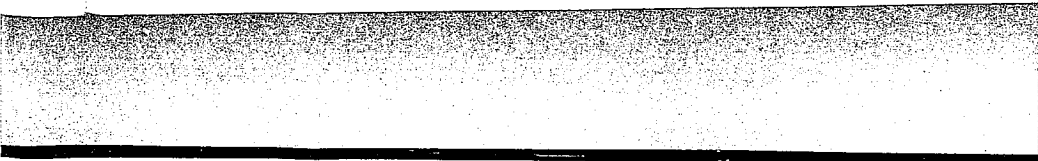


Satan, Cantor
& Infinity

Mind-Boggling Puzzles

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HYPERGAME

"Do you know the paradox *hypergame*?" asked the Sorcerer one day.

Neither Annabelle nor Alexander had heard of it.

"It is a lovely paradox created in the eighties by the mathematician William Zwicker. Besides being a delightful paradox in its own right, it leads to a totally new proof of Cantor's theorem."

"That sounds interesting!" said Annabelle.

"Well, first for the paradox," said the Sorcerer. "We will be discussing games played by just two people. Call a game *normal* if it has to terminate in a finite number of moves. An obvious example is tic-tac-toe; it must end in at most nine moves. Chess is also a normal game; the fifty-move rule ensures that the game cannot go on forever. Checkers is also a normal game. Every card game that I know is normal. Positional games, if played on an infinite board, might be nonnormal."

"Now, here is *hypergame*: The first move in *hypergame* is to declare what *normal* game should be played. Suppose, for example, that one of you is playing against me and I have the first move. Then I must declare what normal game should be played. I might say: 'Let's play chess,' in which case you have the first move in chess and we keep playing until the

game is terminated. Or, instead, I might say: 'Let's play checkers.' Then you make the first move in checkers, and we continue playing until the checker game is terminated. Or I might say: 'Let's play tic-tac-toe'—I can choose *any* normal game I like. But I am not allowed to choose a game that is not normal; I must choose a *normal* game.

"Now the problem is this: Is hypergame normal or not?"

The two thought about this for a while and came to the conclusion that the game must be normal.

"Why?" asked the Sorcerer.

"Because," they explained, "whatever normal game is chosen, that game must eventually terminate, since it is normal. This terminates the hypergame that is played. And so regardless of what normal game is chosen, the process has got to terminate. Thus, hypergame must be normal."

"So far, so good," said the Sorcerer, "but then a problem arises. Now that it is established that hypergame is normal, and since I may select *any* normal game on my first move, then I can say: 'Let's play hypergame.' Then you can say: 'OK, let's play hypergame.' And then I can say: 'OK, let's play hypergame,' and this process can go on indefinitely. Thus, hypergame does *not* have to terminate, which means that hypergame is not normal after all! And yet, you have proved that it is normal! This is a paradox."

Neither Annabelle nor Alexander could solve it.

"The whole point," said the Sorcerer, "is that the general notion of a game is not well defined. Given a set S of well-defined games, one can indeed define a hypergame *of that set* S , but this hypergame cannot itself be one of the games of S .

"Now, someone—I think Hegel—once defined a paradox as a truth standing on its head. Very often what first comes out as a paradox gets modified and leads to an important truth. And so it is with Zwicker's paradox of hypergame. A modification of

the argument establishes an interesting theorem that yields a completely new proof of Cantor's theorem.

"Recall briefly Cantor's proof. We are given a set A and to each element x of A is associated a subset of A denoted S_x . The idea is to construct a subset C of A that is different from S_x for every x . Cantor took C to be the set of all elements x of A such that x doesn't belong to S_x . Now, what Zwicker did was to find an entirely different set Z that is distinct from every one of the sets S_x . It, like Cantor's argument, shows that it is impossible to put A into a 1 to 1 correspondence with the set of all subsets of A , but the new set Z got by Zwicker is entirely different from the set C got by Cantor. Here is what Zwicker did.

"Once the correspondence (that assigns to each x in A the subset S_x) is given, we define a *path* to be any finite or infinite sequence x, y, z, \dots of elements of A such that for each term w of the sequence, either w is the last term, or the next term is an element of S_w . Thus, a *path* is generated as follows: Start with an arbitrary element x of A . If S_x is empty, that's the end of the path. If not, pick some element y of S_x . We then have the two-term sequence (x, y) . If S_y is empty, that's the end of the path. If not, pick some element z of S_y , and we then have the three-term sequence (x, y, z) . If S_z is empty, that's the end of the path, but if S_z is nonempty, pick some element w and make it the fourth term of the path, and keep on going in this manner until you either come to some S_v that is empty, in which case the path ends, or else keep going without stop, thus generating an infinite path. [For example, if y is an element of S_x and x is an element of S_y , then $(x, y, x, y, x, y, \dots)$ would be an infinite path. Or if x happens to be in S_x , then (x, x, x, x, \dots) would be an infinite path.] Now, given any x , either there does or there doesn't exist an infinite path that starts with x . Now define x to be *normal* if there exists no infinite path starting with x . Thus, if x is normal, then *every* possible path starting with x must

terminate. Now let Z be the set of all the normal elements. We then have

Theorem Z—(Zwicker's theorem). The set Z is different from S_x for every x .

“The proof,” said the Sorcerer “is an obvious modification of the argument establishing the paradox in hypergame.”

• 1 •

Give the proof of Zwicker's theorem.

“Notice,” said the Sorcerer, “that Zwicker's set Z bears no relation to Cantor's set C . The set of *normal* elements bears no significant relation to the set of x 's that don't belong to S_x .

“Cantor's proof relies essentially on the notion of negation. C is the set of all x 's such that x does *not* belong to S_x . Zwicker's proof is not based on negation; it is based on the notion of *finiteness* instead.”

“It seems to me,” said Annabelle, “that the notion of negation is secretly hidden in Zwicker's proof. He defines x to be normal if there exists *no* infinite path starting with x . Isn't that an implicit use of negation?”

“That's a clever observation,” said the Sorcerer, “but that use of negation is not really essential. We could have simply defined x to be normal if *all* paths starting with x are finite.”

Solution

1. We are to show that the set Z of normal elements cannot be S_x for any x . Equivalently, we are to show that for no x is

that x is such that S_x is the set of all normal elements. We then get a contradiction as follows:

We will first show that x must be normal. Well, consider any path starting with x . If S_x happens to be empty, the path stops right there with x (since any second term y must be a member of S_x). So we suppose S_x to be nonempty. Then the second term y of the path must be chosen from S_x , hence must be normal (since only normal elements are in S_x). Since y is normal, then every path starting with y must terminate; hence every path starting with (x, y, \dots) must terminate, and so x must be normal.

Since x is normal and S_x is the set of *all* normal elements, then x must be a member of S_x . Hence, there is the infinite path (x, x, x, \dots) just like the infinite game ("Let's play hypergame," "Let's play hypergame," "Let's play hypergame," \dots), and we thus get a contradiction.

Thus the set Z of all normal elements must be different from every S_x .