1. Let \( A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \end{bmatrix} \). Consider the system \( A[x_1, x_2, x_3, x_4]^T = [a, b, c]^T \).

(a) Use row reduction to determine the general solution for this system. For what values of \( a, b, c \) do there exist solutions to this system of equations? Give the vector form of the solution to
\[ A[x_1, x_2, x_3, x_4]^T = [1, 1, 1]^T. \]
Find a basis for the null space \( N(A) \).

(b) Using your work in part (a), give the row reduced echelon form \( R \) for the matrix \( A \). What is the rank of \( A \)? What is the dimension of the column space of \( A \)? Give a basis for the column space of \( A \) that is a subset of the columns of \( A \).

(c) Let \( \text{Col}(A) \) denote the column space of \( A \). Find a basis for \( \text{Col}(A)^\perp \). (Hint: Use the basis for \( \text{Col}(A) \) that you found in part (b).)

2. (a) Let 
\[ M = \begin{bmatrix} t & 1 \\ 0 & t \end{bmatrix}. \]
Prove by induction that 
\[ M^n = \begin{bmatrix} t^n & nt^{n-1} \\ 0 & t^n \end{bmatrix}. \]
Thinking of each entry of \( M^n \) as a function of \( t \), show that 
\[ d(M^n)/dt = nM^{n-1} \]
for \( n \geq 1 \) where \( M^0 = I \) denotes the \( 2 \times 2 \) identity matrix.

(b) Consider the oriented graph \( G \) whose incidence matrix is 
\[ A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}. \]
This means that there are two vertices in the graph, labeled 1 and 2. There is an oriented edge from 1 to 2 and each vertex has two oriented loops from itself, to itself. Draw a diagram of the graph \( G \). How many oriented paths of length 137 are there in \( G \) starting at vertex 1 and ending at vertex 2?

3. Let \( A \) be an \( n \times n \) matrix. Define the trace of \( A \) by the formula \( \text{tr}(A) = \sum_{i=1}^{n} a_{ii} \). That is, the trace of a matrix is the sum of the diagonal entries of the matrix. Recall that for \( n \times n \) matrices \( A \) and \( B \), \( \text{tr}(AB) = \text{tr}(BA) \).
(a) Prove that for two \( n \times n \) matrices \( A \) and \( B \), \( \text{tr}(A+B) = \text{tr}(A) + \text{tr}(B) \).

(b) Prove that for any \( n \times n \) matrix \( A \), \( \text{tr}(A) = \text{tr}(A^T) \).

(c) Show that there do not exist \( n \times n \) (\( n \neq 0 \)) matrices \( A \) and \( B \) such that \( AB - BA = I_n \) where \( I_n \) is the \( n \times n \) identity matrix. (Hint, take the trace of the left-hand side and take the trace of the right-hand side. Show that they cannot be equal.)

4. Let \( S = [1/\sqrt{2}, \cos(x), \cos(2x), \cdots, \cos(nx), \sin(x), \sin(2x), \cdots, \sin(nx)] \).

We know that \( S \) is an orthonormal set of vectors in \( C[-\pi, \pi] \) with inner product defined by \( \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx \). Thus \( S \) can be taken as the basis of its span \( W = \text{Span}(S) \).

(a) What is the dimension of the subspace \( W \) (defined above) of \( C[-\pi, \pi] \)?

(b) Determine the best least squares approximation to \( h(x) = x \) by a function from the subspace \( W \). Hint: You will need to know the integrals \( \int x\sin(kx)dx \) and \( \int x\cos(kx)dx \). You can look these up, or do them by integration by parts.

5. In this problem and the subsequent problems we are concerned with the following question: Given an \( n \times n \) matrix \( A \), does there exist a non-zero vector \( v \in \mathbb{R}^n \) such that \( Av = \lambda v \) for some real number \( \lambda \)? If \( Av = \lambda v \) then

\[
Av - \lambda v = 0.
\]

Hence

\[
Av - \lambda Iv = 0
\]

where \( I \) is the \( n \times n \) identity matrix. Hence

\[
(A - \lambda I)v = 0.
\]

In order for this system to have a non-zero solution \( v \) (for some choice of value for \( \lambda \)) we need that the determinant

\[
\text{Det}(A - \lambda I) = 0.
\]

We say that

\[
C_A(x) = \text{Det}(A - xI)
\]

is the characteristic polynomial for \( A \) and we call solutions \( \lambda \) to the equation \( C_A(x) = 0 \) the eigenvalues of the matrix \( A \). A non-zero vector such that \( Av = \lambda v \) is said to be an eigenvector of \( A \) belonging to the eigenvalue \( \lambda \).
(a) Let \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear transformation whose matrix in the standard basis is
\[
A = \begin{bmatrix}
4 & 1 \\
-2 & 1
\end{bmatrix}.
\]
Show that \( C_A(x) = x^2 - 5x + 6 \), and find the roots of \( C_A(x) = 0 \). Let \( \lambda_1 \) and \( \lambda_2 \) denote these two roots. Find an eigenvector \( v_1 \) belonging to \( \lambda_1 \). Find an eigenvector \( v_2 \) belonging to \( \lambda_2 \). Show that \( v_1 \) and \( v_2 \) are linearly independent. Thus \( E = [v_1, v_2] \) is a basis for \( \mathbb{R}^2 \). Find \( B \), the matrix of the linear transformation \( L \) in the basis \( E \). Find an invertible matrix \( P \) such that \( B = P^{-1}AP \).

(b) Let
\[
M = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]
Show that \( C_M(x) = x^2 - (a + d)x + (ad - bc) \).

(c) Give an example of a \( 2 \times 2 \) matrix that has no real eigenvalues.

(d) Give an example of a \( 2 \times 2 \) matrix whose characteristic polynomial is \( (\lambda - 5)^2 \), and such that the space of eigenvectors with eigenvalue 5 is one-dimensional. Give a second example with the same characteristic polynomial, and such that the space of eigenvectors with eigenvalue 5 is two dimensional. Hint: Consider the properties of the following two matrices.
\[
M = \begin{bmatrix}
7 & 1 \\
0 & 7
\end{bmatrix},
\]
and
\[
M = \begin{bmatrix}
7 & 0 \\
0 & 7
\end{bmatrix}.
\]

6. Let
\[
A = \begin{bmatrix}
2 & 1 & 1 \\
0 & 3 & 1 \\
0 & 0 & 4
\end{bmatrix}.
\]

(a) Compute the characteristic polynomial \( C_A(x) \).

(b) Find the eigenvalues of \( A \) and determine an eigenvector for each eigenvalue.

(c) Use the eigenvectors in part (b) to form a basis for \( \mathbb{R}^3 \) with respect to which the linear transformation corresponding to \( A \) is diagonal. Find an invertible matrix \( P \) such that \( P^{-1}AP \) is diagonal.
7. In this problem we apply eigenvectors and eigenvalues to the solution of a system of differential equations. Suppose you are asked to solve

\[ y'_1 = ay_1 + by_2 \]
\[ y'_2 = cy_1 + dy_2 \]

where \( y_1 \) and \( y_2 \) are differentiable functions of a variable \( t \) and \( y' \) denotes \( dy/dt \). Then try solutions in the form

\[ y_1 = x_1e^{\lambda t}, \]
\[ y_2 = x_2e^{\lambda t}, \]

where \( x_1 \) and \( x_2 \) are constant real numbers and \( \lambda \) is also a constant real number. Note that

\[ y'_1 = \lambda x_1e^{\lambda t}, \]
\[ y'_2 = \lambda x_2e^{\lambda t}. \]

Thus we are attempting to solve

\[ \lambda x_1e^{\lambda t} = ax_1e^{\lambda t} + bx_2e^{\lambda t}, \]
\[ \lambda x_2e^{\lambda t} = cx_1e^{\lambda t} + dx_2e^{\lambda t}. \]

But since \( e^{\lambda t} \neq 0 \), this is the same as trying to solve

\[ \lambda x_1 = ax_1 + bx_2, \]
\[ \lambda x_2 = cx_1 + dx_2. \]

With \( v = (x_1, x_2)^T \), this is the eigenvalue problem for the matrix

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \]

That is, we are looking for the eigenvalues of this matrix and for eigenvectors that belong to them.

Find solutions to the differential system

\[ y'_1 = y_1 + y_2 \]
\[ y'_2 = -2y_1 + 4y_2. \]

by using the method described above. You should find two different fundamental solutions to the system corresponding to two distinct eigenvalues of the matrix

\[ \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}. \]