Sample Final Exam - Math 310 - Linear Algebra - Spring 2003

Remark. There will be some duplication between the problems on this list and problems on previous sample exams and exams. The present form of this sample exam is not complete.

Questions

1. Suppose that \( \{v_1, v_2, ..., v_n\} \) is a subset of vectors in a vector space \( V \) of dimension \( m \geq n \). Let \( T : V \rightarrow V \) be any linear transformation from \( V \) to itself. Prove that if \( \{T(v_1), T(v_2), ..., T(v_n)\} \) is linearly independent, then \( \{v_1, v_2, ..., v_n\} \) is linearly independent.

2. Suppose that \( \{v_1, v_2, ..., v_n\} \) is a subset of linearly independent vectors in a vector space \( V \) of dimension \( m \geq n \). Let \( T : V \rightarrow V \) be any linear transformation from \( V \) to itself. Prove that if \( T \) is invertible then \( \{T(v_1), T(v_2), ..., T(v_n)\} \) is linearly independent.

3. Suppose that \( \{v_1, v_2, v_3\} \) is a subset of non-zero vectors in a vector space \( V \) of dimension \( m \geq 3 \) and that \( T \) is a linear transformation such that \( Tv_i = \lambda_i v_i \) for constants \( \lambda_i \) such that \( \lambda_i \neq \lambda_j \) when \( i \neq j \). Prove that \( \{v_1, v_2, v_3\} \) is linearly independent.

4. Let \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 8 \end{bmatrix} \). Find a basis for the row space of \( A \) and its dimension. Find a basis for the column space of \( A \) and its dimension. Find a matrix \( B \) such that the range of \( B \) equals the null space of \( A \), i.e. \( \text{Range}(B) = R(B) = N(A) \) where \( R(B) = \{Bx | x^T = (x_1, x_2, x_3, \cdots, x_k) \in R^k \} \). Here \( k \) is the dimension of the null space \( N(A) \).

5. (a) Let \( D \) be the linear transformation defined from the vector space \( P_4 \) of polynomials of degree less than 4 to the vector space \( P_3 \) by the equation \( D(f) = df/dx \), where this denotes the usual derivative in calculus. Find the matrix of \( D \) in the bases \( \{1, x, x^2, x^3\} \) of \( P_4 \) and \( \{1, x, x^2\} \) of \( P_3 \). Let \( U \) be the linear transformation defined from \( P_3 \) to \( P_4 \) defined by the formula \( U(f) = xf \). Find the matrix of \( U \) with respect to the bases \( \{1, x, x^2\} \) of \( P_3 \) and \( \{1, x, x^2, x^3\} \) of \( P_4 \). Note that the composition \( DU : P_3 \rightarrow P_3 \). Show that \( DU = I + QE \) where \( I \) denotes the identity transformation of \( P_3 \), \( Q \) denotes the linear transformation defined on \( P_3 \) by \( Q(1) = x, Q(x) = x^2, Q(x^3) = 0 \) and \( E \) denotes the matrix of the restriction of \( D \) to \( P_3 \).

(b) Let \( P_\infty \) denote the vector space of all polynomials (with no restriction on the degree). Then \( D(f) = df/dx \) and \( U(f) = xf \) are linear transformations from \( P_\infty \) to itself. Show that \( DU = I + UD \) is an identity for these transformations of \( P_\infty \).
(c) Let $V$ be a finite dimensional vector space of dimension greater than zero. Let $A$ and $B$ be linear transformations from $V$ to $V$. Let $I$ denote the identity transformation of $V$ to itself. Show that the equation

$$AB - BA = I$$

cannot hold for any choice of the matrices $A$ and $B$.

6. (a) Is the set of vectors \[ \begin{bmatrix} x \\ y \end{bmatrix} \] with $x^2 - y^2 = 0$ a subspace of $\mathbb{R}^2$? Explain how to write this set of vectors as the union of two non-zero subspaces of $\mathbb{R}^2$.

(b) Let $U$ and $V$ be distinct non-zero subspaces of $\mathbb{R}^3$. Explain under exactly what circumstances $U \cup V$ is a subspace of $\mathbb{R}^3$. Prove that for any two subspaces $U$ and $V$, their intersection $U \cap V$ is a subspace of $\mathbb{R}^3$.

7. Suppose that $\{\varepsilon_1, \varepsilon_2\}$ is the standard basis for $\mathbb{R}^2$. Let

$$v_1 = \varepsilon_1 + \varepsilon_2$$

$$v_2 = \varepsilon_1 - \varepsilon_2.$$

Show that $\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$ and find the transition matrix from this basis to the standard basis. Suppose that there is a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$T(v_1) = 2v_1 + 3v_2, \quad T(v_2) = v_1 + v_2.$$

Find a matrix $A$ such that $T(v) = Av$ for all $v$ in $\mathbb{R}^2$.

8. Let $A$ be an $n \times n$ matrix. Let $P$ be an invertible $n \times n$ matrix. Prove that $A$ and $B = P^{-1}AP$ have the same set of eigenvalues.

9. Let $V$ be the vector space of differentiable functions from the real numbers to the real numbers as functions of $t$. Let $F = e^t$, $G = te^t$ and $H = (t^2/2)e^t$. Prove that $B = \{F, G, H\}$ is a linearly independent subset of $V$. Let $W$ be the subspace of $V$ spanned by $F$, $G$ and $H$. Let $D = d/dt$ be the differentiation operator. Find the matrix of $D$ as a linear mapping of $W$ to $W$ with respect to the basis $B$.

10. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find the characteristic polynomial $C_A(\lambda)$ of $A$.

Find the roots of $C_A(\lambda) = 0$. Find the eigenspaces corresponding to each eigenvalue of $A$. 

2
11. Let \( A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \) where \( a \) and \( b \) are real numbers.

(a) Find the characteristic polynomial \( C_A(\lambda) \) of \( A \). Find the roots of \( C_A(\lambda) = 0 \). Find the eigenspaces of \( A \). Find a basis for \( \mathbb{R}^2 \) consisting in eigenvectors for \( A \). Using this basis show explicitly how \( A = PDP^{-1} \) for an invertible matrix \( P \) and a diagonal matrix \( D \). Use this result to obtain an explicit formula for the matrix \( A^N \) for an arbitrary positive integer \( N \).

(b) Let \( f(\lambda) = C_A(\lambda) \) be the characteristic polynomial for the matrix \( A \) above. Show that \( f(A) = 0 \) where here 0 denotes the zero matrix. For example, if

\[
B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

then the characteristic polynomial of \( B \) is \( \lambda^2 + 1 \) and \( B^2 + I = 0 \) where \( I \) is the two by two identity matrix.

(c) Solve the linear system of differential equations

\[
\begin{align*}
x' &= ax + by \\
y' &= bx + ay.
\end{align*}
\]

12. Solve the linear system of differential equations

\[
\begin{align*}
x' &= x + y \\
y' &= -x + y.
\end{align*}
\]

Remember that if the characteristic equation for a system has only complex roots, you can solve the system over the complex numbers and then use the complex solutions to get real solutions.

13. Consider the linear system of differential equations

\[
\begin{align*}
x' &= 3x + y \\
y' &= 3y.
\end{align*}
\]

Solve this system by first writing it as a matrix system with the matrix

\[
A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}
\]

and finding for \( v = (x, y)^T \) that \( v = e^{A\tau} c \) is a solution to the differential equation where \( c = (c_1, c_2)^T \) is a constant vector. Compute an explicit formula for \( e^{A\tau} \) and use this to solve the equation.

14. A matrix \( H \) over the complex numbers is said to be Hermitian if \( H^* = H \) where \( H^* \) denotes the conjugate transpose of \( H \). Show that a Hermitian matrix must have real eigenvalues. Illustrate your result by direct calculation for the matrix

\[
A = \begin{bmatrix} a & c + di \\ c - di & b \end{bmatrix}
\]

where \( a, b, c, d \) are real and \( i^2 = -1 \).
15. Use the Gram Schmidt process to find an orthogonal basis for the subspace of $R^4$ spanned by the vectors

\[ v_1 = (1, 1, 1, 1)^T, \]
\[ v_2 = (1, 1, 1, 0)^T, \]
\[ v_3 = (1, 1, 0, 0)^T. \]

16. This problem is a series of exercises about perpendicularity, projection and the least squares problem. We will augment this with class discussion.

(a) Let $\langle v, w \rangle = v^T w$ denote the standard dot product of vectors in $R^n$. Recall that two vectors $v, w \in R^n$ are said to be perpendicular if $\langle v, w \rangle = 0$. Given a subspace $W$ of $R^n$, we call $W^\perp$ the set of vectors that are perpendicular to all vectors in $W$. That is

\[ W^\perp = \{ v \in R^n | w \in W \implies \langle v, w \rangle = 0 \}. \]

Prove that $W^\perp$ is a subspace of $R^n$.

(b) Let $A$ be an $m \times n$ matrix so that you can view $A$ as a linear transformation from $R^n$ to $R^m$. Let $\ker(A) = \{ v \in R^n | Av = 0 \}$. Let $R(A) = \{ Av | v \in R^n \}$. Show that $\ker(A)$ is a subspace of $R^n$ and that $R(A)$ is a subspace of $R^m$.

(c) Let $A$ be as in part (b). Let $\text{Row}(A)$ denote the row space of $A$. That is, $\text{Row}(A)$ is the subspace of $R^n$ spanned by the rows of $A$. Prove that $\ker(A)^\perp = \text{Row}(A)$.

(d) Let $A$ be as in (b). Suppose that $m \geq n$ and that $A$ has rank $n$. Prove that $\ker(A) = \{ 0 \}$. Prove that $\ker(A^T A) = \{ 0 \}$. Note that $A^T A$ is an $n \times n$ matrix. Prove that $A^T A$ is invertible. Give examples of matrices $A$ of this sort.

(e) Let $A$ be as in part (d) so $\text{rank}(A) = n$ and $A^T A$ is invertible. Let $P = A (A^T A)^{-1} A^T$. Let $W = R(A)$. Show that $P R^n = W$ and that $P(v) = 0$ if and only if $v \in W^\perp$. Show that $P^2 = P$. Thus $P$ is the perpendicular projection of $R^n$ to the subspace $W = R(A)$.

(f) If $A$ is as in (d) and (e) we can consider equations like $Ax = v$ where $v$ is not in the subspace $W = R(A)$. Such equations have no solution, but the best approximation to them is to consider the corresponding equation $Ax = P v$ where $P$ is the perpendicular projection of $R^n$ to $W$. We have just seen that $P = A (A^T A)^{-1} A^T$. Prove that the solutions to $Ax = P v$ are the same as the solutions to $(A^T A)x = A^T v$. Do some specific numerical examples to illustrate this result.
Solutions

1. \( c_1 v_1 + \cdots + c_n v_n = 0 \) implies that \( c_1 T(v_1) + \cdots + c_n T(v_n) = 0 \), which implies that \( c_1 = c_2 = \cdots = c_n = 0 \).

2. \( c_1 T(v_1) + \cdots + c_n T(v_n) = 0 \) implies that \( c_1 T^{-1}(v_1) + \cdots + c_n T^{-1}(v_n) = 0 \), which implies that \( c_1 T(v_1) + \cdots + c_n T(v_n) = 0 \). Since \( \{v_1, \cdots, v_n\} \) is linearly independent, this implies that \( c_1 = c_2 = \cdots = c_n = 0 \).

3. Done in class.

4. The matrix \( A \) row reduces to

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

From this it follows that the row space has dimension two, with basis \( \{(1, 2, 3, 0)^T, (0, 0, 0, 1)^T\} \). It also follows from the row reduction that the column space has dimension two and has a basis consisting in the first column and the fourth column of the original matrix \( A \). Writing out the solution to this system we find for \( A(x, y, z, w)^T = (0, 0, 0, 0)^T \) that \( (x, y, z, w)^T = (-2y, -3z, y, z, 0)^T = B(y, z)^T \) where

\[
B = \begin{bmatrix}
-2 & -3 \\
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

5. (a) Note that \( D1 = 0, Dx = 1, Dx^2 = 2x, Dx^3 = 3x^2 \) while \( U1 = x, Ux = x^2, Ux^2 = x^3 \). You will find that

\[
[D] = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix},
\]

while

\[
[U] = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

From this it follows that

\[
[D][U] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix} = I + \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}.
\]
On the other hand we have $Q: P_3 \rightarrow P_3$ with matrix

$$[Q] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and $E: P_3 \rightarrow P_3$ with matrix

$$[E] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this it follows immediately that $[D][U] = I + [Q][E]$.

(b) Letting $D$ and $U$ be linear mappings of $P_\infty$ to itself, we have

$$DU(f) = D(xf) = f + xdf/dx = f + xD(f) = f + UD(f).$$

Thus $DU = I + UD$ in $P_\infty$.

(c) Let $tr: V \rightarrow V$ be the trace function defined by $tr(T) = \Sigma_{i=1}^{n} [T]_{ii}$ where $n = dim(V)$ and $\{v_1, \ldots, v_n\}$ is any basis of $V$. $T$ is a linear transformation of $V$ to $V$ and $[T]$ is the matrix of $T$ in this basis. The facts about the trace are that it is well-defined, independent of the choice of basis, and that

$$tr(A + B) = tr(A) + tr(B)$$

and

$$tr(AB) = tr(BA)$$

for any linear transformations $A$ and $B$ of $V$ to $V$. But we then have that

$$tr(AB - BA) = tr(AB) - tr(BA) = tr(AB) - tr(AB) = 0$$

while

$$tr(I) = n.$$ 

Since $n$ is not equal to 0 we conclude that

$$AB - BA \neq I.$$ 

6. The set of vectors of the form $(x, y)^T$ such that $x^2 - y^2 = 0$ is not a subspace. It is a union of the two subspaces $\{(x, x)\}$ and $\{(x, -x)\}$. The rest of this problem was discussed fully in class.

7. Done many times in class.
8. Suppose that $Av = \lambda v$. Let $w = P^{-1}v$. Then $P^{-1}APw = P^{-1}AP^{-1}v = P^{-1}P^{-1}Av = \lambda P^{-1}v = \lambda w$. Note that $v \neq 0$ if and only if $w \neq 0$. Thus whenever $A$ has an eigenvector with a given eigenvalue, then $P^{-1}AP$ also has an eigenvector with the same eigenvalue. The correspondence goes both ways, and this shows that the two matrices $A$ and $P^{-1}AP$ have the same eigenvalues.

9. You can prove that $B = \{F, G, H\}$ is linearly independent by computing the Wronskian. We have $F = e^t$, $G = te^t$ and $H = (t^2/2)e^t$. Thus $DF = e^t$, $DG = e^t + te^t$, $DH = te^t + (t^2/2)e^t$ and $D^2F = e^t$, $D^2G = 2e^t + te^t$, $D^2H = e^t + 2te^t + (t^2/2)e^t$. From this we see that the Wronskian is the determinant

$$
\begin{vmatrix}
\ e^t & te^t & (t^2/2)e^t \\
\ e^t & e^t + te^t & te^t + (t^2/2)e^t \\
\ e^t & 2e^t + te^t & e^t + 2te^t + (t^2/2)e^t \\
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\ e^t & te^t & (t^2/2)e^t \\
\ 0 & e^t + te^t & te^t + (t^2/2)e^t \\
\ 0 & 2e^t & e^t + 2te^t \\
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\ e^t & te^t & (t^2/2)e^t \\
\ 0 & e^t & te^t \\
\ 0 & 0 & e^t \\
\end{vmatrix}
= e^{3t} \neq 0.
$$

This shows linear independence of the three functions. Note that

$$
DF = F,
$$

$$
DG = F + G,
$$

$$
DH = G + H.
$$

From this it follows that the matrix of $D$ in the basis $\{F, G, H\}$ is

$$
[D] = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}.
$$

10. Done in class.

11. The characteristic polynomial is

$$
f(\lambda) = C_A(\lambda) = \lambda^2 - 2a\lambda + (a^2 - b^2) = (\lambda - (a + bi))(\lambda - (a - bi)).
$$

The rest of part (a) was gone over in class. For part (b), we have

$$
A = \begin{bmatrix}
\ a & b \\
\ b & a \\
\end{bmatrix}
$$

7
and

\[ A^2 = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{bmatrix}. \]

Thus

\[
\begin{align*}
f(A) &= A^2 - 2aA + (a^2 - b^2) = \\
&= \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{bmatrix} - 2a \begin{bmatrix} a & b \\ b & a \end{bmatrix} + (a^2 - b^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]

This completes part (b). Part (c) was done in class.

12. The matrix of this system is

\[ A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \]

with characteristic polynomial

\[ C_A(\lambda) = \lambda^2 - 2\lambda + 2 \]

with roots

\[ \lambda = 1 \pm i. \]

One complex solution is

\[ z(t) = e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{(1+i)t} \\ ie^{(1+i)t} \end{bmatrix} = \begin{bmatrix} e^t \cos(t) + ie^t \sin(t) \\ -e^t \sin(t) + ie^t \cos(t) \end{bmatrix}. \]

Thus

\[ \text{Re}(z(t)) = \begin{bmatrix} e^t \cos(t) \\ -e^t \sin(t) \end{bmatrix} \]

and

\[ \text{Im}(z(t)) = \begin{bmatrix} e^t \sin(t) \\ e^t \cos(t) \end{bmatrix}. \]

Each of \( \text{Re}(z(t)) \) and \( \text{Im}(z(t)) \) is a real vector solution to the differential equation, and the general real solution is given by \( c_1 \text{Re}(z(t)) + c_2 \text{Im}(z(t)) \) where \( c_1 \) and \( c_2 \) are arbitrary real constants.

13. In order to find

\[ e^{At} = I + At + A^2 t^2/2! + A^3 t^3/3! + \cdots, \]

we need to find the powers \( A^n \) for \( n = 1, 2, 3, \ldots \). If

\[ A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \]
then it is easy to see that

\[ A^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}. \]

From this we see that

\[ e^{At} = \sum_{n=0}^\infty A^n t^n/n! = \sum_{n=0}^\infty \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} t^n/n! = \]

\[ \begin{bmatrix} \sum_{n=0}^\infty \lambda^n t^n/n! & \sum_{n=0}^\infty n\lambda^{n-1} t^n/n! \\ 0 & \sum_{n=0}^\infty \lambda^n t^n/n! \end{bmatrix} = \]

Hence

\[ e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}. \]

From this we conclude that the general solution to the differential equation

\[ x' = \lambda x + y \]
\[ y' = \lambda y, \]

is

\[ (c_1 e^{\lambda t} + c_2 te^{\lambda t}, c_2 e^{\lambda t})^T. \]

In the given problem \( \lambda = 3 \).

14. Let \( A^* \) be the conjugate transpose of the matrix \( A \). That is, \( A^*_{ij} = \bar{a}_{ji} \). Then it is not hard to check that for a matrix products \( (AB)^* = B^*A^* \) and that for a vector \( v \) we have \( v^*v = \bar{v}_1v_1 + \cdots + \bar{v}_nv_n = |v_1|^2 + \cdots + |v_n|^2 \). This shows that \( v^*v \) is real and non-zero whenever the vector \( v \) is non-zero. Now suppose that \( Hv = \lambda v \) for a non-zero vector \( v \). Then \( v^*Hv = \lambda v^*v \).

Hence \( \lambda v^*v = (v^*Hv)^* = v^*H^*v^* = v^*H^*v \), since \( H^* = H \). Thus we have \( \lambda v^*v = \lambda v^*v \). Since \( v^*v \) is real and non-zero, it follows that \( \lambda = \lambda \). Hence \( \lambda \) is real. Hence all the eigenvalues of a Hermitian matrix are real. We leave it for you to work out the eigenvalues for the \( 2 \times 2 \) case.

15. This was done in class.

16. (a) Note that \( <v + v', w> = <v, w> + <v', w> \) and that \( <kv, w> = k<v, w> \). That \( W^\perp \) is a subspace follows easily from these properties.

(b) We leave this for you to write out. (c) If \( v \) belongs to \( Ker(A) \) then it follows from the definition of the kernel that each row vector in \( A \) has dot product zero with \( v \). This means that \( Row(A) \) is a subset of \( Ker(A)^\perp \). In fact, \( Ker(A) \) is defined to be exactly those vectors that are perpendicular to each row of \( A \). It follows from this that the row space \( Row(A) \) is equal to \( Ker(A)^\perp \). (d) \( Ker(A) = 0 \) follows from the rank condition and the form of the row reduction of \( A \). If \( A^TAv = 0 \) then
\( v \in \text{Ker}(A^T) = \text{Row}(A^T) = \text{Col}(A) = \{0\} \) since \( A \) has maximal rank. The invertibility of a square matrix is equivalent to its having a zero null space (i.e., equivalent to its having a vanishing kernel). Since \( \text{NullSpace}(A^T A) = \text{Ker}(A^T A) = 0 \) it follows that \( A^T A \) is invertible. (e)

\[
P^2 = [A(A^T A)^{-1} A^T]^2 = A((A^T A)^{-1} A^T A)(A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P.
\]

Since \( \text{Ker}(A) = 0 \) we have \( P v = 0 \) if and only if \( A(A^T A)^{-1} A^T v = 0 \) if and only if \( A((A^T A)^{-1} A^T v) = 0 \) if and only if \( A^T v = 0 \). But this last condition is equivalent to saying that \( v \) is perpendicular to the range of \( A \) \( \langle v, Aw \rangle = v^T A w = (A^T v)^T w = 0 \) for all \( w \) if and only if \( A^T v = 0 \). (f) We leave the last part for discussion at the review session.