-1** and call **I{+1,-1}** an *iterant*. There are two i*terant views*: **[+1,-1]** and **[-1,+1]**. These, seen as points of view of alternating process will become the square roots of negative unity. We introduce a temporal shift operator η such that

$$[a,b]\eta = \eta[b,a] \quad \text{and} \quad \eta\eta = 1$$

so that concatenated observations can include a time step of one-half period of the process **...abababab...** . We combine iterant views term-by-term as in **[a,b][c,d] = [ac,bd]**. Then we have, with i = **[1,-1]**η (**i** is view/operator),

$$ii = [1,-1]\eta\ [1,-1]\eta = [1,-1][-1,1]\eta\ \eta = [-1,-1] = -1.$$

This gives rise to a new process-oriented construction of the complex numbers, quaternions, and in fact of all of matrix algebra.

We relate this point of view to thinking about the role of complex numbers in quantum mechanics and the role of temporal shift operators in discrete physics, that begins with the understanding that temporal shift operators allow discrete calculus to be represented in a non-commutative (Lie algebraic) context where all derivatives are represented by commutators. (Section XIV.)

We also relate these ideas of reflexivity and fixed points to left or right distributive non-associative algebras and their relationships with knot theory in Section VI. We relate this with approaches to wholeness in physics and philosophy such as the work of Barbara Piechosinska [16]. A *magma* is a non-associative algebra with a single binary operation that is left-associative:

$$a*(b*c) = (a*b)*(a*c).$$

Note that this axiom says that every element A of the magma is a structure preserving mapping of the magma to itself:

$$A(x*y) = (A*x)*(A*y).$$

The notion of a magma is another view of what should be a self-reflexive domain. We raise questions about the relationship of magmas and reflexive domains and, in Section VI, illustrate the remarkable and deep relationships among magmas and knots and braids.

II. Objects as Tokens for Eigenbehaviours

In his paper "Objects as Tokens for Eigenbehaviours" [5] von Foerster suggests that we think seriously about the mathematical structure behind the constructivist doctrine that *perceived worlds are worlds created by the observer.* At first glance such a statement appears to be nothing more than solipsism. At second glance, the statement appears to be a tautology, for who else can create the rich subjectivity of the immediate impression of the senses? At third glance, something more is needed. In that paper he suggests that the familiar objects of our experience are the fixed points of operators. These operators *are* the structure of our perception. To the extent that the operators are shared, there is no solipsism in this point of view. It is the beginning of a mathematics of second order cybernetics.

Consider the relationship between an observer O and an "object" A. The key point about the observer and the object is that "the object remains in constant form with respect to the observer". This constancy of form does not preclude motion or change of shape. Form is more malleable than the geometry of Euclid. In fact, ultimately the form of an "object" is the form of the distinction that "it" makes in the space of our perception. In any attempt to speak absolutely about the nature of form we take the form of distinction for the form. (paraphrasing Spencer-Brown [3]). It is the form of distinction that remains constant and produces an apparent object for the observer. How can you write an equation for this? The simplest route is to write

$$\mathbf{O(A) = A.}$$

The object **A** is a fixed point for the observer **O**. The object is an eigenform. We must emphasize that this is the most schematic possible description of the condition of the observer in relation to an object **A.** We only record that the observer as an actor (operator) manages through his acting to leave the (form of) the object unchanged. This can be a recognition of the symmetry of the object but it also can be a description of how the observer, searching for an object, makes that object up (like a good fairy tale) from the very ingredients that are the observer herself. This is the situation that Heinz von Foerster has been most interested in studying. As he puts it, if you give a person an undecideable problem, then the answer that he gives you is a description of himself. And so, by working on hard and undecideable problems we go deeply into the discovery of who we really are. All this is symbolized in the little equation **O(A) = A.**

And what about this matter of the object as a token for eigenbehaviour? This is the crucial step. We forget about the object and focus on the observer. We attempt to "solve" the equation **O(A) = A** with **A** as the unknown. Not only do we admit that the "inner" structure of the object is unknown, we adhere to whatever knowledge we have of the observer and attempt to find what such an observer could observe based upon that structure.

We can start anew from the dictum that the perceiver and the perceived arise together in the condition of observation. This is a stance that insists on mutuality (neither perceiver nor the perceived causes the other). A distinction has emerged and with it a world with an observer and an observed. The distinction is itself an eigenform.

III. Compresence and Coalescence

We, identify the world in terms of how we shape it. We shape the world in response to how it changes us. We change the world and the world changes us. Objects arise as tokens of a behavior that leads to seemingly unchanging forms. Forms are seen to be unchanging through their invariance under our attempts to change, to shape them.

For an observer there are two primary modes of perception -- *compresence* and *coalesence*. Compresence connotes the coexistence of separate entities together in one including space. Coalesence connotes the one space holding, in perception, the observer and the observed, inseparable in an unbroken wholeness. Coalesence is the constant condition of our awareness. Coalesence is the world taken in simplicity. Compresence is the world taken in apparent multiplicity.

This distinction of compresence and coalesence, drawn by Henri Bortoft [2], can act as a compass in traversing the domains of object and reference. *Eigenform is a first step towards a mathematical description of coalesence.* In the world of eigenform the observer and the observed are one in a process that recursively gives rise to each.

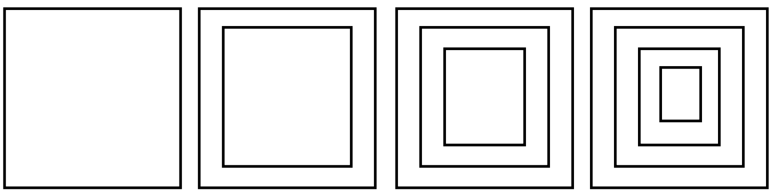
IV. The Eigenform Model

We have seen how the concept of an object has evolved to make what we call objects (and the objective world) processes that are interdependent with the actions of observers. The notion of a fixed object has become a notion of a process that produces the apparent stability of the object. This process can be simplified in a model to become a recursive process where a rule or rules are applied time and time again. The resulting object of such a process is the *eigenform* of the process, and the process itself is the *eigenbehaviour.*

In this way we have a model for thinking about object as token for eigenbehaviour. This model examines the result of a simple recursive process carried to its limit. For example, suppose that

$$F(X) = \boxed{\quad X \quad}$$

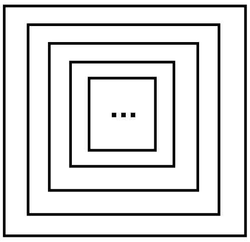
That is, each step in the process encloses the results of the previous step within a box. Here is an illustration of the first few steps of the process applied to an empty box X:



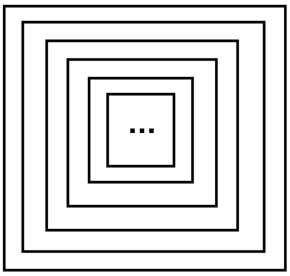
X F(X) F(F(X)) F(F(F(X)))

If we continue this process, then successive nests of boxes resemble one another, and in the limit of infinitely many boxes, we find that

$X=F(F(F(\ldots)))=$



$F(X) =$



$= X$

the infinite nest of boxes is invariant under the addition of one more surrounding box. Hence this infinite nest of boxes is a fixed point for the recursion. In other words, if X denotes the infinite nest of boxes, then

$$\mathbf{X = F(X).}$$

This equation is a description of a state of affairs. The form of an infinite nest of boxes is invariant under the operation of adding one more surrounding box. The infinite nest of boxes is one of the simplest eigenforms.

In the process of observation, we interact with ourselves and with the world to produce stabilities that become the objects of our perception. These objects, like the infinite nest of boxes, may go beyond the specific properties of the world in which we operate. They attain their stability through the limiting process that goes outside the immediate world of individual actions. We make an imaginative leap to complete such objects to become tokens for eigenbehaviours. It is impossible to make an infinite nest of boxes. We do not make it. We *imagine* it. And in imagining that infinite nest of boxes, we arrive at the eigenform.

The leap of imagination to the infinite eigenform is a model of the human ability to create signs and symbols. In the case of the eigenform **X** with **X = F(X), X** can be regarded as the name of the process itself or as the name of the limit process. Note that if you are told that

$$\mathbf{X = F(X)},$$

then substituting **F(X)** for **X**, you can write

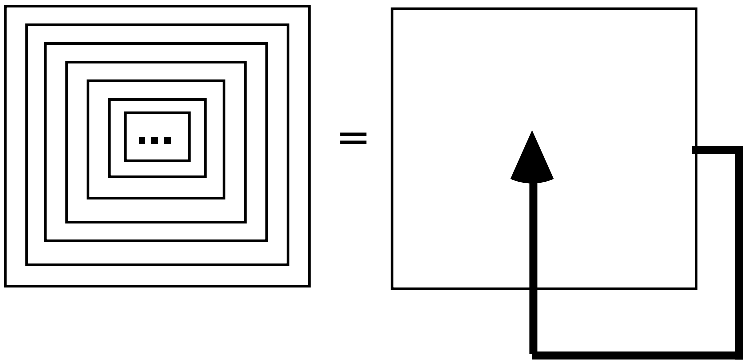
$$\mathbf{X = F(F(X)).}$$

Substituting again and again, you have

$$\mathbf{X = F(F(F(X))) = F(F(F(F(X)))) = F(F(F(F(F(X))))) = ...}$$

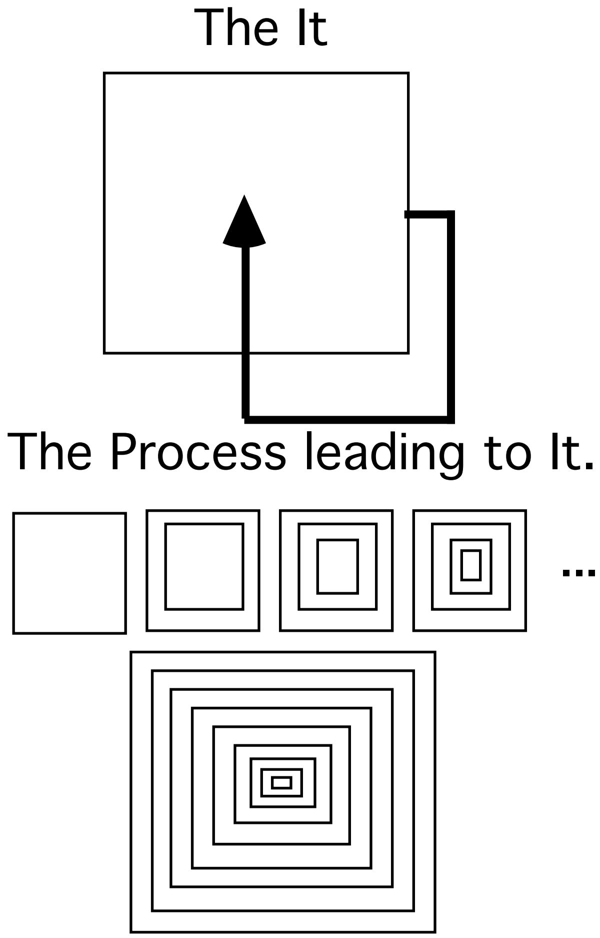
The process arises from the symbolic expression of its eigenform. In this view *the eigenform is an implicate order for the process that generates it.*

Sometimes one stylizes the structure by indicating where the eigenform **X** reenters its own indicational space by an arrow or other graphical device. See the picture below for the case of the nested boxes.



Does the infinite nest of boxes exist? Certainly it does not exist in this page or anywhere in the physical world with which we are familiar. The infinite nest of boxes exists in the imagination. It is a symbolic entity.

Eigenform is the imagined boundary in the reciprocal relationship of the object (the "It") and the process leading to the object (the process leading to "It"). In the diagram below we have indicated these relationships with respect to the eigenform of nested boxes. Note that the "It" is illustrated as a finite approximation (to the infinite limit) that is sufficient to allow an observer to infer/perceive the generating process that underlies it.



Just so, an object in the world (cognitive, physical, ideal,...) provides a conceptual center for the exploration of a skein of relationships related to its context and to the processes that generate it. An object can have varying degrees of reality just as does an eigenform. If we take the suggestion to heart that objects are tokens for eigenbehaviours, then an object in itself is an entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions.

An object is an amphibian between the symbolic and imaginary world of the mind and the complex world of personal experience. The object, when viewed as process, is a dialogue between these worlds. The object when seen as a sign for itself, or in and of itself, is imaginary.

Why are objects apparently solid? Of course you cannot walk through a brick wall even if you think about it differently. I do not mean apparent in the sense of thought alone. I mean apparent in the sense of appearance. The wall appears solid to me because of the actions that I can perform. The wall is quite transparent to a neutrino, and will not even be an eigenform for that neutrino.

This example shows quite sharply how the nature of an object is entailed in the properties of its observer.

The eigenform model can be expressed in quite abstract and general terms. Suppose that we are given a recursion (not necessarily numerical) with the equation

$$\mathbf{X(t+1) = F(X(t)).}$$

Here **X(t)** denotes the condition of observation at time **t.** **X(t)** could be as simple as a set of nested boxes, or as complex as the entire configuration of your body in relation to the known universe at time **t.** Then **F(X(t))** denotes the result of applying the operations symbolized by **F** to the condition at time **t**. You could, for simplicity, assume that **F** is independent of time. Time independence of the recursion **F** will give us simple answers and we can later discuss what will happen if the actions depend upon the time. In the time independent case we can write

$$\mathbf{J = F(F(F(...)))}$$

the infinite concatenation of F upon itself. Then

$$\mathbf{F(J) = J}$$

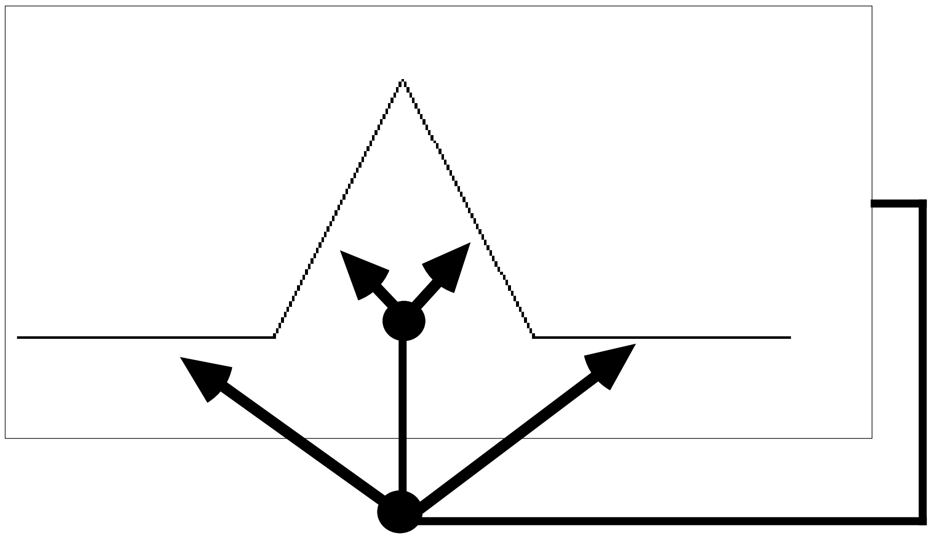
since adding one more **F** to the concatenation changes nothing.

Thus **J,** the infinite concatenation of the operation upon itself leads to a fixed point for **F**. **J** is said to be the eigenform for the recursion **F**. We see that every recursion has an eigenform. Every recursion has an (imaginary) fixed point.

We end this section with one more example. This is the eigenform of the Koch fractal [14]. In this case one can write symbolically the eigenform equation

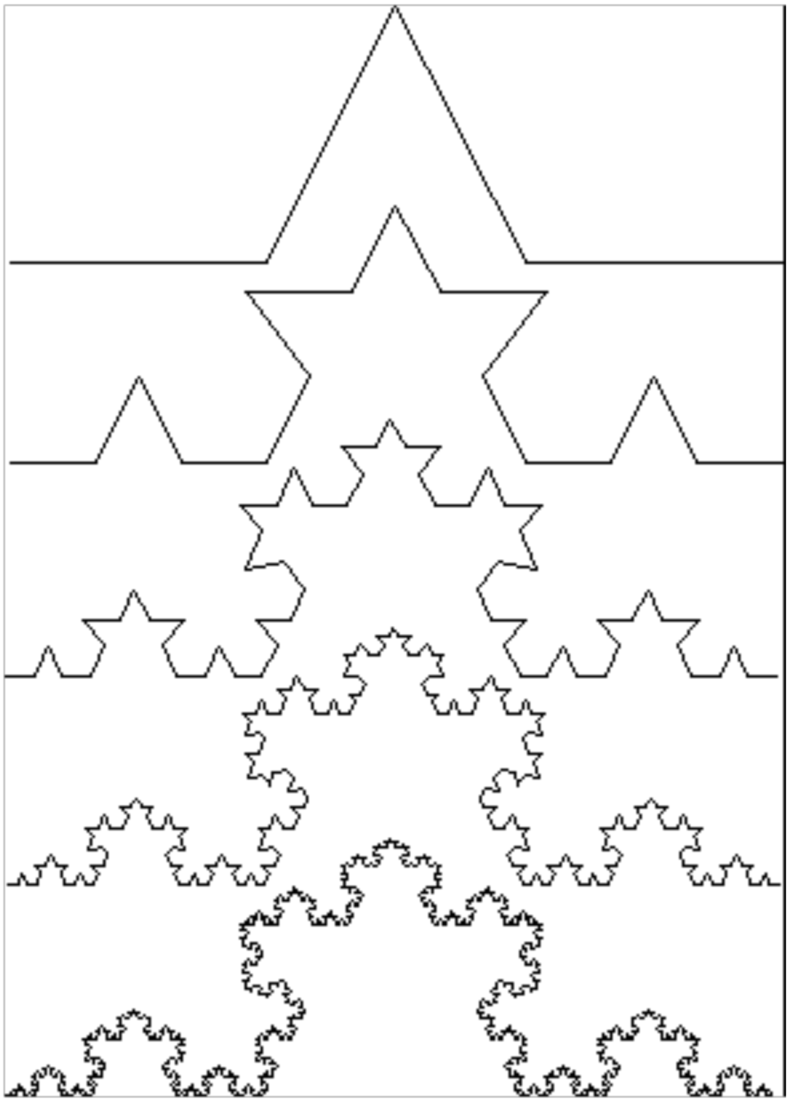
$$\mathbf{K = K\{\ K\ K\ \}K}$$

to indicate that the Koch Fractal reenters its own indicational space four times (that is, it is made up of four copies of itself, each one-third the size of the original. The curly brackets in the center of this equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.



$$K = K \{ K \ K \} K$$

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third its length, arranged according to the pattern of reentry as shown in the figure above. The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.



Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

K = K { K K } K

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

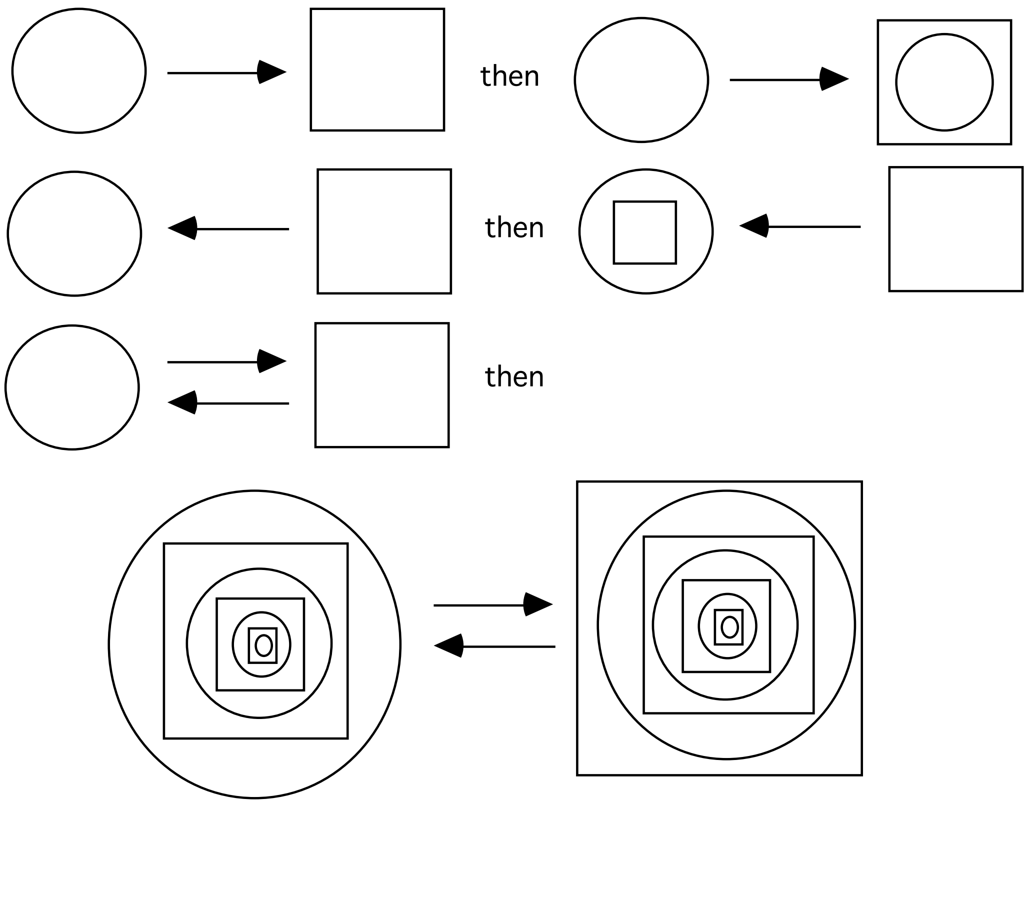
K = K { K K } K
= K { K K } K { K { K K } K K { K K } K } K { K K } K
= ...

We have only performed one line of this skeletal recursion. There are sixteen **K**'s in this second expression just as there are sixteen line segments in the second stage of the geometric recursion. Comparison with this symbolic recursion shows how geometry aids the intuition. The interaction of eigenforms with the geometry of physical, mental, symbolic and spiritual landscapes is an entire subject that is in need of deep exploration.

It is usually thought that the miracle of recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. This is a fine tuning to the point where the action of the perceiver and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet in the mathematical levels, such as number or fractal pattern, part of the process is slowed down to the point where we can begin to apprehend the process. There is a stability in the comparison, in the correspondence that is a process happening at once in the present time. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviours and yet is seen as simple by the perceiving mind. The perceiving mind is itself an eigenform.

Mirror-Mirror

In the next figure we illustrate how an eigenform can arise from a process of mutual reflection. The figure shows a circle with a an arrow pointing to a rectangle and a rectangle with an arrow pointing toward a circle. For this example, we take the rule that an arrow between two entities (P -----> Q) means that the second entity will create an internal image of the first entity (Q will make an image of P). If P -----> Q and Q ------> P, then each entity makes an image of the other. A recursion will ensue. Each of P and Q generates eigenforms in this mutuality.



In this example we can denote the initial forms by C (for circle) and B (for box). We have C -----> B and B -----> C. The rule of imaging is (symbolically):

If P -----> Q then P -----> QP.
If P < ------ Q, then PQ <------ Q.

We start with the mutual reference C < -----> B. This condition of mutual mirroring can be described by two operators C and B:

C(P)= CP corresponds to C -----> P.
B(Q) = BQ corresponds to Q < ----- B.

We are solving the eigenform equations

$$C(Y) = X,$$
$$B(X) = Y.$$

We have the mirror-mirror solution

$$X = BCBCBCBC...,$$
$$Y = CBCBCBCB...,$$

just as in the Figure.

We are quite familiar with this form of mutual mirroring in the physical realm where one can have two facing mirrors, and in the realm of human relations where the complexity of exchange (mutual mirroring) between two individuals leads to the eigenform of their relationship.

V. Reflexive Domains

A reflexive domain D is an arena where actions and processes that transform the domain can also be seen as the elements that compose the domain. Every element of the domain can be seen as a transformation of the domain to itself.

In actual practice an element of a domain may be a person or company (collective of persons) or a physical object or mechanism that is seen to be in action. In actual practice we must note that what are regarded as objects or entities depends upon the way in which observers inside or outside the domain divide their worlds.

It is very difficult to make a detailed mathematical model of such situations. Each actor is an actor in more than one play. His actions undergo separate but related interpretations, depending upon the others with whom he interacts. Mutual feedback of a multiplicity of ongoing processes is not easily described in the Platonic terms of pure mathematics.

Nevertheless, we take as a general principle for a mathematical model that D is a certain set (possibly evolving in time), and we let [D,D] denote a selected collection of mappings from D to D. An element F of [D,D] is a mapping F:D ----->D.

We shall assume that there is a 1-1 correspondence I:D -----> [D,D]. This is the assumption of reflexivity. Every element of the reflexive domain is a transformation of that domain. Each denizen of the reflexive domain has a dual role of actor and actant.

Given an element g in D, I(g):D ----> D is a mapping from D to D, and for every mapping F:D ----> D, there is an element g in D such that I(g) = F. The reflexive domain embodies a perfect correspondence between actions, and entities that are the recipients of these actions.

An important precursor to this notion of reflexive domain in mathematics is the notion of Goedel numbering of texts. One chooses a method to encode a text as a specific natural number (a certain product of prime powers). Then texts that speak about numbers can, in principle speak about other texts and even about themselves. If a text is seen as a transformation on the field of numbers, then that text is itself a number (its Goedelian code) and so can be transforming itself. The precision of this idea enabled Goedel to construct mathematical systems that could talk about their own properties without contradiction and he showed that all sufficiently rich mathematical systems have this property. In this way, these systems become self-limiting due to the possibility of statements whose coded meaning becomes "This statement has no proof in the system of mathematics in which it is written," while the surface meaning of the same statement is a discussion of the properties of certain numerical relations. The domain of numerical relations appears innocuous, and yet it sows the seeds of its own limitations through this ability to reflect itself through the mirror of the Goedel coding.

The Goedelian example is not just a piece of mathematics. It is a reflection with mathematical precision of the condition of our language, thought and action. We are always equipped to comment on our own doings and in so doing to create new language about our old language and new language about our worlds. All our apparent well-thought-out and directed actions in worlds that seem to extend outward from us in an objective way are fraught with the circularity not just of our meta-comments, but also with the circular return of the consequences of those actions and the influence of our very theories of the world on the properties of that world itself.

We now prove a fundamental theorem about reflexive domains. We show that every mapping F:D -----> D has a fixed point p, an element p in D such that F(p) = p. What does this mean? It means that there is another way, in a reflexive domain, to associate a point to a transformation. The point can be seen as the fixed point of a transformation and in that way, the points of the domain disappear into the self-referential nature of the transformations.

Let me tender persuasions. Suppose that p = F(p). Then we can regard this equation as an expression of p in terms of F and itself and write

$$
\begin{aligned}
p &= F(p) \\
&= F(F(p) \\
&= F(F(F(p))) \\
&=F(F(F(F(p))))
\end{aligned}
$$

and continue in this fashion until the appearance of p on the right hand side is lost in the depths of the composition of F upon itself.

$$p = F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(...)))))))))))))))))))).$$

The infinite composition of F upon itself is invariant under one more composition with F and so F(p) = p is consistent with this process.

To show that an entity p is a fixed point for a process F is to show that p can be confused with the infinite concatenation of F upon itself. This is an image of the way objects become tokens for eigenbehaviours in the language of Heinz von Foerster [ ]. Later in this paper we will discuss the production of many examples of such eigenforms, fixed points of repeated transformations. For now, here is the proof of the fixed point theorem for reflexive domains.

Fixed Point Theorem. Let D be a reflexive domain with actor/actant correspondence F:D -----> [D,D]. Then every F in [D,D] has fixed point. That is, there exists a p in D such that F(p) = p.

Proof. Define G:D ----> D by the equation
Gx = F(I(x)x)for each x in D.
Since I:D -----> [D,D] is a 1-1 correspondence,
we know that G = I(g) for some g in D.
Hence Gx = I(g)x = F(I(x)x) for all x in D.
Therefore, letting x = g,
I(g)g = F(I(g)g) and so p = I(g)g is a fixed point for F.
Q.E.D.

We shall discuss this proof and its meaning right now in a series of remarks, and later in the paper in regard to examples that will be constructed.

Remark1.
Suppose that we reduce the notational complexity of our description of the reflexive domain by simply saying that for any two entities g and x in the domain there is a new entity gx that is the result of the interaction of g and x. (We think of gx as I(g)x = I(g) applied to x.) In mathematical terms, we define

$$gx = I(g)x.$$

Then the proof of the fixed point theorem appears in a simpler form: We define Gx = F(xx) and note that GG = F(GG). Thus GG is the fixed point for F!

I like to call G " F's Gremlin".

According to Webster [Webster's New Collegiate Dictionary , G. C. Merriam Publishers (1956)] a gremlin is "One of the impish foot-high gnomes whimsically blamed by airmen for interfering with motors, instruments, machine guns, etc.; hence any like disruptive elf."

This is an apt description of our G. At first G looks quite harmless. Applying G to any A we just apply A to itself and apply F to the result. GA = F(AA). The dangerous mixture is comes when it is possible to apply G to itself! Then GG = F((GG) and GG is sitting right in there surrounded by F and you cannot stop the action. Off goes the recursion

GG = F(GG)
= F(F(GG))
= F(F(F(F(GG))))
= F(F(F(F(F(F(F(F(GG))))))))

The diabolical nature of the Gremlin is that he represents a process that once started, is hard to stop. Such are the processes by which we make the world into a field of tokens and symbols and forget the behaviours and processes and reflexive spaces from which they came. Fixed points and self-references are the unavoidable fruits of reflexivity, and reflexivity is the natural condition in a universe where there is no complete separation of part from the whole.

Remark 2.
A reflexive domain is a place where actions and events coincide. An action as a mapping of the whole space, because there is no intrinsic separation of the local and the global. Feedback is an attempt to handle the lack of separation of part and whole by describing their mutual influence.

When we define a new element g of D via gx = F(x) for any mapping F:D---->D, and we have a notion of combination of elements of D: a,b -----> ab, then we can define gx = F(xx) and so get gg = F(gg). Here we have not made a big separation between the elements of D and the mappings, since each element g of D gives the mapping I(g)x = gx. But in fact, we could define ab = I(a)b in a reflexive domain.

Whenever anyone comes up with a transformation, we make that transformation into an element of the domain by the definition gx = F(x). We transmute verbs to nouns. The reflexive domain evolves.

The space is not given a priori. The space evolves in relation to actions and definitions. The road unfolds before us as we travel.

Remark 3.

We create languages for evolving concepts. The outer reaches of set theory (and category theory) lead to clear concepts, but these concepts are not themselves sets or categories. A good example is the famous Russellian concept of sets that are not members of themselves. Russell's concept is not a set. Another example is the concept of set itself. There is no set that is the set of all sets. This very limitation on the notion of set is its opening. It shows us that set theory is an evolving language.

Language and concepts expand in time.

Here is a transformation on sets: $F(X) = \{X\}$. The transform of a set X is the singleton set whose member is X. If X is not a member of itself, then F(X) is also not a member of itself. But a fixed point of the transformation F is an entity U such that $\{U\} = U$. We have shown that within the domain of sets that are not members of themselves, there is no fixed point for the transformation X -----> {X}. This fragment of set theory (sets that are not members of themselves) is not yet a reflexive domain. We shall at least allow sets that are members of themselves if we wish to have a set theory with reflexivity.

Remark 4.

Transcendence

The leap to infinity via self-reference.
The production of the finity of a new level of infinity.
The completion of an incompletion.
The emergence of eternity from the world of time.

How then is observation different from action? If observation is a form of recursion coupled with the production of the finity of the limiting form, then observation is a transcendence to a new level. The model of observation as simple eigen-vector must be shifted to observation as the production of eigenform. It is not enough to produce eigenform. The fixed point is itself an active element and can itself engage in transformation

In the creation of spaces of conversation for human beings, we partake of a reflexivity of action and apparent object, where it is seen that every local manifestation of process, every seemingly fixed entity in a moving world is an indicator of global transformation. The local and the global intertwine in a reflexive and cybernetic unity.

Retuning (returning/tuning/retuning) to thoughts of reflexivity.
One creates by going outside oneself, but the creation returns in the form of a conversation with one's self. There is a feedback loop between the person/designer and the world that she makes. Each acts in the creation of the other. Priorities may be assigned, but it is the loop that interests us, and the possibility of stability (or at least temporal persistence) of what is created in that loop.

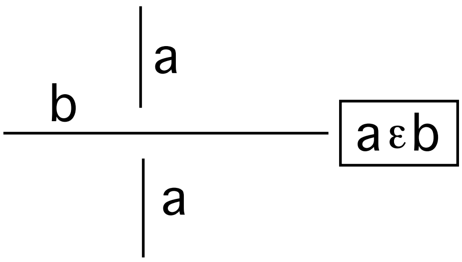
VI. Knot Sets, Topological Eigenforms, Quandles and Right and Left Distributivity

We shall use knot and link diagrams to represent sets. More about this point of view can be found in the author's paper "Knot Logic" [9]. In this notation the eigenset Ω satisfying the equation

$$\Omega = \{\Omega\}$$

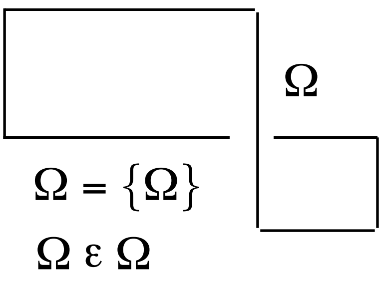
is a topological curl. If you travel along the curl you can start as a member and find that after a while you have become the container. Further travel takes you back to being a member in an infinite round. In the topological realm Ω does not have any associated paradox. This section is intended as an introduction to the idea of *topological eigenforms*, a subject that we shall develop more fully elsewhere.

Set theory is about an asymmetric relation called *membership.* We write **a Ɛ S** to say that **a** is a member of the set **S.** In this section we shall diagram the membership relation as follows:

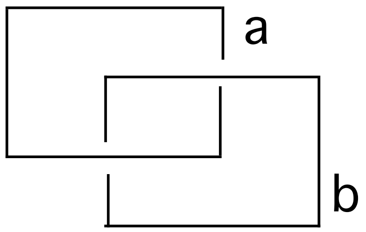


This is *knot-set notation.* In this notation, if b goes once under a, we write a={b}. If b goes twice under a, we write a={b,b}. This means that the "sets" are multi-sets, allowing more than one appearance of a member. For a deeper analysis of the knot-set structure see [KL].

This knot-set notation allows us to have sets that are members of themselves,

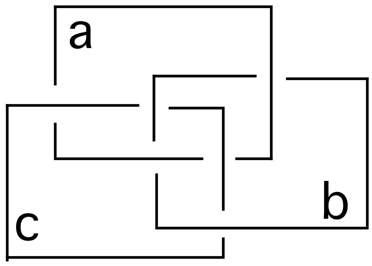


and sets can be members of each other.



a={b}
b={a}

Here a mutual relationship of **a** and **b** is diagrammed as topological linking.



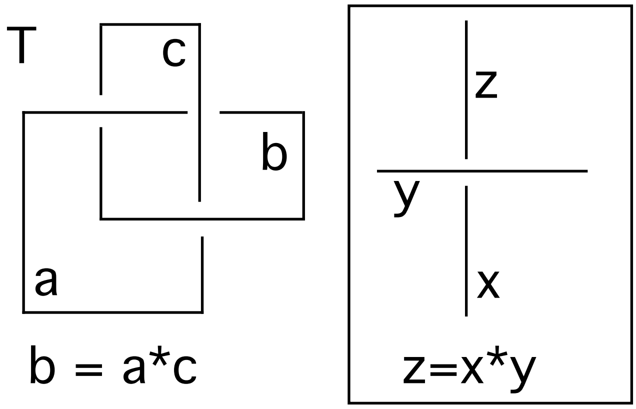
a = {b,b}
b = {c,c}
c = {a,a}

Here are the *Borromean Rings.* The Rings have the property that if you remove any one of them, then the other two are topologically unlinked. They form a topological tripartite relation. Their knot-set is described by the three equations in the diagram. Thus we see that this representative knot-set is a "scissors-paper-stone" pattern. Each component of the Rings lies over one other component, in a cyclic pattern.

To go beyond this first level of knot set theory we need to examine the formal structure of the relationships among the arcs on a link diagram.

Quandles and Colorings of Knot Diagrams

There is an approach to studying knots and links that is very close to our knot sets, but starts from a rather different premise. In this approach each arc of the diagram receives a label or "color". An arc of the diagram is a continuous curve in the diagram that starts at one undercrossing and ends at another undercrossing. For example, the trefoil diagram below has three arcs.



$$b = a*c$$
$$c = b*a$$
$$a = c*b$$

Each arc corresponds to an element of a "Trefoil Color Algebra" **IQ(T)** where **T** denotes the trefoil knot. We have that **IQ(T)** is generated by colors **a**,**b** and **c** with the relations

$$\mathbf{a*a = a,}$$
$$\mathbf{b*b = b,}$$
$$\mathbf{c*c = c,}$$
$$\mathbf{a*b = b*a = c,}$$
$$\mathbf{b*c = c*b = a,}$$
$$\mathbf{a*c = c*a = b.}$$

Each of these relations in the diagram above is a description of one of the crossings in **T**. The full set of relations describes the coloring rules for an algebra that contains these relations and allows any two elements to be combined to a third element. This three-element algebra is particularly simple. If two colors are different, they combine to form the remaining third color. If two colors are the same, they combine to form the same color.

When we take an algebra of this sort, we want its coloring structure to be invariant under the Reidemeister moves (illustrated below). This means that when you make a new diagram from the old diagram by a topological move, the resulting new diagram inherits a unique coloring from the old diagram. Then one can see from this that the trefoil must be knotted since all diagrams topologically equivalent to it will carry three colors, while an unknotted diagram can carry only one color.

As the next diagram shows, invariance of the coloring rules under the Reidemeister moves implies the following *global relations on the algebra:*

$$\mathbf{x*x = x}$$
$$\mathbf{(x*y)*y = x}$$
$$\mathbf{(x*y)*z = (x*z)*(y*z)}$$

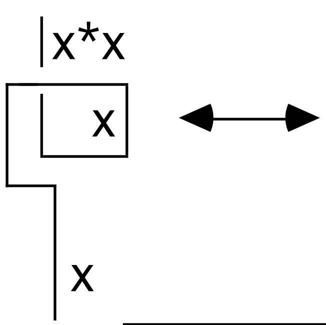
for any **x**, **y** and **z** in the algebra (set of colors) **IQ(T)**.

An algebra that satisfies these rules is called an *Involutory Quandle* [9], hence the initials **IQ**. Perhaps the most remarkable property of the quandle is its right-distributive law corresponding to the third Reidemeister move, as illustrated below. The reader will be interested to observe that in a multiplicative group **G**, the following operation satisfies all the axioms for the quandle: $\mathbf{g*h = hg^{-1}h}$.

In an additive and commutative version of this axiom we can write $\mathbf{a*b = 2b - a}$. Here the models that are most useful to the knot theorist are to take **a** and **b** to be elements of the integers **Z** or elements of the modular number system $\mathbf{Z/dZ = Z_d}$ for some appropriate modulus **d.** The knot being analyzed restricts the modular possibilities. In the case of the trefoil knot the only possibility is $\mathbf{d = 3}$, and in the case of the Figure Eight knot (shown after the Reidemeister moves below) the only possibility is $\mathbf{d = 5}$.

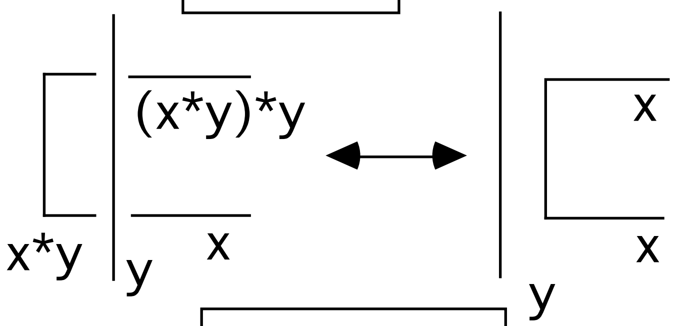
This analysis then shows that there cannot be any sequence of Reidemeister moves connecting the Trefoil and the Figure Eight. They are distinct knot types.

I.



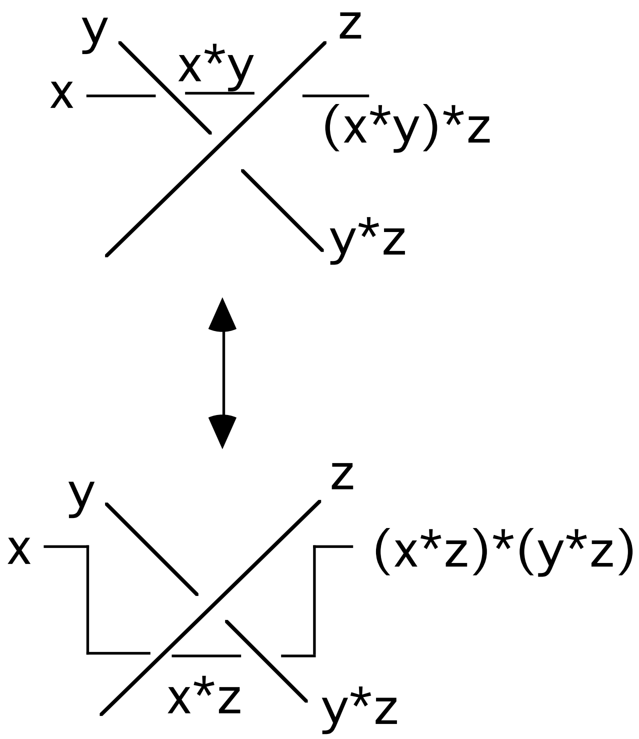
$$\boxed{x*x=x}$$

II.



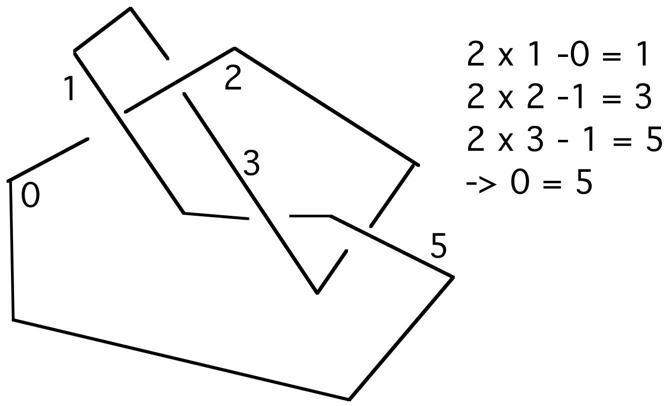
$$\boxed{(x*y)*y = x}$$

III.



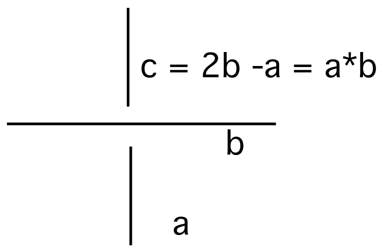
$$\boxed{(x*y)*z = (x*z)*(y*z)}$$

Here is the example for the Figure Eight Knot.



Z/5Z = {0,1,2,3,4} with 0 = 5.

We have shown how an attempt to label the arcs of the knot according to the quandle rule



a*b = 2b -a, leads to a labeling of the Figure Eight knot in **Z/5Z**. In our illustration we have shown that there is a compatible coloring using four out of the five elements of **Z/5Z**. If you apply Reidemeister moves to the diagram for the Figure Eight knot you will see that other versions of the knot require all five colors. It is interesting to prove that there is no diagram of the Figure Eight knot that can be colored in less than four colors.

It should be noted that the knot diagrams give a remarkable picture of non-associative algebra structure and that each arc-label **a** in a diagram is both an element of the algebra and a transformation of the algebra to itself via the mapping $\mathbf{O_a(x) = x*a}$. Note that the right distributivity of this operation has the equation

$$\mathbf{O_a(x*y) = (x*y)*a = (x*a)*(y*a) = O_a(x)*O_a(y)}$$

That is, we have

$$\mathbf{O_a(x*y) = O_a(x)*O_a(y).}$$

The right distributive law tells us that e*ach quandle operation is a quandle homomorphism*. That is, each quandle operation is a structure preserving mapping of the quandle to itself. This is an underlying algebraic meaning of the third Reidemeister move. Since the mappings $\mathbf{O_a}$ are invertible, we see that any quandle **Q** is in 1-1 correspondence with a certain collection of automorphisms of itself. In this sense a quandle is a reflexive domain with a limitation on the allowable collection of self-mappings. In fact we have a very simple fixed point theorem for quandles since

$$\mathbf{O_a(a) = a*a = a.}$$

Every element of the quandle is fixed by its own automorphism. Since we take **[Q,Q]** to be the set of mappings of **Q** to itself of the form $\mathbf{O_a(x) = x*a}$**,** we see that any quandle is a reflexive domain of a restricted sort. (Not every set theoretic mapping of **Q** to **Q** is realized in the above manner.)

How far is the quandle from being a reflexive space in the full sense of the word? Lets look at the fixed point construction. We define **G(x) = (x*x)*F** for a given element **F** of the quandle . Is it then the case that **(x*x)*F = x*g** for some g in the quandle? The answer is yes, but for very simple reason: We have **x*x = x** so that **(x*x)*F = x*F** and consequently **(F*F)*F = F*F**. In fact **F*F = F** so **F** is already its own fixed point. We see therefore that in a quandle the fixed point theorem is satisfied automatically due to the axiom **x*x = x** for all **x.**

On the other hand if **F:Q ---->Q i**s an arbitrary mapping from **Q** to **Q,** then we cannot expect that **F** will have a fixed point. Suppose, for example, we define **F(x) = (x*(a*x)** and use the Trefoil quandle. Then

$$\mathbf{F(a) = (a*(a*a)) = a,}$$
$$\mathbf{F(b) = (b*(a*b)) = b*c = a,}$$
$$\mathbf{F(c) = (c*(a*c)) = c*b = a.}$$

Thus F has no fixed point, verifying that the Trefoil quandle is not a full reflexive domain.

Left Distributivity

We have written the quandle as a right-distributive structure with inveritible elements. It is mathematically equivalent to use the formalism of a left distributive operation. In left distributive formalism we have **A*(b*c) = (A*b)*(A*c)**. This corresponds exactly to the interpretation that each element **A** in **Q** is a mapping of **Q** to **Q** where the mapping **A[x] = A*x** is a structure preserving mapping from **Q** to **Q**.

$$\mathbf{A[b*c] = A[b]*A[c].}$$

We can ask of a domain that *every element of the domain is itself a structure preserving mapping of that domain.* This is very similar to the requirement of reflexivity and, as we have seen in the case of quandles, can often be realized for small structures such as the Trefoil quandle.

We call a domain M with an operation * that is left distributive a *magma*. Magmas are more general than the link diagrammatic quandles. We take only the analog of the third Reidemeister move and do not assume any other axioms. Even so there is much structure here. A magma with no other relations than left-distributivity is called a *free magma.*

The search for structure preserving mappings can occur in rarefied contexts. See for example the work of Laver and Dehornoy [9] who studied mappings of set theory to itself that would preserve all definable structure in the theory. Dehornoy realized that many of the problems he studied in relation to set theory were accessible in more concrete ways via the use of knots and braids. Thus the knots and braids become a language for understanding for formal properties of self-embedded structure.

Structure preserving mappings of set theory must begin as the identity mapping since the relations of sets are quite rigid at the beginning. (You would not be able to map an empty set to a set that was not empty for example, and so the empty set would have to go to itself.) The existence of non-trivial structure preserving mappings of set theory questions the boundaries of definability and involves the postulation of sets of very large size. See [16] for a good exposition of the philosophical issues about such embeddings and for an approach to wholeness in physics that is based on these ideas.

It is worth making a remark here about sets. Consider the collection Aleph of all sets whose members are themselves sets and such that any investigation into membership will just reveal more sets as members. Typical elements of Aleph are the empty set { }, the set whose member is the empty set { { } } and of course various curious constructs that have infinitely many members such as

$$\{ \{ \} , \{\{ \}\}, \{\{\{ \}\}\}, \{\{\{\{ \}\}\}\}, ... \}$$

and we may even consider sets that are members of themselves (eigen-sets!) such as

$$\{ \{ \{ \{ \{ ... \} \} \} \} \}.$$

The key thing to understand about Aleph as a class of sets is that any member of Aleph is, by definition, a subset of Aleph. And any subset of Aleph is by definition a member of Aleph. This is a beautiful property of the class Aleph, and it is a paradoxical property if we imagine that Aleph is a set! For if Aleph is a set, then we have just shown that Aleph is in 1-1 correspondence with the set of subsets P(Aleph) of Aleph. If X is any set then we denote the set of subsets of X by P(X). Cantor's Theorem (proved here in Section VIII and related in that section o the fixed point theory of reflexive domains) tells us that f*or any set X, P(X) is larger than X.*

This means that there cannot be a 1-1 correspondence between Aleph and P(Aleph) if Aleph is a set.

We can only conclude that Aleph is not a set. It is a class, to give it a name. It is an unbroken wholeness whose particularities we can always consider, but whose totality will always elude us. The way that the totality of Aleph eludes us is right before our eyes. Any particular element of Aleph is a set and it is a collection of sets as well. But we cannot complete Aleph. Any attempt to approximate Aleph as a set will always have some subsets that have not been tallied inside itself and so the set of subsets of the approximation will grow beyond that approximation to a new and larger domain of sets. Philosophically, this observation of the unreachability of Aleph, the set of all sets, as a set itself is very interesting and important. We see here how a perfectly clear mathematical concept may always remain outside the bounds of the formalities to which it refers and yet that concept is indeed composed of these formalities. It is the leading presence of the ultimately huge and unattainable Aleph that leads us to consider exceeding large sets in the pursuit of a flexibility in self-embeddings of set theory. At the end of Section VIII we take an alternative view of Aleph and consider what would have to change if Aleph were admitted to be a set.

Enough said about the abstract reaches of the magma. We should not expect that any given structure is a reflexive space. But it is possible to create languages that can expand indefinitely and thus partake of the ideal of reflexivity.

VII. Church and Curry

In this section we point out how the notion of a reflexive domain first appeared in the work of Alonzo Church and Haskell Curry [1] in the 1930's. This method is commonly called the "lambda calculus". The key to lambda calculus is the construction of a self-reflexive language, a language that can refer and operate upon itself. In this way eigenforms can be woven into the context of languages that are their own metalanguages, hence into the context of natural language and observing systems.

In the Church-Curry language (the lambda calculus), there are two basic rules:

1. Naming. *If you have an expression in the symbols in lambda calculus then there is always a single word in the language that encodes this expression.* The application of this word has the same effect as the application of the expression itself.

2. Reflexivity. *Given any two words A and B in the lambda calculus, there is permission to form their concatenation AB, with the interpretation that A operates upon or qualifies B.* In this way, every word in the lambda calculus is both an operator and an operand. The calculus is inherently self-reflexive.

Here is an example. Let **GA** denote the process that creates two copies of **A** and puts them in a box.

$$GA = \boxed{AA}$$

In lambda calculus we are allowed to apply **G** to itself. The result is two copies of **G** next to one another, inside the box.

$$GG = \boxed{GG}$$

This equation about **GG** exhibits **GG** directly as a solution to the eigenform equation

$$X = \boxed{X}$$

thus producing the eigenform without an infinite limiting process.

More generally, we wish to find the eigenform for a process **F**. We want to find a **J** so that **F(J) = J.** We create an operator **G** with the property that

GX = F(XX)

for any **X**. When **G** operates on **X, G** makes a duplicate of **X** and allows **X** to act on its duplicate. Now comes the kicker. Let **G** act on herself and look!

GG = F(GG)

So **GG** is a fixed point for **F**.

We have solved the eigenform problem without the excursion to infinity. If you reflect on this magic trick of Church and Curry you will see that it has come directly from the postulates of **Naming** and **Reflexivity** that we have discussed above. These notions, *that there should be a name for everything,* and that *words can be applied to the description and production of other words*, allow the language to refer to itself and to produce itself from itself. The Church-Curry construction was devised for mathematical logic, but it is fundamental to the logic of logic, the linguistics of linguistics and the cybernetics of cybernetics.

I like to call the construction of the intermediate operator **G**, the "gremlin" (See [10].) Gremlins seem innocent. They just duplicate entities that they meet, and set up an operation of the duplicate on the duplicand. But when you let a gremlin meet a gremlin then strange things can happen. It is a bit like the story of the sorcerer's apprentice. A recursion may happen whether you like it or not.

An eigenform must be placed in a context in order for it to have human meaning. The struggle on the mathematical side is to control recursions, bending them to desired ends. The struggle on the human side is to cognize a world sensibly and communicate well and effectively with others. For each of us, there is a continual manufacture of eigenforms (tokens for eigenbehaviour). Such tokens will not pass as the currency of communication unless we achieve mutuality as well. Mutuality itself is a higher eigenform. As with all eigenforms, the abstract version exists. Realization happens in the course of time.

VIII. Cantor's Diagonal Argument and Russell's Paradox

Let **AB** mean that *B is a member of A.*

Cantor's Theorem. Let **S** be any set (**S** can be finite or infinite).
Let **P(S)** be the set of subsets of **S**. Then **P(S)** is bigger than **S** in the sense that for any mapping **F: S -----> P(S)** there will be subsets **C** of **S** (hence elements of **F(S)**) that are not of the form **F(a)** for any **a** in **S**. In short ,the power set **P(S)** of any set **S** is larger than **S**.

Proof. Suppose that you were given a way to associate to each element **x** of a set **S** a subset **F(x)** of **S**. Then we can ask whether **x** is a member of **F(x)**. Either it is or it isn't. So lets form the set of all **x** such that **x** is not a member of **F(x)**. Call this new set **C.** We have the defining equation for **C** :

$$\mathbf{Cx = \sim F(x)x}.$$

Is **C = F(a)** for some a in **S**?
If **C = F(a)** then for all **x** we have
F(a)x = ~F(x)x.
Take **x = a.** Then
F(a)a = ~F(a)a.

This says that a is a member of **F(a)** if and only if **a** is not a member of **F(a)**. This shows that indeed **C** cannot be of the form **F(a)**, and we have proved that the set of subsets of a set is always larger than the set itself.

Note that in the usual language,

$$\mathbf{C = \{ x \text{ in } X \mid x \text{ is not a member of } F(x) \}.}$$

Note the problem that the assumption that **C = F(a)** gave us. If **C = F(a)**, then **F(a)a = ~F(a)a**. We would have a fixed point for negation. But there is no fixed point for negation in classical logic! If we had enlarged the truth set to

$$\mathbf{\{T, F, I\}}$$

where **~I=I** is an eigenform for negation, then **F(a)a** would have value **I**. What does this mean? It means that the index **a** of the set **F(a)** corresponding to it would have an oscillating membership value. The element **a** would be like Groucho Marx who declared that he would not join any club that would have him as a member. We would be propelled into sets that vary in time.

Note that our proof of Cantor's Theorem has exactly the same form as our earlier proof of the existence of fixed points for a reflexive space.

he mapping **F:X -----> P(X)** takes the role of the 1-1 correspondence between **D** and **[D,D]**. The reader will enjoy thinking about this analogy. In the Cantor Theorem we have used the non-existence of a fixed point for negation to deduce a difference between and set **X** and its powerset **P(X)**. In the study of a reflexive domain we have shown the existence of fixed points, but we have seen that such domains must be open to new elements and new transformations.

There are many points of view about Cantor's Theorem. Lets start again by considering the assemblage (we shall not call it a set) **Aleph** of all sets whose members are sets that are members of **Aleph**. That is, a set **S** is a member of **Aleph** if every member of **S** is a set and when you look at the members of the members, they too are sets, and this process of finding sets continues to all depths. We allow the possibility of infinite depth of membership and hence the possibility of self-membership for sets in **Aleph**. Note that **Aleph** is a natural concept - the concept of sets that are made up from sets. But by definition, any set **S t**hat is a member of **Aleph** is also a subset of **Aleph**. And by definition, any subset of **Aleph** is a member of **Aleph!** Thus **Aleph** is *identical* with **P(Aleph)**. *According to Cantor's Theorem, **Aleph** is not a set.*

What is the contradiction that Cantor's Theorem produces for **Aleph?** Cantor forms *C = {x in **Aleph**| x is not a member of x }* since we can take **F:Aleph -----> P(Aleph)** to be the identity mapping. But is this a contradiction?! It would be a contradiction if we knew that **C** is a set. Then **C** would be a member of itself if and only if it was not a member of itself.

But **C** is not a set! **C i**s itself a contradiction. **C** is the Russell paradox. We have that **C** is a member of **C** if and only if **C** is not a member of **C.** Cantor's process applied to **Aleph** produces a set that is supposed to be a new subset of **Aleph**, but in fact it is a paradoxical set. We could take the point of view that this shows that there are cases where the Cantor definition *C = {x in X| x is not in F(x)}* leads to an undefined set, a set for which one cannot actually decide on the membership of certain elements. In that viewpoint, **Aleph** may be considered an example of a set to which Cantor's Theorem does not apply.

We say, how did this happen? Isn't it always clear whether or not **x** is in **F(x)**? You would think so. But in the case at hand we have **F(x) = x** and the question becomes: *does **x** belong to **x**?* And then we see that as far as **C** itself is concerned this question creates an iterant, an oscillation, a paradox. By applying Cantor's argument to **Aleph**, *we have found iterants and imaginary values at the very heart of set theory.*

The notion that we can always specify a set by a definition in the form **S = {x | P(x) }** where **P(x)** is a logical proposition is *naive.* The propositional statement provides a criterion of distinction, but it is possible that this criterion will be circular or undecideable. So we have to keep attending to what we define, and find out when it makes the sense. Why should such things be automatic?

IX. The Secret

What is the simplest language that is capable of self-reference? We are all familiar with the abilities of natural language to refer to itself. Why this very sentence is an example of self-referentiality. The American dollar bill declares "This bill is legal tender.". The sentence that you are now reading declares that you, the reader, are complicit in its own act of reference. But what is the simplest language that can refer to itself?

The simplest language would have a simple alphabet. Let us say it has only the letter R. The words in this language will be all strings of R's. Call the language LS. The words in LS are the following:

R,
RR,
RRR,
RRRR,
and so on.

Two words are equal if they have the same number of letter R's. E*ach word makes a meaningful statement of reference via the rule:*

If X is a word in LS, then RX refers to XX.
RX refers to XX, the repetition of X.

Thus RRR refers to RRRR (not to itself), and R refers to the empty word.

There is a word in LS that refers to itself. Can you find it?

Lets see.
RX refers to XX.
So we need XX = RX if RX would refer to RX.
If XX = RX, then X = R.
So we need X = R.
And RR refers to itself.

The little language LS looks like a pedantic triviality, but it is actually at the root of reflexivity, Godel's incompleteness Theorem, recursion theory, Russell's paradox and the notion of self-observing and self-referring systems. It seems paradoxical that what looks like a trick of repeating a symbol can be so important. The trick is more than just a trick.

Just to show you how this works, consider Russell's paradox again. Russell asks us to consider the set of all sets that are not members of themselves. Lets call this set B for "Bertrand Russell". Lets write YX to mean "X is a member of Y". And write ~YX to mean "X is not a member of Y". OK?

Then Russell's set is defined by the equation

$$BX = \sim XX.$$

(= means "if and only if" in a logical context) Read it out loud:

"X is a member of B if and only if X is not a member of X". Exactly. What about B? Is B a member of B? Try it. Let X = B. Then

$$BB = \sim BB.$$

"B is a member of B if and only if B is not a member of B."

This is the Russell paradox. You see that in the form $BX = \sim XX$ the Russell paradox is an instance (in a slightly more complex language) of exactly our LS trick of self-reference.

The Russell paradox continues to act as a mystery at the center of our attempts to relate syntax and semantics. In that center is a little trick of syntactical repetition. I would like to think that when we eventually discover the true secret of the universe it will turn out to be this simple.

The snake bites its tail. The Universe is constructed in such a way that it can refer to itself. In so doing, the Universe must divide itself into a part that refers and part to which it refers, a part that sees and a part that is seen.

Let us say that R is the part that refers and U is the referent. The divided universe is RX and RX = U and RX refers to U (itself). Our solution suggests that the Universe divides itself into two identical parts each of which refers to the universe as a whole. This is

$$RR.$$

In other words, the universe can pretend that it is two and then let itself refer to the two, and find that it has in the process referred only to the one, that is itself.

The Universe plays hide and seek with herself, pretending to divide herself into two when she is really only one. And that is the secret of the Universe and that is the universal source of our trick of self-reference.

X. The World of Recursive Emergence and Creativity

We have repeatedly insisted that a formal fixed point or eigenform is associated with any transformation T in any domain where infinite composition of transformations is possible. Thus we make $E = T(T(T(T(T(...)))))$ and find that $E = T(E)$. This is the symbolic fixed point that sometimes corresponds to a stability in the original domain of the recursion. We have also seen that one can take a seed z for the recursion and repeatedly form

$$z, T(z), T(T(z)), T(T(T(z))), ...$$

in a temporal sequence or recursive process. Then the finite products of this process can exhibit similarity to the infinite eigenform, and they can also exhibit novelty and emergence structure in ways that are most surprising. It is this appearance of creativity and novelty in recursive process that makes reflexivity more than abstract mathematics and more than a philosophical idea.

The purpose of this last section is to exhibit an example involving cellular automata that illustrates these ideas and gives us a platform for thought. In this example, we are using an algorithm that I call 7-Life. It is a variant of the Life automaton of John H. Conway. Conway's automaton is governed by the rule B3/S23 which means that a white square in the grid is born (B) when it has 3 neighbors and it survives (S) when it has exactly 2 or 3 neighbors. Life has the property that there are many intriguing formations and processes, but statistically most configurations dies out to a collection of isolated static patterns (still lifes) and oscillating patterns that do not grow or interact outside themselves.

7-Life has the rule B37/S23 and has many of the properties of Life, plus the phenomenon that many starting configurations grow, self-interact and produce streams of *gliders*. The gliders are five-square formations (occurring in Life as well) that occur spontaneously and regenerate themselves, appearing to move along diagonal directions in the process. The most striking property of 7-Life is the long term persistence of such self-interacting configurations, growing slowly in complexity over time.

In the Figures 1,2, and 3 we indicate the result of applying the 7-Life algorithm to a simple and not-quite symmetrical starting configuration shown in Figure 1. In Figure 2 we see the result of 33911 iterations of the process. We now have a galaxy of complex interactions. The small entities radiating away from the galaxy are gliders, as described above, and if a reader were to watch the process using a computer program, he or she would see a teeming, seemingly random mass of activity. Then in Figure 3 we see that after 49281 iterations something new has emerged. It seems that a highly patterned *dragon* is emerging from the chaos of the complex process. The tip of this dragon moves forward relentlessly. The body of the dragon interacts with the glider radiation and begins to roil in the chaotic process. So far, the growing tip of the dragon has not interacted with any gliders.

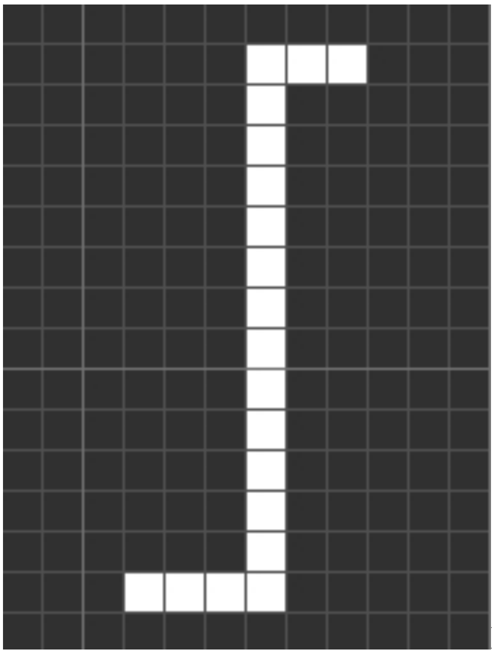


Figure 1. The Starting Configuration

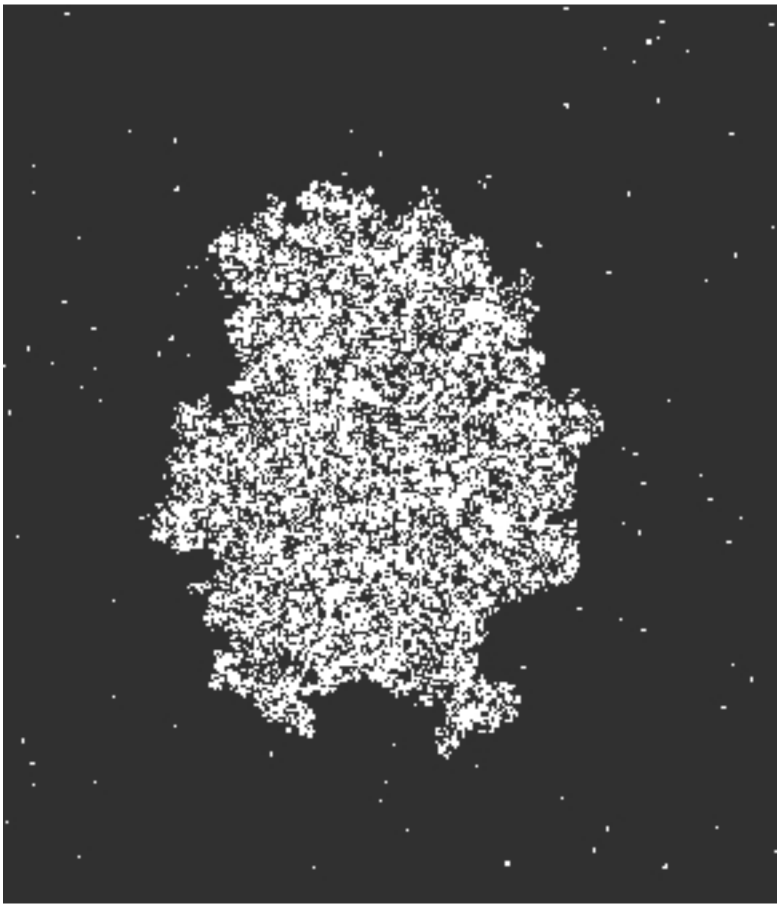


Figure 2. After 33911 Iterations

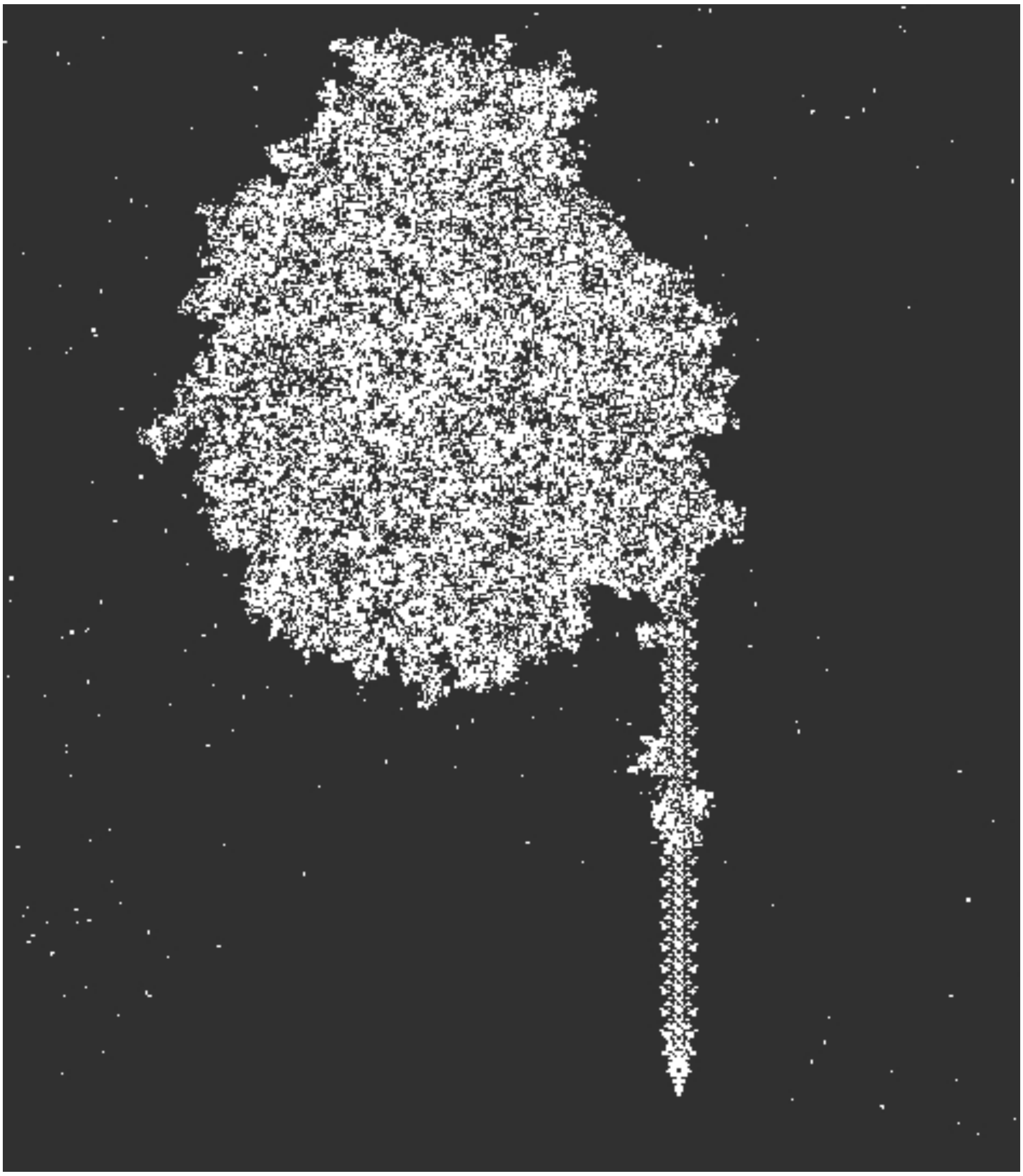


Figure 3. After 49281 Iterations

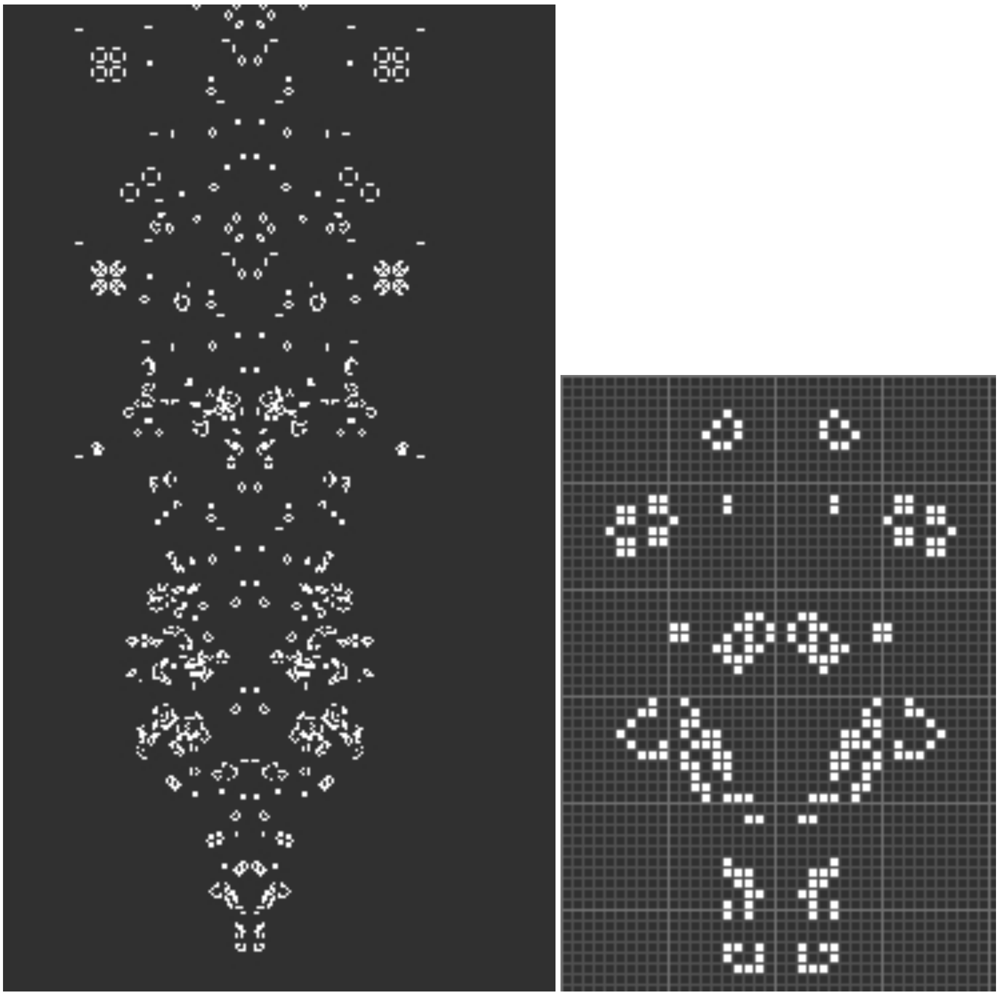


Figure 4. The Growing Tip

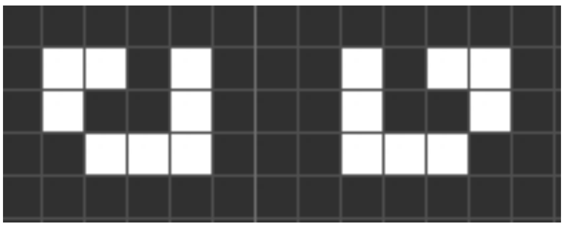


Figure 5. The Generating Tip GG

Figures 4 gives closeups of the tip of the dragon and Figure 5 isolates the generator, GG, of the dragon itself. This configuration GG of 16 squares in mirror symmetry, when placed on an otherwise blank lattice will generate the dragon in the 7-Life algorithm.

What has happened is that this 16-square generator GG has appeared the course of the complex interactions, and it has had enough room to move forward in its own pattern -- forming the dragon behind it and periodically regenerating itself. The generator of the dragon, GG, is not our invention. GG is a natural consequence of the complex process of 7-Life. GG emerges, but with much lower probability than the gliders. The result is an appearance of novelty and creativity in the complex process as it happens over time. We can only speculate what more complex entities would eventually emerge in 7-Life over many more iterations.

Just so does DNA emerge from the complex process of the world of the earth and sun.

We see from this example that eigenforms that are processes, such as the self-generating GG, c*an and will emerge of their own accord* from complex systems based on recursion. In this sense, such systems begin to generate their own reflexive spaces. The novel and self-reproducing forms that emerge from them can be seen in a similar light.

All these observations are made by an observer. The observer is clever only in the distinctions that he or she makes, and that is enough to found an entire universe.

XI. In Zermelo's Bar

The section is a multi-logue about the attempts to solve the equation of the observer in relation to his/her observation. We first encounter Mr. D, who has solved his own equation in such a way that he has no head and instead has a great open space of possibility where his head was supposed to be. This requires a drink to ingest and we go to Zermelo's Bar, where we find two mathematicians arguing over the solution to an equation whose solution is the Golden Ratio, a proportion well known to the Greeks. The mathematicians are a little hard to follow, but their discussion turns on all the essential issues of recursion, reality and infinity that we will need for this adventure. Then Dr. Von F appears in the bar (we think you can guess who this is) and explains the nature of eigenforms. He is followed by a character named Charlie and Dr. CC, a linguist and logician, then by Dr. HM, a biologist. Later there appears a physicist, Dr. JB and finally Dr. R himself, the source of the self-referential paradox. We hope that you will join in on this discussion yourself.

Infinite Recursion and Its Relatives

Our problem is to solve the equation

$$\mathbf{O(A) = A}$$

for **A** in terms of **O**.

For example, suppose that the observer **O** is Mr. D, a man who insists that he has no head. We interview him. Well Mr. D, why do you say that you have no head? Mr. D. replies. Oh it is so simple, you will see at once what I mean. In fact, consider what you yourself see. Look directly around. Do you see your head? No. You see and feel a great open space of perception where your head is supposed to be, and a flow of thoughts and feelings. But no head! The body comes in. Shoulders, arms, legs, shoes and the world. But no head. Instead of a head there is a great teeming void of perception. Once I realized this, I knew that the relationship of a self to reality was indeed deep and mysterious.

As we can see, Mr. D has discovered that what is constant for his visual observer is a body without a head. He has solved the problem of finding himself as a solution of the equation of himself in terms of himself. Perhaps we need a drink.

We walk into Zermelo's Bar and two mathematicians appear on the scene. One says to other: How do you solve this equation? I want a positive real solution.

$$\mathbf{1 + 1/A = A.}$$

The second one says: Nothing to it, we multiply both sides by the unknown **A** and rewrite as

$$\mathbf{A + 1 = A^2.}$$

Then, solving the quadratic equation, we find that

$$\mathbf{A = (1 + \sqrt{5})/2.}$$

The first mathematician says: Nice tricks you have there, but I prefer infinite reentry of the equation into itself. Look here:

If $\mathbf{A = 1 + 1/A}$, then

$$\begin{gathered}\mathbf{A =}\\ \mathbf{1 + 1/A =}\\ \mathbf{1+1/(1+1/A) =}\\ \mathbf{1+1/(1+1/(1+1/A)) =}\\ \mathbf{1+1/(1+1/(1+1/(1+1/A)))}\end{gathered}$$

and I will take this reentry process to infinity and obtain the form

$$\mathbf{A = 1+ 1/(1+1/(1+1/(1+1/(1+ 1/(1+1/(1+...)))))).}$$

The second mathematician then says: Well I like your method. We can combine our answers and write a beautiful formula!

$$\mathbf{(1+\sqrt{5})/2 \ =}$$

$$\mathbf{1+ 1/(1+1/(1+1/(1+1/(1+ 1/(1+1/(1+...))))))}$$

Why do you like this formula? says the second guy. Well, says the first guy, the left hand side is a definite irrational number and it is easy to see by squaring it that it satisfies the equation $\mathbf{A^2 = A + 1}$ as we wanted it. But irrational numbers have a curiously tenuous existence unless you know a way to calculate approximations for them. On the other hand, your right hand side can be regarded as the limit of the fractions

$$\mathbf{1 = 1/1}$$

$$1+1/1 = 2/1 = 2$$
$$1+1/(1+1/1) = 3/2$$
$$1+1/(1+1/(1+1/1)) = 5/3$$
$$1+1/(1+1/(1+1/(1+1/1))) = 8/5$$
$$1+1/(1+1/(1+1/(1+1/(1+1/1)))) = 13/8$$
$$1+1/(1+1/(1+1/(1+1/(1+1/(1+1/1))))) = 21/13$$

with the first few terms of this limit being

$$(1+\sqrt{5})/2 = 1.618...$$

On top of this your infinite formula actually does reenter itself as an infinite expression it really is of the form

$$A = 1 + 1/A.$$

The first guy comes back with: Well it sounds to me like you really believe in the "actual" infinity of the terms on the right-hand side. I also like to imagine that they are all there existing together in space with no time.

Right ! says the second guy. We know that this is an idealization, but it lets us actually reason to correct answers and to put them in an aesthetically pleasing form.

The bartender is listening to all of this, and he leans over and says: You guys have to meet a couple of others on this score. There is Dr. Von F and Dr. CC. They both have some ideas very similar to yours. Hey, here is Dr. Von F now. Dr. Von F, could you tell these fellows about your eigenforms?

Jah! Of course! It is all very simple. We just combine this notion of recursion with the most general possible situation. Suppose we have any observer **O** and we wish to find a fixed point for her. Well then we just let the observer act without limit as in

$$A = O(O(O(O(O(O(O(O(...)))))))).$$

After infinity, one more application of **O** does not change the result and we have

$$O(A) = A.$$

This is very simple, no? And it shows how we make objects. These objects are the tokens of our repeated behaviors in shaping a form from nothing but our own operations. As I have said before, the human identity is precisely the fixed point of such a recursion. "I am the observed link between myself and observing myself." [2]

The first mathematician makes a comment: What you are doing is a precise generalization of my infinite continued fraction! If I had defined

$$O(A) = 1 + 1/A$$

then we would have

$$\mathbf{O(O(O(...))) = 1+ 1/(1+1/(1+1/(1+...)))}.$$

But I am puzzled by your approach, for it would seem that you are willing that your solution **A** will have no relation with how the process starts, and also it may not related to the original domain in which it was constructed! For example, in my mathematics, I could consider the operator

$$\mathbf{O(A) = -1/A}$$

and this operator does not have a fixed point in the real numbers, but if we take **A=i** where $\mathbf{i^2 = -1}$ (the simplest imaginary number), then **O(i) = i.** Are you suggesting that

$$\mathbf{i = -1/-1/-1/... \quad ?}$$

Dr. Von F replies: Jah, Jah! This is very important! The fixed point can be a construction that breaks ground into an entirely new domain! Actually, I am mainly interested in those fixed points that do break new ground. We are looking for the places where new structures emerge. In your mathematics you have illustrated this in two ways. In the first recursion, the values converge to an irrational number (the golden ratio). All the finite approximations are rational fractions (ratios of Fibonacci numbers) but in the limit of the infinite eigenform, you arrive at this beautiful new irrational number! And in your second example all the finite approximations oscillate like a buzzer, or a paradox, between positive unity and negative unity, but the eigenform is a true representative of the imaginary square root on minus one! And don't forget that this "imaginary" quantity is fundamental to both logic and physics. The fully general eigenforms are fundamental to the ontology of the world.

Suddenly the door to Zermelo's Bar opens and in walks a character that everyone calls "Charlie." Charlie! say the barkeep, where have you been? We have a good discussion on signs going here. You have to hear this stuff. Charlie says, Well I heard just about everything Dr. Von F said as I admit here to a bit of eavesdropping on the other side of the door! These eigenforms of Von F are quite familiar to me as I have thought continuously along these lines for many years. You see, any sign once you look at it in the context of its reference and the continuous expansion of its interpretant becomes a growing complex of signs referring to other signs, growing until the references close on themselves and, as Dr. Von F correctly describes, these closures are the eigenforms, the tokens for apparently stable behaviors. As the complex of signs grows, the complex itself is a sign and as the closures occur, that sign becomes a sign for itself. We humans are in our very nature such signs for ourselves.

Dr. Von F says: Well I always say that I am the observed link between myself and observing myself. I am a sign for myself!

At this point Dr. CC chimes in: But Dr. Von F and Charlie, this excursion to recursion and infinity seems quite excessive! It is all right for mathematicians to imagine such a thing, but we humans exist in language and the finiteness of expressions. Surely you do not suggest that this profligate composition of the operator and expansion of sign complexes actually happens!

Well, Dr. CC, says Von F, I am really a physicist and well aware of the speed of physical process in relation to the very slow pace of our verbal thought. Surely you have stood between two facing mirrors and seen the near-instantaneous tunnel of reflections created by light bouncing back and forth between the mirrors. Yes, I am seriously suggesting that the self-composition of the observer is carried to high orders. These orders are sufficiently large and accomplished with such a high speed that they appear infinite in the eyes of the observer. Now you may detect the beginning of a paradoxical flight here. The very observer who is too slow to detect the difference between a large number and infinity is yet so quick and subtle that he/she can produce this flight to infinity. But I beg your pardon, this is still a matter of the interaction of slow thought and fast action. Wave your arm back and forth rapidly in front of your eyes. For all practical purposes the arm appears to be in two places at the same time! You do not deny that it is "you" that moves the arm, and it is "you" that perceives it.

I simply go further and suggest that every perception is based on such an illusion of permanency, based on the self composition of your self. You do it all and you are surprised at the result. You do it all, but you can not perceive all that you do!

Charlie adds: I agree but do not have to rest on physics. Our shortsighted view of our own nature arises from the difficulty in reckoning that our true nature is as signs for ourselves. It is only at the limit of eigenbehaviours that such signs appear simple. We partake of the complexity of the universe.

Dr CC replies: Ah Charlie and Dr. Von F, I have been working in the linguistic and logical realm and you will see that our points of view are mutually supporting. For I imagine the structure of the observer as a big network of communicating entities. These entities have so much interrelation among themselves that their identities begin to merge into one identity and that is the apparent identity of the self.

Charlie interrupts with: Yes! That is the essence of continuity.

Dr. CC continues. I agree! The infinity in my view is not with any one of them, but with the aggregate of them that has become so large as to begin to merge into a continuity.

But let me explain: If A and B are entities in my "community of the self", then they can interact with each other and with themselves. These processes of interaction produce new entitles who exist at the same level as the original entities. Can you imagine this? Of course you can, you are such an entity. For example, I suggest to you that you are the self that thinks kindly of others, that you satisfy the equation **SX = KX** where **S** is "you" and **KX** is the being "thinks kindly of **X**". *Then that entity **S*** *exists.* In the world of language, *every definable entity exists*. The consequence is that **S** might even think kindly of herself as in **SS = KS**. That **S** can think kindly of herself is, in this linguistic world, dependent on the condition that the kindly thinking observer is an observer at the same level as any other observer. Now there are many such entities. Watch this magic trick. Let

$$\mathbf{GX = O(XX).}$$

The gentility **G** is the observer who observes an entity observing herself. What happens when G observes herself? Then **G** observes herself observing herself and we have a fixed point, an eigenform!

$$\mathbf{GG = O(GG).}$$

I have constructed the eigenform without the infinite composition of the observer upon herself. Of course once this self-reflexive construction comes into the being of language then it runs automatically to the level of practical infinity and produces your recursion.

$$\mathbf{GG = O(GG) = O(O(GG)) = O(O(O(GG))) = \ldots}$$

I believe my linguistic construction provides the context for your observer's self interaction. The true infinity in my world is a distributed infinity of beings each coming into being as a name for a process of observation. This continues without end and is the basis of the coincidence of the language and the metalanguage in this world.

At this point Dr. HM, a biologist, walks into the room. He remarks: I see that you have been discussing the stability of perceptions from physical and linguistic principles. Let me tell you how I see these matters in my domain. The beings you talk about are biological, not just logical. They exist in the evolutionary flow of coordinations of coordinations that give rise to the mutual patternings that you call "language" and "thought". It is not at all surprising that each such being, coordinated with the others in the deep flow of its history in biological time will appear layered like an onion with the actions of each on each. The long time history of mutual interaction and coordination will generate the appearance of the eigenforms. But there is no "disembodied observer" who generates these forms from some abstract place. In biology there is no problem of mind (abstract observer) and body. They are one. Mind and observer both refer to the conversational domain that arises in the construction of the coordination of coordinations that is language. The disembodied observer is a fantasy that is convenient for the mathematician or the physicist. In the biological realm all forms are generated through time in an organic way.

And finally, Dr. JB enters the room, a very theoretical physicist. He says: Ah it is not surprising, but you all have the business of objects and eigenforms quite wrong. Let me start with the views of the biologist Dr. HM. You see, there is no time. None. Time is an illusion. Of course in order to tell you about this insight I shall have to use words that appear to describe states in time. That is my fate to be so projected into language. You must forgive me.

Each moment of being is eternal, beyond time. I prefer to call such moments "time capsules." Each moment contains that possibility that it can be interpreted in terms of a "history", a story of events leading up to the "present moment" that constitutes the time capsule as a whole. But this history is a pattern in eternity. That the history can be told with some coherence and that we manage to tell the story of "past events" leads us to believe that these past events "actually happened". But in fact what has happened is happening now and only now in the eternity of the time capsule whose richness derives from the superposition of its quantum states.

At this point the bartender chimes in: I'll drink to that. Time is a grand illusion and a wee scotch from my bar will convince ye o' that in less time than it takes to wink an eye!

All well and good, says Dr. R, who just walked into the bar, but as I was telling my friend Frege, if there is one thing that will give us trouble it is this notion of eternity and the non-existence of time. For as I told Gottlob just the other day, you have only to imagine the timeless reality of the set of all sets that are not members of themselves and you will have to leave logic behind! I gave up long ago my travails on this issue with Professor Whitehead. We tried to make logic go first and it was a disaster. Now I let logic run along behind and there is no problem at all. As far as fixed points are concerned my favorite is Omega, the set whose only member is Omega herself. You see that the act of set formation is nothing but an act of reflection. Omega finds herself in reflecting on herself.

Dr. CC retorts: Well, Russell, I hardly expected you to capitulate your position on logic. Your Type is hardly likely to just slip away. I prefer to make a specimen of your famous set in the following way. I let **AB** mean that "B is a member of A". Then I define your set of all sets that are not members of themselves" by the equation

$$\mathbf{Rx = \sim xx.}$$

Then we can pin the specimen to the board by substituting **R** for **x** as in

$$\mathbf{RR = \sim RR.}$$

This **RR** is a fixed point for negation. It is neither true nor false. I do not leave logic behind. I imagine new states of logical discourse that are beyond the true and the false. Your set performs this transition to imaginary Boolean values.

Now Dr. HM says: Well I see you fellows are beginning to foment an argument. I feel that I must point out to you that logical paradox occurs only in the domain of language. There is no such matter as the paradox of the Russell set in the natural domain. In the natural domain, all apparent contradictions are only antimonies in the eyes of some observer. Nature herself runs in the single valued logic of the evolutionary flow. This is why I emphasize that it is only in the linguistic domain of coordinations of coordinations that the eigenforms arise. At the biological level there are processes that can be seen as recursions, but this seeing is already at the level of the coordinations. There is no mystery in this, but it is necessary to round out the mathematical models with the prolific play and dynamics of the underlying biology. In this sense biology is prior to physics as well as cognition.

At this point a tremor shakes the bar and the lights go out. I am sorry folks, the bartender says from the darkness, but this is another one of our natural events in the single valued logical flow of biological time -- a small earthquake. I will have to ask you to leave now for your own safety. And so the discussion ended, unfinished but perhaps that was for the best.

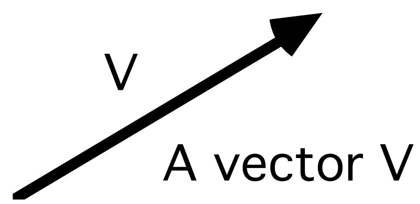
A Remark

The story in this section presents a number of different points of view about the cybernetics of fixed points. Fixed points can be produced by infinite recursion, by direct self-reference, through the linguistics of lambda calculus, and by approximation to infinites. Mr. D is a fictionalized version of Douglas Harding the man who indeed realized that he did not have a head, and had the courage to write about it. The good Drs. at the bar represent these points of view and are thinly disguised representatives of the viewpoints of Heinz von Foerster, Alonzo Church and Haskell Curry (Dr. CC), Humberto Maturana and the physicist Julian Barbour. Charlie represents the American mathematical philosopher Charles Sanders Peirce. All this is only the beginning. The most famous fixed point of them all is the Universe herself, acted here by the bartender.

XII. Quantum Physics, Eigenvalue and Eigenform

There are two reasons for including a discussion of quantum mechanics in this essay. On the one hand the quantum mechanics has been a powerful force in asking us to rethink our notions of objects and causality. On the other hand, von Foerster's notion of eigenform is an outgrowth of his background as a quantum physicist. We should ask what eigenforms might have to do with quantum theory and with the quantum world.

In this section we meet the concurrence of the view of object as token for eigenbehaviour and the observation postulate of quantum mechanics. In quantum mechanics observation is modeled not by eigenform but by its mathematical relative the eigenvector. The reader should recall that a *vector* is a quantity with magnitude and direction, often pictured as an arrow in the plane or in three dimensional space.



In quantum physics [11], the state of a physical system is modeled by a vector in a high-dimensional space, called a Hilbert space. As time goes on the vector rotates in this high dimensional space. Observable quantities correspond to (linear) operators **H** on these vectors **v** that have the property that the application of **H** to **v** results in a new vector that is a multiple of **v** by a real factor **λ.** (An operator is said to be linear if H(av +w) = aH(v) + H(w) for vectors v and w, and any number a. Linearity is usually a simplifying assumption in mathematical models, but it is an essential feature of quantum mechanics.)

In symbols this has the form

$$\mathbf{Hv} = \boldsymbol{\lambda}\mathbf{v}.$$

One says that **v** is an *eigenvector* for the operator **H**, and that **λ** is the *eigenvalue.* The constant **λ** is the quantity that is observed (for example the energy of an electron). These are particular properties of the mathematical context of quantum mechanics. The **λ** can be eliminated by replacing **H** by **G = H/λ** (when **λ** is non zero) so that

$$\mathbf{Gv = (H/\boldsymbol{\lambda})v = (Hv)/\boldsymbol{\lambda} = \boldsymbol{\lambda}v/k = v.}$$

Thus

$$\mathbf{Gv = v.}$$

In quantum mechanics observation is founded on the production of eigenvectors ***v*** *with* **Gv = v** *where* ***v*** *is a vector in a Hilbert space and* ***G*** *is a linear operator on that space.*

Many of the strange and fascinating properties of quantum mechanics emanate directly from this model of observation. In order to observe a quantum state, its vector is projected into an eigenvector for that particular mode of observation. By projecting the vector into that mode and not another, one manages to make the observation, but at the cost of losing information about the other possibilities inherent in the vector. This is the source, in the mathematical model, of the complementarities that allow exact determination of the position of a particle at the expense of nearly complete uncertainty about its momentum (or vice versa the determination of momentum at the expense of knowledge of the position).

Observation and quantum evolution (the determinate rotation of the state vector in the high dimensional Hilbert space) are interlocked. Each observation discontinuously projects the state vector to an eigenvector. The intervals between observations allow the continuous evolution of the state vector. This tapestry of interaction of the continuous and the discrete is the basis for the quantum mechanical description of the world.

The theory of eigenforms is a sweeping generalization of quantum mechanics that creates a context for understanding the remarkable effectiveness of that theory. If indeed the world of objects is a world of tokens for eigenbehaviours, and if physics demands forms of observations that give numerical results, then a simplest example of such observation is the observable in the quantum mechanical model.

Is the quantum model, in its details, a consequence of general principles about systems? This is an exploration that needs to be made. We can only ask the question here. But the mysteries of the interpretation of quantum mechanics all hinge on an assumption of a world external to the quantum language. Thinking in terms of eigenform we can begin to look at how the physics of objects emerges from the model itself.

Where are the eigenforms in quantum physics? They are in the mathematics itself. For example, we have the simplest wave-function

$$\varphi(x,t) = e^{i(kx - \omega t)}.$$

Since we know that the function $E(x) = e^x$ is an eigenform for operation of differentiation with respect to x, $\varphi(x,t)$ is a special multiple eigenform from which the energy can be extracted by temporal differentiation, and the momentum can be extracted by spatial differentiation. We see in $\varphi(x,t)$ the complexity of an individual who presents many possible sides to the world. $\varphi(x,t)$ is an eigenform for more than one operator. It is this internal complexity that is mirrored in the uncertainty relations of Heisenberg and the complementarily of Bohr. The eigenforms themselves, as wave-functions, are *inside* the mathematical model, on the *other side* of that which can be observed by the physicist.

We have seen eigenforms as the constructs of the observer, and in that sense they are on the side of the observer, even if the process that generates them is outside the realm of his perception. This suggests that we think again about the nature of the wave function in quantum mechanics. Is it also a construct of the observer? To see quantum mechanics and the world in terms of eigenforms requires a turning around, a shift of perception where indeed we shall find that the distinction between model and reality has disappeared into the world of appearance.

This is a reversal of epistemology, a complete turning of the world upside down. Eigenform has tricked us into considering the world of our experience and finding that it is our world, generated by our actions. It has become objective through the self-generated stabilities of those actions.

A Quick Review of Quantum Mechanics

DeBroglie hypothesized two fundamental relationships: between energy and frequency, and between momentum and wave number. These relationships are summarized in the equations

$$E = \hbar w,$$
$$P = \hbar k,$$

where **E** denotes the energy associated with a wave and **p** denotes the momentum associated with the wave**.** Here $\hbar = h/2\pi$ where **h** is Planck's constant.

Schrödinger answered the question: *Where is the wave equation for DeBroglie's waves?* Writing an elementary wave in complex form

$$\psi = \psi(x,t) = \exp(i(kx - wt)),$$

we see that we can extract DeBroglie's energy and momentum by differentiating:

$$i\hbar\partial\Psi/\partial t = E\psi \quad \text{and} \quad -i\hbar\partial\Psi/\partial x = p\psi.$$

This led Schrödinger to postulate *the identification of dynamical variables with operators* so that the first equation ,

$$i\hbar\partial\Psi/\partial t = E\Psi,$$

is promoted to the status of an equation of motion while the second equation becomes the definition of momentum as an operator:

$$p = -ih\partial/\partial x\ .$$

Once **p** is identified as an operator, the numerical value of momentum is associated with an eigenvalue of this operator, just as in the example above. In our example $p\psi = \hbar k\psi$.

In this formulation, the position operator is just multiplication by **x** itself. Once we have fixed specific operators for position and momentum, the operators for other physical quantities can be expressed in terms of them. We obtain the energy operator by substitution of the momentum operator in the classical formula for the energy:

$$E = (1/2)mv^2 + V$$
$$E = p^2/2m + V$$
$$E = -(\hbar^2/2m)\partial^2/\partial x^2 + V.$$

Here **V** is the potential energy, and its corresponding operator depends upon the details of the application.

With this operator identification for **E**, Schrödinger's equation

$$i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2 + V\psi$$

is an equation in the first derivatives of time and in second derivatives of space. In this form of the theory one considers general solutions to the differential equation and this in turn leads to excellent results in a myriad of applications.

In quantum theory, observation is modeled by the concept of eigenvalues for corresponding operators. *The quantum model of an observation is a projection of the wave function into an eigenstate.*

An energy spectrum **{Ek}** corresponds to wave functions ψ satisfying the Schrödinger equation, such that there are constants Ek with $\mathbf{E}\psi = Ek\psi$. An *observable* (such as energy) **E** is a Hermitian operator on a Hilbert space of wavefunctions. Since Hermitian operators have real eigenvalues, this provides the link with measurement for the quantum theory.

It is important to notice that there is no mechanism postulated in this theory for how a wave function is "sent" into an eigenstate by an observable. Just as mathematical logic need not demand causality behind an implication between propositions, the logic of quantum mechanics does not demand a specified cause behind an observation. This absence of an assumption of causality in logic does not obviate the possibility of causality in the world. Similarly, the absence of causality in quantum observation does not obviate causality in the physical world. Nevertheless, the debate over the interpretation of quantum theory has often led its participants into asserting that causality has been demolished in physics.

Note that the operators for position and momentum satisfy the equation $\mathbf{xp - px = \hbar i}$. This corresponds directly to the equation obtained by Heisenberg, on other grounds, that dynamical variables can no longer necessarily commute with one another. In this way, the points of view of DeBroglie, Schrödinger and Heisenberg came together, and quantum mechanics was born. In the course of this development, interpretations varied widely. Eventually, physicists came to regard the wave function not as a generalized wave packet, but as a carrier of information about possible observations. In this way of thinking $\psi^*\psi$ (ψ^* denotes the complex conjugate of ψ) represents the probability of finding the "particle" (A particle is an observable with local spatial characteristics.) at a given point in spacetime. Strictly speaking, it is the spatial integral of $\psi^*\psi$ that is interpreted as a total probability with $\psi^*\psi$ the probability density. This way of thinking is supported by the fact that the total spatial integral is time-invariant as a consequence of Schrodinger's equation!

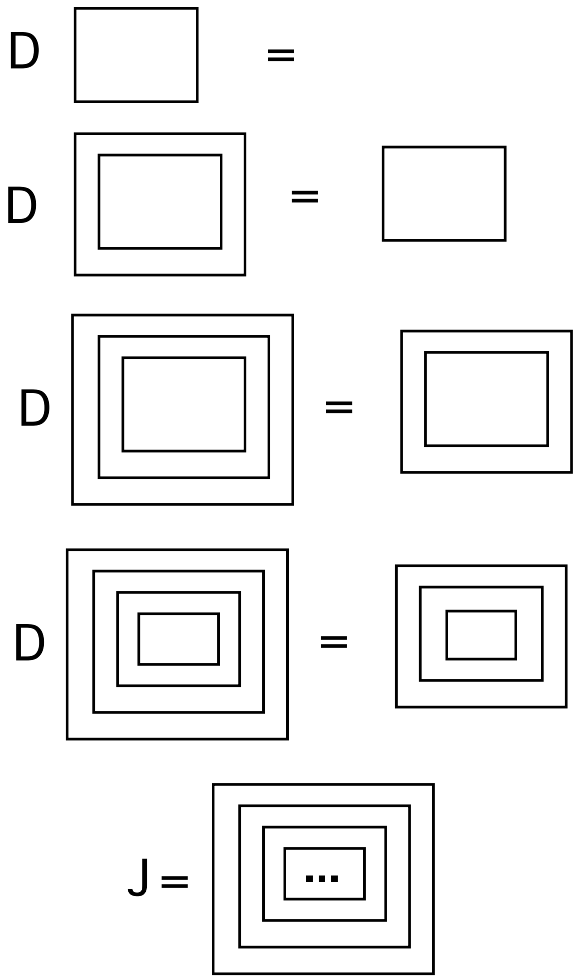
XIII. Iterants, Complex Numbers and Quantum Mechanics

We have seen that there are indeed eigenforms in quantum mechanics.

The eigenforms in quantum mechanics are the mathematical functions such as e^x *that are invariant under operators such as* $\boldsymbol{D = d/dx}$.

But we wish to examine the possibly deep relationship between recursion, reflexive spaces and the properties of the quantum world. The hint we have received from the theory of the quantum is that we should begin with the mathematics which is replete with eigenforms. In fact, this hint seems very rich when we consider that **i**, the square root of minus one, is a key eigenform in our panoply of eigenforms and it is a key ingredient in quantum mechanics.

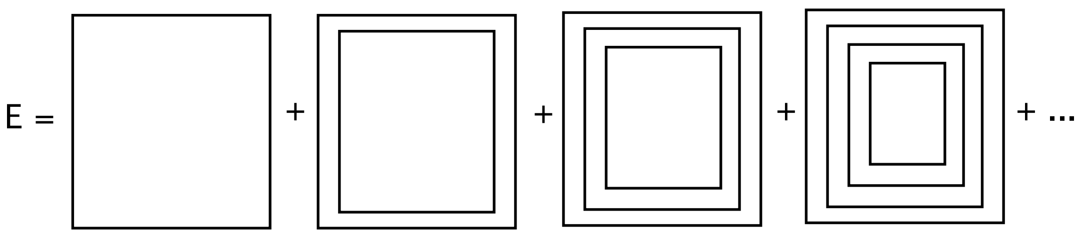
Lets begin by looking at the simpler case of differentiation. Consider an operator **D** that *removes* a box from around **X**.



$$DJ = J$$

Our familiar infinite nest of boxes is an eigenform for the "differentiation" operator **D**.

But we can go further. Consider an infinite series E of nested boxes as shown below.



Then extending **D** formally so that **D(X + Y) = D(X) + D(Y)**, we see that **D(E) = E** since **D** shifts the first box to void, the second box to the first box, the third box to the second box and so on.

Calculus and the Mathematics of Eigenforms

The exponential function is invariant under differentiation. Thus it is an eigenform for the operator **D=d/dt:**

$$\mathbf{D(exp(t)) = exp(t)} \text{ where } \mathbf{D=d/dt}.$$

In fact,

$$\exp(t) = 1 + t/1! + t^2/2! + t^3/3! + \ldots$$

where

$$D1 = 0,$$

$$Dt^{(n+1)}/(n+1)! = t^n/n!$$

from which it follows that

$$D(\exp(t)) = \exp(t).$$

If we think of the exponential function as a nest of boxes, each of which corresponds to one of the terms $t^n/n!$, then we see that the invariance of the nest of boxes **E** (above) under the formal differentiation operator has exactly the form of the invariance of **exp(t)** under differentiation in the calculus.

Another simple example of this sort is the series

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots$$

Here we can write

$$S = 1 + x(1 + x + x^2 + x^3 + x^4 + x^5 + \ldots).$$

Thus

$$S = 1 + x S$$

and so **S** is an eigenform for the operator

$$T(A) = 1 + xA.$$

Now **i** is a close relative of this operator. If we define

$$\mathbf{R(A) = -1/A}$$

then **R(i) = i** since $\mathbf{i^2 = -1}$ is equivalent to **i = -1/i.**

Using the infinite recursion we would then write (see the discussion in Zermelo's Bar)

$$\mathbf{i = -1/-1/-1/-1/-1/...} \, ,$$

making **i** an infinite reentry form for the operator **R.** Lets choose a notation for abbreviating such forms. We will write **i = [-1/*]** where * denotes the reentry of the whole form into that place in the right-hand part of the expression. Ok?

Similarly, if **F(x) = 1 + 1/x,** then the eigenform would be **[1 + 1/*]** and we could write

$$\mathbf{(1+\sqrt{5})/2 = [1 + 1/*].}$$

With this in place we can now consider wave functions in quantum mechanics such as

$$\mathbf{\psi(x,t) = exp(i(kx - wt)) = exp([-1/*]\ (kx - wt))}$$

and we can consider classical formulas in mathematics such as Euler's formula

$$\mathbf{exp([-1/*]\varphi) = cos(\varphi) + [-1/*]\ sin(\varphi)}$$

in this light. Really, we must start here with Euler's formula, for this formula is the key relation between complex numbers, **i** and waves and periodicity.

We have to return to the finite nature of **[-1/*]**. This eigenform is an oscillator between **-1** and **+1**. It is only **i** in its idealization or in its appropriate synchronization that it has the property that **i = -1/i**. As a real oscillator, the equation **R(i) = -1/i** tells us that when **i** is **1**, then i is transformed to **-1** and when **i** is **-1** then **i** is transformed to **+1**. There is no fixed point in the real domain. The eigenform is achieved by leaving the real domain for a new and larger domain. We know that this larger domain can be conceptualized as the plane with Euclidean rotational geometry, but we want to here explore the larger domain in terms of eigenforms.

We are now going to do this exploration, but we have to warn the reader: *We find that **i** itself is a fundamental example of a discrete physical process, and it is in the "microworld" of such discrete physical processes that not only quantum mechanics, but also classical mechanics is born.*

Iterants and Iterant Views

In order to think about i, consider an infinite oscillation between **+1** and **-1**:

$$\mathbf{... -1,+1,-1,+1,-1,+1,-1,+1,...}$$

This oscillation can be seen in two distinct ways. It can we seen as **[-1,+1]** (a repetition in this order) or as **[+1,-1]** (a repetition in the opposite order). This suggests regarding an infinite alternation such as

... a,b,a,b,a,b,a,b,a,b,a,b,a,b,...

as an entity that can be seen in two possible ways, indicated by the ordered pairs **[a,b]** and **[c,d]**. We shall call the infinite alternation of **a** and **b** the *iterant* of **a** and **b** and denote it by **I{a,b}**. Just as with a set **{a,b}**, the iterant is independent of the order of **a** and **b**. We have **I{a,b} = I{b,a}**, but there are two distinct views of any iterant and these are denoted by **[a,b]** and **[b,a]**.

The key to iterants is that two representatives of an iterant can *by themselves* appear identical, but *taken together* are seen to be different. For example, consider

... a,b,a,b,a,b,a,b,a,b,a,b,a,b,...

and also consider

...b,a,b,a,b,a,b,a,b,a,b,a,b,a,...

There is no way to tell the difference between these two iterants except by a direct comparison as shown below

... a,b,a,b,a,b,a,b,a,b,a,b,a,b,...
... b,a,b,a,b,a,b,a,b,a,b,a,b,a,...

In the direct comparison we see that if one of them is **[a,b]**, then the other one should be **[b,a]**. Still, there is no reason to assign one of them to be **[a,b]** and the other **[b,a]**. It is a strictly relative matter. The two iterants are entangled (to borrow a term from quantum mechanics) and if one of them is observed to be **[a,b]**, then the other is necessarily observed to be **[b,a]**.

Lets go back to the square root of minus one as an oscillatory eigenform.

... -1,+1,-1,+1,-1,+1,-1,+1,...

What is the operation **R(x) = -1/x** in this case? We usually think of a starting value and then the new operation shifts everything by one value with **R(+1) = -1** and **R(-1) = +1**. Thus would suggest that

R(... -1,+1,-1,+1,-1,+1,-1,...) = ... +1,-1,+1,-1,+1,-1,+1,...

and these sequences will be different when we compare, them even though they are identical as individual iterants.

... -1,+1,-1,+1,-1,+1,-1,+1,...
... +1,-1,+1,-1,+1,-1,+1,-1,...

However, we would like to take the eigenform/iterant concept and make a more finite algebraic model by using the iterant views **[-1,+1]** and **[+1,-1]**. Certainly we should consider the transform **P[a,b] = [b,a]** and we take

$$\mathbf{-[a,b] = [-a, -b],}$$

so that

$$\mathbf{-P[a,b] = [-b,-a]}.$$

Then

$$\mathbf{-P[1,-1] = [1,-1].}$$

In this sense the operation **-P** has eigenforms **[1,-1]** and **[-1,1]**. You can think of **P** as the shift by one-half of a period in the process

...ababababab.... .

Then **[-1,1]** is an eigenform for the operator that combines negation and shift.

We will take a shorthand for the operator **P** via

$$\mathbf{P[a,b] = [a,b]' = [b,a].}$$

If **x=[a,b]** then **x' = [b,a]**.

We can add and multiply iterant views by the combinations

$$\mathbf{[a,b][c,d] = [ac,bd],}$$
$$\mathbf{[a,b] + [c,d] = [a+c, b+ d],}$$
$$\mathbf{k[a,b] = [k,a,k,b] \text{ when k is a number.}}$$

We take **1 = [1,1]** and **-1 = [-1,-1].** This is a natural algebra of iterant views, but note that **[-1,+1][-1,+1] = [1,1] = 1**, so we do not yet have the square root of minus one.

Consider **[a,b]** as representative of a process of observation of the iterant **I{a,b}.** **[a,b]** is an *iterant view.* We wish to combine **[a,b]** and **[c,d]** as *processes of observation.* Suppose that observing **I{a,b}** requires a step in time. That being the case, **[a,b]** will have shifted to **[b,a]** in the course of the single time step. We need an algebraic structure to handle the temporality. To this end, we introduce an operator η with the property that

$$\mathbf{[a,b]}\eta = \eta\,\mathbf{[b,a]} \text{ with } \eta^2 = \eta\eta = \mathbf{1}$$

where **1** means the identity operator. You can think of η as a *temporal shift operator* that can act on a sequence of individual observations. The algebra generated by iterant views and the operator η is taken to be associative.

Here the interpretation is that **XY** denotes "first observe **X**, then observe **Y**". Thus $\mathbf{X\eta Y\eta} = \mathbf{XY'\eta\eta} = \mathbf{XY'}$ and we see that **Y** has been shifted by the presence of the operator η, just in accord with our temporal interpretation above.

We can now have a theory where **i** and its conjugate **-i** correspond to the two views of the iterant **I{-1,+1}.** Let $\mathbf{i} = [1,-1]\eta$ and $\mathbf{-i} = [-1,1]\eta$. We get a square roots of minus one:

$$\mathbf{ii = [1,-1]\eta[1,-1]\eta = [1,-1][-1,1]\eta\eta = [-1,-1] = -[1,1] = -1.}$$

The square roots of minus one are iterant views coupled with temporal shift operators. Not so simple, but not so complex either! If **e = [1,-1]** then **e' = [-1,1] = -e** and **ee = [1,1] = 1** with **ee' = -1.**

$$\mathbf{i = e\eta}$$
$$\mathbf{ii = e\eta e\eta = ee'\eta\eta = ee' = -1}$$

With this definition of i, we have an algebraic interpretation of complex numbers that allows one to think of them as observations of discrete processes.

This algebra contains more than just the complex numbers. With **x = [a,b]** and **y = [c,d]**, consider the products $\mathbf{(x\eta)(y\eta)}$ and $\mathbf{(y\eta)(x\eta)}$**:**

$$\mathbf{(x\eta)(y\eta) = [a,b]\eta[c,d]\eta = [a,b][d,c] = [ad,bc]}$$

$$\mathbf{(y\eta)(x\eta) = [c,d]\eta[a,b]\eta = [c,d][b,a] = [cb,da].}$$

Thus

$$\mathbf{(x\eta)(y\eta) - (y\eta)(x\eta)}$$
$$\mathbf{= [ad-bc, -(ad-bc)]}$$
$$\mathbf{= (ad -bc)[1,-1].}$$

Thus

$$\mathbf{x\eta y\eta - y\eta x\eta = (ad -bc)i\,\eta.}$$

We see that, with temporal shifts, the algebra of observations is non-commutative. Note that for these processes, represented by vectors **[a,b]**, the commutator $\mathbf{x\eta y\eta - y\eta x\eta = (ad -bc)i\eta}$ is given by the determinant of the matrix corresponding to two process vectors, and hence will be non-zero whenever the two process vectors are non-zero and represent different spatial rays in the plane.

There is more. *The full algebra of iterant views can be taken to be generated by elements of the form*

$$\mathbf{[a,b] + [c,d]\eta}$$

and it is not hard to see that this is isomorphic with 2 x 2 matrix algebra with the correspondence given by the diagram below.

$$[a,b] + [c,d]\eta \longleftrightarrow \boxed{\begin{matrix} a & c \\ d & b \end{matrix}}$$

We see from this excursion that there is a full interpretation for the complex numbers (and indeed matrix algebra) as an observational system taking into account time shifts for underlying iterant processes.

Let $\mathbf{A = [a,b]}$ and $\mathbf{B = [c,d]}$ and let $\mathbf{C = [r,s], D = [t,u].}$ With $\mathbf{A' = [b,a]}$, we have

$$\mathbf{(A + B\eta)(C + D\eta) = (AC + BD') + (AD + BC')\eta.}$$

This writes 2 x 2 matrix algebra in the form of a hypercomplex number system. From the point of view of iterants, the sum $\mathbf{[a,b] + [b,c]\eta}$ can be regarded as a superposition of two types of observation of the iterants $\mathbf{I\{a,b\}}$ and $\mathbf{I\{c,d\}}$. The operator-view $\mathbf{[c,d]\eta}$ includes the shift that will move the viewpoint from $\mathbf{[c,d]}$ to $\mathbf{[d,c]}$, while $\mathbf{[a,b]}$ does not contain this shift. Thus a shift of viewpoint on $\mathbf{[c,d]}$ in this superposition does not affect the values of $\mathbf{[a,b]}$. One can think of the corresponding process as having the form shown below.

... a a a a a a a a a a a a a a a ...
... c d c d c d c d c d c d c d ...
... b b b b b b b b b b b b b b ...

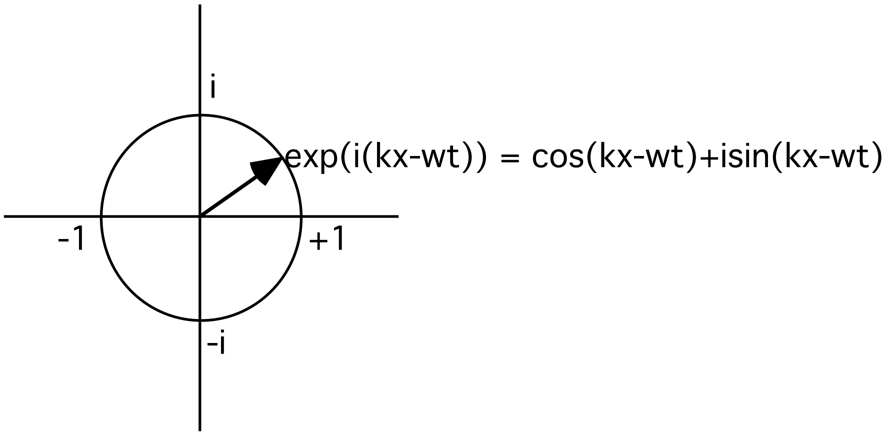
The snapshot **[c,d]** changes to **[d,c]** in the horizontal time-shift while the vertical snapshot **[a,b]** remains invariant under the shift. It is interesting to note that in the spatial explication of the process we can imagine the horizontal oscillation corresponding to $\mathbf{[c,d]\eta}$ as making a boundary (like a frieze pattern), while the vertical iterant parts a and b mark the two sides of that boundary.

Returning to Quantum Mechanics

You can regard $\mathbf{\psi(x,t) = exp(i(kx - wt))}$ as containing a micro-oscillatory system with the special synchronizations of the iterant view $\mathbf{i = [+1,-1]\eta}$. It is these synchronizations that make the big eigenform of the exponential $\mathbf{\psi(x,t)}$ work correctly with respect to differentiation, allowing it to create the appearance of rotational behavior, wave behavior and the semblance of the continuum. Note that

$$\mathbf{exp(i\varphi) = cos(\varphi) + i\, sin(\varphi)}$$

in this way of thinking is an infinite series involving powers of **i**. The exponential is synchronized via **i** to separate out its classical trigonometric parts. In the parts we have $\mathbf{cos(\varphi) + i\, sin(\varphi) = [cos(\varphi), cos(\varphi)] + [sin(\varphi), -sin(\varphi)]\eta}$, a superposition of the constant cosine iterant and the oscillating sine iterant. Euler's formula is the result of a synchronization of iterant process. One can blend the classical geometrical view of the complex numbers with the iterant view by thinking of a point that orbits the origin of the complex plane, intersecting the real axis periodically and producing, in the real axis, a periodic oscillation in relation to its orbital movement in the higher dimensional space.



The diagram above is the familiar depiction of a vector in the complex plane that represents the phase of a wave-function. I hope that the reader can now look at this picture in a new way, seeing $\mathbf{i = [+1,-1]\eta}$ as a discrete oscillation with built-in time shift and the exponential as a process oscillating between **cos(kx-wt) + sin(kx-wt)** and **cos(kx-wt)-sin(kx-wt)**. The exponential function takes the simple oscillation between **+(kx-wt)** and **-(kx-wt)** and converts it by a complex of observations of this discrete process to the trigonometric wave-forms. All this goes on beneath the surface of the Schrodinger equation. This is the production of the eigenforms from which may be extracted the energy, position and momentum.

Higher Orders of Iterant Structure. What works for 2 x 2 matrices generalizes to n x n matrix algebra, but then the operations on a vector $\mathbf{[x_1,x_2,...,x_n]}$ constitute all permutations of n objects. That is a generating element of iterant algebra is now of the form $\mathbf{x\,\sigma = [x_1,x_2,...,x_n]\sigma}$ where σ is an element of the symmetric group $\mathbf{S_n}$. The iterant algebra is the linear span of all elements $\mathbf{x\,\sigma}$, and we take the rule of multiplication as

$$\mathbf{x\,\sigma\ y\,\tau = xy^{\sigma}\ \sigma\tau}$$

where $\mathbf{y^{\sigma}}$ denotes the vector obtained from y by permuting its coordinates via σ ; **xy** is the vector whose k-th coordinate is the product of the k-th coordinate of x and the k-th coordinate of y ; $\sigma\tau$ is the composition of the two permutations σ and τ .

Hamilton's Quaternions
Here is an example. Hamilton's Quaternions are generated by the iterant views

$$\mathbf{I = [+1,-1,-1,+1]\sigma}$$
$$\mathbf{J= [+1,+1,-1,-1]\lambda}$$
$$\mathbf{K= [+1,-1,+1,-1]\tau}$$

where

$$\sigma = (12)(34)$$
$$\lambda = (13)(24)$$
$$\tau = (14)(23).$$

Here we represent the permutations as products of transpositions. One can verify that

$$I^2 = J^2 = K^2 = IJK = -1.$$

For example,

$$\begin{aligned} I^2 &= [+1,-1,-1,+1]\sigma\ [+1,-1,-1,+1]\sigma \\ &= [+1,-1,-1,+1][-1,+1,+1,-1]\sigma\ \sigma \\ &= [-1,-1,-1,-1] \\ &= -1. \end{aligned}$$

and

$$\begin{aligned} IJ &= [+1,-1,-1,+1]\sigma\ [+1,+1,-1,-1]\lambda \\ &= [+1,-1,-1,+1][+1,+1,-1,-1]\ \sigma\ \lambda \\ &= [+1,-1,+1,-1]\ (12)(34)(13)(24) \\ &= [+1,-1,+1,-1]\ (14)(23) \\ &= [+1,-1,+1,-1]\ \tau. \end{aligned}$$

In a sequel to this paper, we will investigate this iterant approach to the Quaternions and other algebras related to fundamental physics. For now it suffices to point out that the quaternions of the form $\mathbf{a + bI + cJ + dK}$ with $\mathbf{a^2 + b^2 + c^2 + d^2 = 1}$ (a,b,c,d real numbers) constitute the group **SU(2)**, ubiquitous in physics and fundamental to quantum theory. Thus the formal structure of all processes in quantum mechanics can be represented as actions of iterant viewpoints.

Nevertheless, we must note that making an iterant interpretation of an entity like $\mathbf{I = [+1,-1,-1,+1]\sigma}$ is a conceptually natural departure from our original period two iterant notion. Now we are considering iterants such as **I{+1,-1,-1,+1}** where the iterant is a multi-set and the permutation group acts to produce all possible orderings of that multi-set. The iterant itself is not an oscillation. It represents an implicate form that can be seen in any of its possible orders. Once seen, these orders are subject to permutations that produce the possible views of the iterant. Algebraic structures such as the quaternions appear in the explication of such implicate forms.

The reader will also note that we have moved into a different conceptual domain from the original emphasis in this paper on eigenform in relation to recursion.

Indeed, each generating quaternion is an eigenform for the transformation $\mathbf{R(x) = -1/x}$.

The richness of the quaternions arises from the closed algebra that arises with its infinity of eigenforms that satisfy this equation, all of the form $\mathbf{U = aI + bJ + cK}$ where $\mathbf{a^2 + b^2 + c^2 = 1}$. This kind of significant extra structure in the eigenforms comes from paying attention to specific aspects of implicate and explicate structure, relationships with geometry and ideas and inputs from the perceptual, conceptual and physical worlds. Just as with our earlier examples (with cellular automata) of phenomena arising in the course of the recursion, we see the same phenomena here in the evolution of mathematical and theoretical physical structures in the course of the recursion that constitutes scientific conversation.

Quaternions and SU(2) Using Complex Number Iterants

Since complex numbers commute with one another, we could consider iterants whose values are in the complex numbers. This is just like considering matrices whose entries are complex numbers. For this purpose we shall allow given a version of **i** that commutes with the iterant shift operator η. Let this commuting **i** be denoted by ι (iota). Then we are assuming that

$$\iota^2 = -1$$
$$\eta\,\iota = \iota\,\eta$$
$$\eta^2 = +1.$$

We then consider iterant views of the form $\mathbf{[a + b\iota, c + d\iota]}$ and $\mathbf{[a + b\iota\ , c + d\iota\ ]\eta = \eta\ [c + d\iota\ , a + b\iota\ ]}$. In particular, we have $\mathbf{e = [1,-1]}$, and $\mathbf{i = e\eta}$ is quite distinct from ι. Note, as before, that $e\eta = -\eta e$ and that $\mathbf{e^2 = 1}$. Now let

$$\mathbf{I = \iota e}$$
$$\mathbf{J = e\eta}$$
$$\mathbf{K = \iota\eta}\ .$$

We have used the commuting version of the square root of minus one in these definitions, and indeed we find the Quaternions once more.

$$\mathbf{I^2 = \iota e\, \iota e = \iota\, \iota\ e\, e = (-1)(+1) = -1,}$$
$$\mathbf{J^2 = e\eta\ e\eta = e\,(-e)\,\eta\eta = -1,}$$
$$\mathbf{K^2 = \iota\eta\ \iota\eta = \iota\ \iota\ \eta\eta = -1,}$$
$$\mathbf{IJK = \iota e\ e\eta\ \ \iota\eta = \iota\ 1\ \iota\ \eta\ \eta = \iota\ \iota = -1.}$$

Thus

$$\mathbf{I^2 = J^2 = K^2 = IJK = -1.}$$

This must look a bit cryptic at first glance, but the construction shows how the structure of the quaternions comes directly from the non-commutative structure of our period two iterants. In other, words, quaternions can be represented by 2 x 2 matrices. This is the way it has been presented in standard language. The group **SU(2)** of 2 x 2 unitary matrices of determinant one is isomorphic to the quaternions of length one.

$$\boxed{\begin{matrix} z & w \\ -\overline{w} & \overline{z} \end{matrix}} = [\, z \,,\, \overline{z} \,] + [\, w \,,\, -\overline{w} \,]\eta$$

In the equation above, we indicate the matrix form of an element of **SU(2)** and its corresponding complex valued iterant. You can easily verify that

1: z = 1, w = 0,
I: z = ι, w = 0,
J: z = 0, w = 1,
K: z = 0, w = ι.

This gives the generators of the quaternions as we have indicated them above and also as generators of **SU(2)**.

Similarly, **H = [a,b] + [c + di, c-di]η** represents a Hermitian 2 x 2 matrix and hence an observable for quantum processes mediated by **SU(2)**. Hermitian matrices have real eigenvalues. It is curious how certain key iterant combinations turn out to be essential for the relations with quantum observation.

XIV. Time Series and Discrete Physics

In this section we shall use the convention (outside of iterants) that successive observations, first A and then B will be denoted BA rather than AB. This is to follow previous conventions that we have used. We continue to interpret iterant observation sequences in the opposite order as in the previous section. This section is based on our work in [20] but takes a different interpretation of the meaning of the diffusion equation in relation to quantum mechanics.

We have just reformulated the complex numbers and expanded the context of matrix algebra to an interpretation of i as an oscillatory process and matrix elements as combined spatial and temporal oscillatory processes (in the sense that **[a,b]** is not affected in its order by a time step, while **[a,b]η** includes the time dynamic in its interactive capability, and 2 x 2 matrix algebra is the algebra of iterant views **[a,b] + [c,d]η**). We now consider elementary discrete physics in one dimension. Consider a time series of positions **x(t), t = 0, Δt, 2Δt, 3Δt, ...** . We can define the velocity v(t) by the formula **v(t) = (v(t + Δ) - v(t))/Δt = Dx(t)** where **D** denotes this discrete derivative. In order to obtain v(t) we need at least one tick **Δt** of the discrete clock. Just as in the iterant algebra, we need a time-shift operator to handle the fact that once we have observed **v(t)**, the time has moved up by one tick. Thus we shall add an operator **J** that in this context accomplishes the time shift:

$$\mathbf{x(t)J = Jx(t+\Delta t).}$$

We then *redefine* the derivative to include this shift:

$$\mathbf{Dx(t) = J(x(t+\Delta) - x(t))/\Delta t\ .}$$

The result of this definition is that a successive observation of the form **x(Dx)** is distinct from an observation of the form **(Dx)x**. In the first case, we observe the velocity and then **x** is measured at $\mathbf{t + \Delta t}$. In the second case, we measure **x** at **t** and then measure the velocity. Here are the two calculations:

$$\begin{aligned}\mathbf{x(Dx)} &= \mathbf{x(t)\ (J(x(t + \Delta) - x(t))/\Delta t\ )}\\ &= \mathbf{(J/\Delta)(x(t + \Delta t))(x(t + \Delta t) - x(t))}\\ &= \mathbf{(J/\Delta t)(x(t + \Delta t)^2\ - x(t+ \Delta t)x(t)).}\end{aligned}$$

$$\begin{aligned}\mathbf{(Dx)x} &= \mathbf{(J(x(t + \Delta t) - x(t))/\Delta t\ )x(t)}\\ &= \mathbf{(J/\Delta t)(x(t + \Delta t)x(t) - x(t)^2).}\end{aligned}$$

We measure the difference between these two results by taking a commutator **[A,B] = AB - BA** and we get the following formula where we write $\mathbf{\Delta x = x(t+ \Delta t) - x(t).}$

$$\begin{aligned}\mathbf{[x,(Dx)]} &= \mathbf{x(Dx) - (Dx)x}\\ &= \mathbf{(J/\Delta t)(x(t + \Delta t) - x(t))^2}\\ &= \mathbf{J\ (\Delta x)^2/\Delta t}\end{aligned}$$

This final result is worth marking:

$$\mathbf{[x,(Dx)] = J\ (\Delta x)^2/\Delta t.}$$

From this result we see that the commutator of **x** and **Dx** will be constant if $\mathbf{(\Delta x)^2/\Delta t = K}$ is a constant. For a given time-step, this means that $\mathbf{(\Delta x)^2 = K\ \Delta t}$ so that $\mathbf{\Delta x = +\ \sqrt{(K\ \Delta t\ )}}$ or $\mathbf{-\ \sqrt{(K\ \Delta t\ )}}$. In other words,

$$\mathbf{x(t + \Delta t\ ) = x(t) +\ \sqrt{(K\ \Delta t\ )}}\quad \text{or}\quad \mathbf{x(t) - \sqrt{(K\ \Delta t\ )}}.$$

This is a Brownian process with diffusion constant equal to **K.**

Digression on Browian Processes and the Diffusion Equation

Assume, for the purpose of discussion that in the above process, at each next time, it is equally likely to have + or - in the formulas

$$x(t + \Delta t) = x(t) + \sqrt{(K \Delta t)} \quad \text{or} \quad x(t) - \sqrt{(K \Delta t)}.$$

Let **P(x,t)** denote the probability of the particle being at the location x at time t in this process. Then we have

$$P(x, t + \Delta t) = (1/2)(P(x - \Delta x) + P(x + \Delta x)).$$

Hence

$$\begin{gathered}(P(x, t + \Delta t) - P(x,t))/\Delta t) \\ = ((\Delta x)^2/2\Delta t)(P(x - \Delta x) - 2P(x,t) + P(x + \Delta x,t))/(\Delta x)^2) \\ = (K/2)(P(x - \Delta x) - 2P(x,t) + P(x + \Delta x,t))/(\Delta x)^2).\end{gathered}$$

Thus we see that **P(x,t)** satisfies the a discretization of the diffusion equation

$$\partial P/\partial t = (K/2)\partial^2 P/\partial x^2.$$

Of course, this demands comparison with the Schrodinger equation in the form (with zero potential) shown below.

$$i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2$$

In the Schrodinger equation we see that we can rewrite it in the form

$$\partial\psi/\partial t = i(\hbar/2m)\partial^2\psi/\partial x^2$$

Thus, if we were to make a literal comparison with the diffusion equation we would take $K = i(\hbar/m)$ and we would identify

$$(\Delta x)^2/\Delta t = i(\hbar/m).$$

Whence

$$\Delta x = ((1+i)/\sqrt{2}) \sqrt{[(\hbar/m)\Delta t]}$$

and the corresponding Brownian process is

$$x(t + \Delta t) = x + \Delta x \text{ or } x - \Delta x.$$

The process is a step-process along a diagonal line in the complex plane. We are looking at a Brownian process with complex values! What can this possibly mean? Note that if we take this point of view, then **x** is a complex variable and the partial derivative with respect to **x** is taken with respect to this complex variable. In this view of a complexified version of the Schrodinger equation, the solutions for **Δx** as above are real probabilities. We shall have to move the **x** variation to real x to get the usual Schrodinger equation, and this will result in complex valued wave functions in its solutions.

In our context, the complex numbers are themselves oscillating and synchronized processes. We have $\mathbf{i = [1,-1]\eta}$ where η is a shifter satisfying the rules of the last section, and **[1,-1]** is a view of the iterant that oscillates between plus and minus one. Thus we are now observing that solutions to the Schrodinger equation can be construed as Brownian paths in a more complicated discrete space that is populated by both probabilistic and synchronized oscillations. This demands further discussion, which we now undertake.

The first comment that needs to be made is that since in the iterant context $\mathbf{\Delta x}$ is an oscillatory quantity it does make sense to calculate the partial derivatives using the limits as $\mathbf{\Delta x}$ and $\mathbf{\Delta t}$ approach zero, but this means that the interpretation of the Schrodinger equation as a diffusion equation and the wave function as a probability is dependent on this generalization of the derivative. *If we take Δx to be real, then we will get complex solutions to Schrodinger's equation.* In fact we can write

$$\psi(x, t + \Delta t) = (1 - i)\psi(x, t) + (i/2)\psi(x - \Delta x) + (i/2)\psi(x + \Delta x)$$

and then we will have, in the limit,

$$\partial\psi/\partial t = i(\hbar/2m)\partial^2\psi/\partial x^2$$

if we take $(\Delta x)^2/\Delta t = (\hbar/m)$.

It is interesting to compare these two choices. In one case we took

$$(\Delta x)^2/\Delta t = i(\hbar/m)$$

and obtained *a Brownian process with imaginary steps.*

In the other case we took

$$(\Delta x)^2/\Delta t = (\hbar/m)$$

and obtained *a real valued process with imaginary probability weights.* These are complementary points of view about the same structure.

With $(\Delta x)^2/\Delta t = (\hbar/m)$, $\psi(x, t)$ is no longer the classical probability for a simple Brownian process. We can imagine that the coefficients $\mathbf{(1-i)}$ and $\mathbf{(i/2)}$ in the expansion of $\psi(x, t + \Delta t)$ are somehow analogous to probability weights, and that these weights would correspond to the generalized Brownian process where the real-valued particle can move left or right by $\mathbf{\Delta x}$ or just stay put. Note that we have

$$(1 - i) + (i/2) + (i/2) = 1,$$

signaling a direct analogy with probability where the probability values are imaginary. But this must be explored in the iterant epistemology!

Note that **1-i = [1,1] -[1,-1]**η and so at any given time represents either **[1,1] - [1,-1] = [0,2]** or **[1,1] - [-1,1] = [2,0]**. It is very peculiar to try to conceptualize this in terms of probability or amplitudes. Yet we know that in the standard interpretations of quantum mechanics one derives probability from the products of complex numbers and their conjugates. To this end it is worth seeing how the product of **a+bi** and **a-bi** works out:

$$\begin{aligned}(a + bi)(a - bi) &= aa + bia + a(-bi) + (bi)(-bi)\\ &= aa + abi - abi - bbii\\ &= aa - bb(-1)\\ &= aa + bb.\end{aligned}$$

It is really the rotational nature of **exp(it)** that comes in and makes this work. **exp(it)exp(-it) = exp(it - it) = exp(0) = 1** The structure is in the exponent. The additive combinatory properties of the complex numbers are all under the wing of the rotation group.

A fundamental symmetry is at work, and that symmetry is a property of the synchronization of the periodicities of underlying process. The fundamental iterant process of **i** disappears in the multiplication of a complex number by its conjugate. In its place is a pattern of apparent actuality. It is actual just to the extent that one regards **i** as only possibility. On making a reality of **i** itself we have removed the boundary between mathematics and the reality that "it" is supposed to describe. There is no such boundary.

XV. Epilogue

The problem that we have resolved in this paper is the problem to understand the nature of observation in quantum mechanics. In fact, we hope that the problem is seen to disappear the more we enter into the present viewpoint. A viewpoint is only on the periphery. The iterant from which the viewpoint emerges is in a superposition of indistinguishables, and can only be approached by varying the viewpoint until one is released from the particularities that each point of view contains.

It is not just the eigenvalues of Hermitian operators that are the structures of the observation, but rather the eigenforms that populate the mathematical models at all levels. These forms are the indicators of process. Mathematics, instead of being a descriptive symbol system for various algorithms, comes alive as an interrelated orchestration of processes. It is these processes that become the exemplary operators and elements of the mathematics that are put together to form the physical theory. We hope that the reader will be unable, ever again, to look at Schrodinger's equation the same way, after reading this argument.

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