1) [20 pts] Consider the following matrix

\[ A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 1 & 5 \\ 3 & 4 & 1 & 7 \end{bmatrix} \]

as a linear transformation from \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \).

(a) Let \( \text{Col}(A) \) denote the range of \( A \). That is, \( \text{Col}(A) = \{Ax\} \) where \( x \) runs over all vectors in \( \mathbb{R}^4 \). Determine a basis for \( \text{Col}(A) \) and find the dimension of \( \text{Col}(A) \).

(b) Let \( S = \text{Col}(A)^\perp \) be the subspace of \( \mathbb{R}^3 \) orthogonal to the range of \( A \). Find a basis for \( S \). What is the dimension of \( S \)?

(c) Find a basis for the row space of \( A \).

2) [20 pts] Let \( L \) be the linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) given by the following equation:

\[ L(x, y)^T = (x + y, x - y, y - x)^T. \]

(a) Let \( M \) denote the matrix of \( L \) with respect to the standard bases for \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). Determine the matrix \( M \).

(b) Let \( E = [u_1, u_2] = [(1, 1)^T, (1, -1)^T] \) be a new basis for \( \mathbb{R}^2 \) and let \( F = [b_1, b_2, b_3] = [(1, 1, 1)^T, (1, 1, 0)^T, (1, 0, 0)^T] \) be a new basis for \( \mathbb{R}^3 \). Find the matrix \( A \) for \( L \) with respect to these bases.

3) [20 pts] Let \( V \) be the space of real-valued differentiable functions of the variable \( x \) spanned by \( \{e^x, xe^x\} \). Let \( D : V \to V \) be the linear transformation \( d/dx \) (derivative with respect to \( x \)).

(a) Show that \( \{e^x, xe^x\} \) are linearly independent in \( V \). Note that \( V \) is a subspace of the space of all differentiable functions of a real variable \( x \). Addition in \( V \) is addition of functions. Scalar multiplication is the multiplication of a function by that scalar.

(b) By part (a), \( \{e^x, xe^x\} \) is a basis for \( V \). Find the matrix of \( D \) with respect to this basis.
4) [20 pts] Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation whose matrix in the standard basis is
$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$ 
(a) $v_1 = (2, -1)^T$ and $v_2 = (1, -1)^T$. Verify that $E = [v_1, v_2]$ is a basis for $\mathbb{R}^2$.
(b) Find the matrix $B = [L]_E^E$. This is the matrix for $L$ in the basis $E$. Check your answer.

5) [20 pts] (a) Let $\Pi$ be the plane in $\mathbb{R}^3$ defined by the equation $x + y + z = 0$. Find a general formula for the distance of a point $P = (x, y, z)^T$ to the plane. Use your formula to find the distance from $(1, 1, 1)^T$ to the plane.
(b) Let $u$, $v$ and $w$ be three non-zero vectors in $\mathbb{R}^3$ such that each pair $\{u, v\}$, $\{u, w\}$ and $\{v, w\}$ is orthogonal. Show that $[u, v, w]$ is a basis for $\mathbb{R}^3$. Verify this using only the properties given for these vectors. Do not use specific numerical examples.