Give a procedure for deciding which white cycles to retain or throw away.

SEIFERT PAIRING

We now define an algebraic method for measuring the embedding of an oriented surface $F \subset S^3$. Given $F \subset S^3$, and a cycle $a$ on $F$, let $a^\infty$ denote the result of pushing $a$ a very small amount into $S^3 \setminus F$ along the positive normal direction to $F$. Using this, we define the Seifert pairing $\theta : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$ by the formula

$$\theta(a, b) = \text{lk}(a^\infty, b).$$

This is a well-defined, bilinear pairing. It is an invariant of the ambient isotopy class of the embedding $F \subset S^3$.

Seifert invented a version of this pairing in [S]. He used it to investigate branched covering spaces. It has since proved to be extraordinarily useful in both classical and higher-dimensional knot theory.

Example 7.4:

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$\theta$ & $a$ & $b$ \\
\hline
$a$ & $-1$ & 1 \\
$b$ & 0 & $-1$ \\
\hline
\end{tabular}
\end{center}
The surface \( F \) is oriented so that the positive normal points out of the page, toward the reader. For the self-linking \( \theta(a,a) = \text{lk}(a^\ast,a) \), \( a^\ast \) may be represented by a parallel copy of \( a \) along the surface. Thus \( \theta(a,a) \) can be computed from a disk with bands, by counting curls with sign.

Example 7.5:

\[
\begin{array}{c|cc} 
\theta & a & b \\
\hline 
a & -1 & 1 \\
b & 0 & 0 \\
\end{array}
\quad \begin{array}{c|ccc} 
\theta & a+b & b \\
\hline 
a+b & 0 & 1 \\
b & 0 & 0 \\
\end{array}
\]

\[\theta(a+b,a+b) = \theta(a,a) + \theta(b,a) + \theta(b,b) = 0\]

\[
\begin{array}{c|cc} 
\theta' & c & d \\
\hline 
c & 0 & 1 \\
d & 0 & 0 \\
\end{array}
\]

Thus these pairings are isomorphic. In fact, these two embeddings are isotopic:

\[
\sim \quad \sim \quad \sim \quad \sim \]
Exercises. Determine the Seifert pairing for this surface $F$.

SEIFERT PAIRING FOR THE SEIFERT SURFACE

Now let's work out an algorithm for computing the Seifert pairing from a Seifert surface (without pushing it into band-form). Recall that $H_1(F_k)$ is generated by the white cycles. (These are circles encircling white regions in $F_k$.) Thus we must determine how each crossing in the diagram contributes to the Seifert linking number $\theta(a,b)$.

The self-linking contribution is $\theta(a,b) = -\frac{1}{2} = \theta(b,b)$. (Note: The cycles bounding white regions are all oriented compatibly with an orientation for the white region itself.)

\[
\begin{align*}
\theta(a,b) &= +1 \\
\theta(b,a) &= 0 \\
\theta(a,a) &= \theta(b,b) = -\frac{1}{2}
\end{align*}
\]
For example:

\[ \begin{array}{c|cc}
\theta & \alpha & \beta \\
\hline
a & -1 & 1 \\
b & 0 & -1 \\
\end{array} \]

Here \( a \) and \( b \) interact at only one crossing. But we look at two crossings to compute \( \theta(a,b) \) and \( \theta(b,b) \).

**Exercise.** Compute the Seifert pairing for \( F_\alpha \) of Figure 7.1.

**Exercise.** Let \( x \cdot y \) denote intersection number on the surface \( F \). Show that for all \( x, y \in H_1(F) \),

\[ \theta(x,y) - \theta(y,x) = x \cdot y. \]

**Hint:** Do it for Seifert surface first. Then try the general case. To do the general case it helps to have the following description of linking numbers: Let \( \alpha, \beta \subset S^3 \) be two disjoint oriented curves. Let \( B \) be an oriented surface bounding \( \beta \). Isotope \( \alpha \) so that \( \alpha \) intersects \( B \) transversally. Then \( \text{lk}(\alpha, \beta) = \alpha \cdot B \).

[Why is this independent of the choice of \( B \)?]

**Exercise.** Prove, using Seifert (or spanning) surfaces, that this description of linking implies our original description.

Now return to the formula \( \theta(x,y) - \theta(y,x) = x \cdot y \), contemplate

\[ \theta B = \text{boundary of } B = x^* - x^* \]

\[ \theta(x,y) - \theta(y,x) = \text{lk}(x^*,y) - \text{lk}(y^*,x) \]

\[ = \text{lk}(y,x^*) - \text{lk}(y^*,x) \]

\[ = \text{lk}(y,x^*) - \text{lk}(y,x^*) \]

\[ = y \cdot B \]

\[ = x \cdot y. \]
DIFFERENT SURFACES FOR ISOTOPIC KNOTS

A given knot or link can have many different spanning surfaces. For example, two isotopic diagrams will have rather different Seifert surfaces. How are all the different surfaces spanning a knot related to one another?

The answer is, in principle, surprisingly simple. Consider the following way to complicate a spanning surface:

1) Cut out two discs, \( D_1, D_2 \).

2) Take a tube \( S^1 \times I \) and embed it in \( S^3 \) disjointly from the surface, but with the tube boundary attached to \( \partial D_1 \) and \( \partial D_2 \).

This is called doing a \( 1 \)-surgery to the surface.

\[ \begin{array}{c}
\text{before} \\
\text{after}
\end{array} \]

This is a \( 0 \)-surgery. It simplifies the surface (i.e., reduces genus).

These two surgery operations give us different surfaces with the same boundary.

DEFINITION 7.6. Let \( F \) and \( F' \) be oriented surfaces with boundary that are embedded in \( S^3 \). We say that \( F \) and \( F' \) are \( S \)-equivalent (\( F \cong F' \)) if \( F' \) may be obtained from \( F \) by a combination of \( 0 \)-surgery, \( 1 \)-surgery and ambient isotopy.

THEOREM 7.7 [L1]. Let \( F \) and \( F' \) be connected, oriented spanning surfaces for ambient isotopic links \( L, L' \subset S^3 \).

Then \( F \) and \( F' \) are \( S \)-equivalent.

Proof sketch: Let \( \lambda = S^3 \times I \) and suppose that \( \alpha : S^3 \times I \rightarrow S^3 \) is the ambient isotopy from \( L = \alpha(S^1 \times 0) \) to \( L' = \alpha(S^1 \times 1) \). Then we get an embedding of an annulus in \( \lambda' = S^1 \times I \) via \( \lambda' \circ \alpha(S^1 \times 1) \). If we form \( M = (F \cup \partial \lambda) \cup \alpha(S^1 \times 1) \cup (F' \cup \partial \lambda) \), then this is a closed surface embedded in \( S^3 \times I \). One then shows that \( M = \partial W \), where \( W \) is a 3-manifold embedded in \( S^3 \times I \). \( W \) can be...
arranged so that \((S^3 \times \mathbb{R}) \cap W\) has only Morse critical points of type \(z^2 + y^2 - z^2\) or \(-z^2 - y^2 + z^2\). These correspond to the 0-surgeries and 1-surgeries we described earlier.

\[x^2 + y^2 - z^2 < 0\]

\[x^2 + y^2 - z^2 = 0\]

Remark: It may be of interest to look directly at the \(S\)-equivalences between Seifert surfaces for diagrams that are related by Reidemeister moves. For example, is obtained from by the surgery

Now consider the Seifert pairings for \(S\)-equivalent surfaces. Suppose that \(F'\) is obtained from \(F\) by adding a tube. Then \(H_1(F') \cong H_1(F) \oplus \mathbb{Z} \oplus \mathbb{Z}\) where these two

extra factors are generated by a *meridian for the tube* and an element \(b\) that passes once along the tube orier so that \(a \cdot b = 1\).

We then have \(\theta(a, a) = 0\), \(\theta(a, b) = 1\), \(\theta(b, a) = 0\) and \(\theta(a, x) = \theta(x, a) = 0\) for all \(x \in H_1(F)\). Let \(\theta_0\) denote the Seifert pairing for \(F\). Then we have \(\theta = \begin{pmatrix} \theta_0 & 0 & 0 \\ 0 & 0 & 0 \\ b & \beta & 0 \end{pmatrix}\)

where \(\beta\) is a row vector, and \(\alpha\) is a column vector.

Because of the row \((0, 0, 1)\), \(\theta\) becomes on change of ba

\[
\begin{pmatrix}
\theta_0 & 0 & 0 \\
0 & 0 & 0 \\
b & \beta & 0
\end{pmatrix}
\]

An enlargement of this kind is called an \(S\)-equivalence. More generally, two matrices \(\theta\) and \(\phi\) are said to be \(S\)-equivalent if \(\phi\) can be obtained from \(\theta\) by a combination of congruence (\(\theta \rightarrow P \theta P'\) where \(P'\) is the transpose of \(P\), \(P\) invertible over \(\mathbb{Z}\). This corresponds to basis change.) and enlargements and contractions (reversal of enlargement) as above. If \(\theta\) and \(\phi\) are \(S\)-equivalent, we write \(\theta \cong \phi\).
COROLLARY 7.8. Let $K$ and $K'$ be ambient isotopic knots or links with connected spanning surfaces $F$ (for $K$) and $F'$ (for $K'$). Let $\theta$ be the Seifert pairing for $F$ and $\psi$ be the Seifert pairing for $F'$. Then $\theta$ and $\psi$ are $S$-equivalent.

INVARINTS OF $S$-EQUIVALENCE

DEFINITION 7.9. Let $F$ be a connected spanning surface for the knot or link $K$ and $\theta$ the Seifert pairing for $F$. Define

(i) The determinant of $K$, $D(K) = D(\theta+\theta')$ where $D$ denotes determinant.

(ii) The potential function of $K$, $\Omega_K(t) \in \mathbb{Z}[t^{-1}, t]$ by the formula $\Omega_K(t) = D(t^{-1}\theta-t\theta')$.

(iii) The signature of $K$, $\sigma(K) \in \mathbb{Z}$, by $\sigma(K) = \text{Sign}(\theta+\theta')$ where $\text{Sign}$ denotes the signature of this matrix.

(See definition below.)

Of course the gadgets produced in this definition are not going to change under $S$-equivalence! Hence they will be invariants of $K$.

For example, if $\theta = \begin{bmatrix} \theta_0 & 0 & 0 \\ 0 & \theta_0 & 0 \\ \alpha & 0 & 1 \end{bmatrix}$ then $\theta+\theta' = \begin{bmatrix} \theta_0+\theta_0' & 0 & \alpha' \\ 0 & \theta_0 & 0 \\ \alpha & 0 & 1 \end{bmatrix}$ and $D(t^{-1}\theta-t\theta') = D(t^{-1}\theta_0-t\theta_0')$ because

$$D\begin{bmatrix} 0 & t^{-1} \\ -t & 0 \end{bmatrix} = 1.$$ 

For the signature, recall that a symmetric matrix over $\mathbb{Z}$ can be diagonalized through congruence over $\mathbb{Q}$ (the rationals) or over $\mathbb{R}$. Let $e_+$ denote the number of positive diagonal entries, and $e_-$ the number of negative diagonal entries. The signature, $\text{Sign}(M)$, is defined over $\mathbb{Q}$ by the congruence class of $M$. (See [HNN].) Note in particular that $\text{Sign}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0$. From this it follows that $\text{Sign}(\theta)$ is an invariant of its $S$-equivalence class, hence an invariant of $K$. We shall also show that $\sigma(K)$ is an invariant of concordance.

The potential function provides a model for the Casson polynomial:

THEOREM 7.10.

(i) If $K$ and $K'$ are ambient isotopic oriented links, then $\Omega_K(t) = \Omega_{K'}(t)$.

(ii) If $K \sim 0$, then $\Omega_K(t) = 1$.

(iii) If links $K$, $\overline{K}$ and $L$ are related as below, then $\Omega_{K_0} - \Omega_{K_1} = (t-t^{-1})\Omega_L$.

$$K \quad \overline{K} \quad \overline{L}$$
CHAPTER VII

Proof: We have already proved (i) and (ii). Note that $\Omega_K = 0$ if $K$ is a split link. To see this, choose disjoint spanning surfaces for two pieces of the link, and connect these by a tube to form a connected spanning surface $F$.

If $\alpha$ is a meridian of this type, then

$$H_1(F) \cong H_1(F_1) \oplus H_1(F_2) \oplus \mathbb{Z}$$

where $a$ generates the extra copy of $\mathbb{Z}$. Since $\theta(\alpha, x) = \theta(x, \alpha) = 0 \forall x \in H_1(F)$, it follows that $\Omega_K(t) = 0$.

We use this discussion as follows. Consider Seifert surfaces for $K$, $\overline{K}$ and $L$. Locally, they appear as

$$a \quad a'$$

$\overline{F}_K$ $\overline{F}_K$ $F_L$

We see that $H_1(F_K)$ and $H_1(\overline{F}_K)$ will have one more homology generator than $F_L$, unless it should happen $L$ is a split diagram. But in this case $\Omega_L = 0$ while $F_K$ and $\overline{F}_K$ are isotopic by a $2\pi$ twist. Thus $\Omega_K - \Omega_{\overline{K}} = (t - t^{-1})\Omega_L$, proving (iii).

If $L$ is not a split diagram, then the extra generator may be represented as $a$ on $F_K$ and $a'$ on $\overline{F}_K$. See that $\theta(a', a') = \theta(a, a) + 1$. Hence $\theta_K = \left[ \begin{array}{c|c} n+1 & b \\ \hline a & \theta_L \end{array} \right]$, with appropriate choice of bases. I now a straightforward determinant calculation to show $\Omega_K - \Omega_{\overline{K}} = (t - t^{-1})\Omega_L$.

Remark: By our axiomatics, it follows that the Conway polynomial and our potential function are related by the substitution $z = t^{-1}/t$. Thus $\Omega_K(t) = v_K(t^{-1}/t)$. It amusing to solve the reverse. Then

$$t = z + 1/t.$$ 

Hence

$$t = \frac{z + 1}{z^{-1}}.$$ 

Using the notation $[z + 1/z]$ for the continued fraction $z + 1/z$, we have $v_K(z) = \Omega_K([z + 1/z])$. In particular

$$v_K(1) = \Omega_K\left[\frac{1 + \sqrt{5}}{2}\right].$$
We shall return to this subject!

Example: Let $T$ be a trefoil with $\Theta = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Then

$$\Omega_T = D \begin{bmatrix} -t^{-1} & t^{-1} \\ -t & -t^{-1} \end{bmatrix} = (t-t^{-1})^2 + 1 = z^2 + 1.$$ 

This agrees with our previous calculations.

Example: Given a knot $K$, let $K$ denote the numerator of the tangle obtained by running a parallel copy of $K$ with opposite orientation. $K$ is a link of two components.

Since $K$ has a spanning surface that is an annulus, we see that $\Theta = [-1:lk(\hat{K})]$ is a Seifert matrix for $\hat{K}$. Therefore $\Omega_K = (t^{-1} - t)(-1:lk(\hat{K}))$ and hence $v_K = lk(\hat{K})z$. Apparently, in this case the Conway polynomial is much easier to compute using the Seifert pairing. (Compare this discussion with the last exercise of Chapter IV of these notes.)

**TRANSLATING \(v\) AND \(\Omega\).**

Note that $\Omega_K(t) = D(t^{-1} - t)$. Therefore

$$\Omega_K(t^{-1}) = D(t\delta - t^{-1}\delta')$$

$$\Omega_K(t^{-1}) = D(t\delta' - t^{-1}\delta)$$

$$\Omega_K(t) = D(-t^{-1}\delta - \delta').$$

Since $\delta$ is $2g \times 2g$ for knots, $(2g+1) \times (2g+1)$ for 2-component links, we conclude that $\Omega_K(t^{-1}) = (-1)^{\mu+1}\Omega_K(t)$ where $\mu$ is the number of components of $K$.

To obtain a practical method of translation between $\Omega_K$ and $v_K$, we need to write $t^{n+(-1)^n}t^{-n} = T_{n}$ in terms of $z = t - t^{-1}$.

Look at the pattern:

$$t^{2} - t^{-2} = (t - t^{-1})^2 + 2 = z^2 + 2$$

$$t^{3} - t^{-3} = (t - t^{-1})^3 + 3t - 3t^{-1} = z^3 + 3z.$$ 

**Exercise.** Let $T_{n} = t^{n+(-1)^n}t^{-n}$ and $z = t - t^{-1}$. Show that $T_{n+2} = zT_{n+1} + T_{n}$ for $n \geq 0$.

$$t^{n} - t^{-n} = z^n$$

$$t^{2} - t^{-2} = z^2 + 2$$

$$t^{3} - t^{-3} = z^3 + 3z$$

$$t^{4} - t^{-4} = z^4 + 4z^2 + 2$$

$$t^{5} - t^{-5} = z^5 + 5z^3 + 5z$$

$$t^{6} - t^{-6} = z^6 + 6z^4 + 9z^2 + 2.$$ 

Show that the coefficient of $z^n$ in $t^{2n} - t^{-2n}$ is $n^2$.

We can use this exercise to obtain a curious form for the second Conway coefficient $a_2(K)$. For let $K$ be a knot. Then $K$ has potential function in the form
\[ \Omega_K(t) = b_0 + b_1(t^2 + t^{-2}) + b_2(t^4 - t^{-4}) + \cdots + b_n(t^{2n} + t^{-2n}) \]

It follows from our exercise that

\[ a_2(K) = b_1 + 4b_2 + 9b_3 + 16b_4 + \cdots + n^2b_n. \]

Exercise. Compute Seifert pairing, determinant, potential function and signature for the torus knots and links of type (2,n).

Exercise. Compute Seifert pairing, determinant, potential function and signature for the torus knots and links of type (2,n).

Exercise. Choose a knot or link and compute everything can.

Exercise. Let \( K \) be a knot. Show that \( v_K(21) / |v_K(21) = 2^{\sigma(K)} \). Use this in conjunction with the (easily prov) fact \( \sigma_K = \sigma_{K^+} + \sigma_{K^-} \) to show how inductively calculate knot signatures using a skein decomposition (see [C1], [G1]).

Apply this method to the knot \( 9_{42} \) (see the end of Section 19 of Chapter VI in these notes) to show that it has signature 2. This completes our earlier assertion that \( 9_{42} \) is not amphicheiral.

Exercise. Prove that \( \sigma(K^t) = -\sigma(K) \) when \( K \) is a knot and \( K^t \) is its mirror image. Calculate \( \sigma(T) \) and thereby show that \( T = \bigcirc \) and \( T^t = \bigcirc \) are not ambient isotopic.

Exercise. Prove that for knots \( K, K' \),

\[ \sigma(K \# K') = \sigma(K) + \sigma(K'). \]

Use this exercise and the previous exercise to distinguish the granny and the square knot.
THE ALEXANDER POLYNOMIAL AND THE ARF INVARIANT

Recall that we have defined, for a knot $K$, the invariant $A(K) \in \mathbb{Z}_2$ via $A(K) = a_2(K)$ (modulo-2) where $a_2(K)$ is the second Conway coefficient. And we showed (Chapter V) that $A(K) = 0$ for ribbon knots. In this chapter we will show that $A(K)$ is identical with the Arf invariant, $\text{ARF}(K)$, which is the Arf invariant of a mod-2 quadratic form related to $K$.

MOD-2 QUADRATIC FORMS

First recall that a mod-2 quadratic form $q$ is a mapping $q : V \to \mathbb{Z}_2^*$ where $V$ is a $\mathbb{Z}_2$-vector space such that $V$ has a bilinear symmetric pairing $(\cdot, \cdot) : V \times V \to \mathbb{Z}_2$. The mapping $q$ must satisfy the following property:

$$(*) \quad q(x+y) = q(x) + q(y) + (x,y) \text{ for all } x,y \in V.$$

Remark: Over a field of characteristic $\neq 2$ quadratic forms and symmetric bilinear forms are in 1-1 correspondence. Thus if $[,] : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is a symmetric bilinear form, and $\text{char} \mathbb{F} \neq 2$, then we can define $Q(x) = [x,x]/2$ and obtain:

$$Q(x,y) = \frac{1}{2}([x+y,x+y])$$
$$= \frac{1}{2}([x,x]+2[x,y]+[y,y])$$

$$\Rightarrow \quad Q(x,y) = Q(x) + Q(y) + [x,y].$$

In characteristic 2 the situation is subtler, and more the one quadratic form may correspond to a given bilinear form.

Classically, a quadratic form in two variables looks like a quadratic polynomial,

$$Q(x,y) = ax^2 + bxy + cy^2$$

and if $\text{char} \mathbb{F} \neq 2$ then we can write

$$ax^2 + bxy + cy^2 = (x,y) \left[ \begin{array}{cc} a/2 & b/2 \\ b/2 & c \end{array} \right] \begin{array}{c} x \\ y \end{array}$$

and classify the form $ax^2 + bxy + cy^2$ by analyzing the congruence class of the matrix $\left[ \begin{array}{cc} a/2 \\ b/2 & c \end{array} \right]$.

In characteristic 2, there is still a symmetric bilinear form associated with a quadratic polynomial, but now it occurs because $2 = 0$: If $Q(x,y) = ax^2 + bxy + cy^2$, let $v = (x,y)$, $v_1 = (x_1,y_1)$, $v_2 = (x_2,y_2)$. Then

$$Q(v_1,v_2) = a(x_1^2 + x_2^2) + b(x_1 y_1 + x_2 y_2) + c(y_1^2 + y_2^2)$$
$$= a(x_1^2 + x_2^2) + b(x_1 y_1 + x_2 y_2) + c(y_1^2 + y_2^2)$$
$$= Q(v_1) + Q(v_2) + b(x_1 y_1 + x_2 y_2)$$
$$= Q(v_1) + Q(v_2) + v_1 \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} v_2'.$$
The associated symmetric bilinear form has matrix
\[
\begin{bmatrix}
0 & b \\
b & 0
\end{bmatrix} = b \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]
This should remind us of the mod-2 intersection form on the (punctured) torus:

DEFINITION 10.1. Let \( K \subset \mathbb{S}^3 \) be a knot and \( F \) a connected oriented spanning surface for \( K \) with Seifert pairing \( \theta : H_1(F) \times H_1(F) \rightarrow \mathbb{Z} \). Let \( V = H_1(F) \otimes \mathbb{Z}_2 \), \( \overline{\theta} = \theta \) on \( V \), and let \( \langle , \rangle \) denote the mod-2 reduction of the intersection form \( S \) on \( H_1(F) \). The **mod-2 quadratic form** of \( F \) is then defined by \( q(x) = \overline{\theta}(x,x) \) for all \( x \in V \).

Note that

\[
q(x+y) = \overline{\theta}(x+y, x+y) = \overline{\theta}(x,x) + \overline{\theta}(y,y) + \overline{\theta}(x,y) + \overline{\theta}(y,x) \equiv q(x) + q(y) + (\overline{\theta}(x,y) - \overline{\theta}(y,x)) \pmod{2} \\
\equiv q(x) + q(y) + S(x,y) \pmod{2}
\]

Thus, \( q(x+y) = q(x) + q(y) + \langle x,y \rangle \).

We see that with respect to the standard basis (symplectic basis) for the surface it is easy to write the quadratic polynomial that corresponds to the form. Thus:

\[
q_0 = xy \quad x^2+xy \quad xy+y^2 \quad x^2+xy+y^2 = q
\]

We know that the first three surfaces are isotopic, hence the forms \( xy \), \( x^2+xy \) and \( xy+y^2 \) must be isomorphic! Indeed, this is the case. For example \( x^2+xy = x(x+y) \) so is isomorphic to \( xy \) via the change of basis \( x' = x \), \( y' = x+y \).

These four forms are nondegenerate in the sense that the associated bilinear form is nondegenerate. Here it is in matrix form \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]
Nondegeneracy of \( \langle , \rangle \) means that the matrix of \( \langle , \rangle \) is nonsingular.

In fact, we have just shown that there are at most \( t \) isomorphism classes of nondegenerate dimension-two mod-2 forms: \( q_0 = xy \) and \( q_1 = x^2+xy+y^2 \). It is easy to see that \( q_0 \) and \( q_1 \) are not isomorphic. For, if \( V = \mathbb{Z}_2 \times \mathbb{Z} \) then \( q_0 \) takes a majority of elements to 0, while \( q_1 \) takes a majority of elements to 1. Thus we have classified rank-2 forms over \( \mathbb{Z}_2 \).
DEFINITION 10.2. Let \( V \) be a finite dimensional vector space over \( \mathbb{Z}_2 \) and \( q : V \rightarrow \mathbb{Z}_2 \) a nondegenerate quadratic form. The Arf invariant \( \text{ARF}(q) \in \mathbb{Z}_2 \) is defined by the formula

\[
\text{ARF}(q) = \begin{cases} 
0 & \text{if } q \text{ takes a majority of elements to } 0 \\
1 & \text{if } q \text{ takes a majority of elements to } 1 
\end{cases}
\]

Certainly \( \text{ARF} \) is an invariant so long as it is well-defined. Indeed it is well-defined, and this comes about as follows:

(i) Symmetric bilinear forms over \( \mathbb{Z}_2 \) are all (when nondegenerate) sums of forms of type \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).
That is, there is a symplectic basis

\( \{a_i, b_i, \ldots, a_n, b_n\} \) for \( V \) such that

\[ (a_i, b_j) = \delta_{ij}, \quad (a_i, a_j) = (b_i, b_j) = 0 \quad \text{for all} \quad i \quad \text{and} \quad j. \]
This, of course, is given geometrically in our Seifert form case.

(ii) It follows from (i) that any nondegenerate mod-2 quadratic form is a direct sum of the two-dimensional forms. Hence it is a direct sum involving \( q_0 \) and \( q_1 \).

(iii) \( q_1 \oplus q_1 \cong q_0 \oplus q_0 \). This is the basic fact. You can prove it by a basis-change, or you can see it geometrically by taking the connected sum

which has the form \( q_1 \oplus q_1 \) and find the basis change by topological script! Here we can use mod-2 script in the plane so that

\[
\begin{array}{c}
\text{These modifications do not change the mod-2 quadratic form of the corresponding surface.}
\end{array}
\]

You may also think of these script moves as equivalent to consequences of

performed on the bands (compare with pass-equivalence, Chapter V). For then

\[
\text{pass}
\]
\[ q_0 \oplus \cdots \oplus q_0 = \emptyset_0 \quad \text{or} \]
\[ q_1 \oplus q_0 \oplus \cdots \oplus q_0 = \emptyset_1. \]

It is then a counting matter to see that \( \text{ARF}(\emptyset_0) = 0 \) and \( \text{ARF}(\emptyset_1) = 1 \). Thus, we have classified all nondegenerate mod-2 quadratic forms, and shown the utility of the ARF invariant in the process.

(iv) It follows from what we have said, that \( q \oplus q' \) has an ARF invariant whenever \( q \) and \( q' \) have ARF invariants. Furthermore,

\[ \text{ARF}(q \oplus q') = \text{ARF}(q) \oplus \text{ARF}(q'). \]

(v) It can be shown that (do it!) if
\[ \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \]

is a symplectic basis for \( V \)
\[ q : V \rightarrow \mathbb{Z}_2 \]
a mod-2 quadratic form, then

\[ \text{ARF}(q) = \sum_{k=1}^{2^k} q(a_k)q(b_k). \]

This gives an explicit formula for ARF.

Let \( K \subset S^3 \) be a knot. We now define \( \text{ARF}(K) \in \mathbb{Z}_2 \) by the formula \( \text{ARF}(K) = \text{ARF}(q) \) where \( q \) is the mod-2 quadratic form of any spanning surface for \( K \). We leave it as an exercise in \( s \)-equivalence to see that this is an invariant of \( K \).

**Theorem 10.3.** If knots \( K \) and \( \overline{K} \) are related by one crossing change, and \( L \) is the 2-component link obtained
by splicing this crossing, then
\[ \text{ARF}(K) - \text{ARF}(\overline{K}) = \Omega_k(L). \]

**Corollary.** Let \( A(K) \) be the mod-2 reduction of the second Conway coefficient \( a_2(K) \). Then \( A(K) = \text{ARF}(K) \).

**Proof:** Exercise.

**Proof of Theorem.** Also an exercise. Compare on a spanning surface with the curve \( \alpha \) depicted to the left as part of a symplectic basis. Note that you can assume that this appears as part of a band, and that the dual curve \( \beta \) is on another band so that the simplest picture gives:

\[ K \quad L \]

THEOREM 10.4 (Levine [L2]). Let \( K \subset S^3 \) be a knot. Let \( \Delta_K(t) \) be the Alexander polynomial for \( K \). Then
\[ \text{ARF}(K) = 0 \iff \Delta_K(-1) \equiv \pm 1 \pmod{8} \]
\[ \text{ARF}(K) = 1 \iff \Delta_K(-1) \equiv \pm 3 \pmod{8}. \]

**Proof:** \( \Delta_K(z) \) denotes the Conway polynomial. We know (Proposition 9.3) that
\[ \Delta_K(i^{-1} - i^{-1}) = \Delta_K(t). \]

Hence \( \Delta_K(2i) = \Delta_K(-1) \) where \( i = \sqrt{-1} \). Now, for a knot, \( \Delta_K(z) = 1 + a_2 z^2 + a_4 z^4 + \cdots \). Hence \( \Delta_K(2i) \equiv 1 - 4a_2(K) \pmod{8} \). Since \( a_2(K) = \text{ARF}(K) \pmod{2} \), the theorem follows immediately from this.

**Remark:** \( 3^2 \equiv 1 \pmod{8} \).

In order to get a taste of the power of Levine's result for calculating Arf invariants, we now give a brief introduction to Fox's Free Differential Calculus, and its use in computing Alexander polynomials, hence, derivatively, in computing ARF.