Detecting Virtual Knots

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Abstract

This paper is a survey of the theory of virtual knots. We dedicate this paper to the memory of Professor Mario Pezzana.

1 Introduction

This paper is a survey of virtual knot theory, a generalisation of classical knot theory [6], [7]. Here we give the basic definitions, some fundamental properties,a collection of examples and discussion of the work that I and other people have so far contributed to this idea. This paper is an outgrowth of the lectures that I gave at the "Knots in Hellas" conference in the summer of 1998 [12]. It is also a companion version to the paper [7] by this author that introduces this theory of virtual knots. This paper goes beyond the lectures for "Knots in Hellas" by bringing in recent work that detects subtle knotted virtuals and also work relating virtual knot theory to embedded surfaces in four dimensional space. Throughout this paper I shall refer to knots and links by the generic term "knot". In referring to a trivial fundamental group of a knot, I mean that the fundamental group is isomorphic to the integers.

The paper is organised as follows. Section 2 gives the definition of a virtual knot in terms of diagrams and moves on diagrams. Section 3 discusses both the motivation from knots in thickened surfaces and the abstract properties of Gauss codes. Section 3 proves basic results about virtual knots by using reconstruction properties of Gauss codes. Section 4 discusses the fundamental group and the quandle extended for virtual knots. Examples are given of non-trivial virtual knots with trivial (isomorphic to the integers) fundamental group. An example shows that some virtual knots are distinguished from their mirror images by the fundamental group, a very non-classical effect. Section 5 shows how the bracket polynomial (hence the Jones polynomial) extends naturally to virtuals and gives examples of non-trivial virtual knots with trivial Jones polynomial. Examples of infinitely many distinct virtuals with the same fundamental group are verified by using the bracket polynomial. An example is given of a knotted virtual with trivial fundamental group and unit Jones polynomial. It is conjectured that this phenomenon cannot happen with virtuals whose shadow code is classical. Section 6 shows how to construct infinitely many non-trivial virtual knots with unit Jones polynomial. It is an open question whether any of these examples are equivalent to classical knots. In Section 7 we discuss the general framework for the biquandle, an algebraic invariant of virtual knots that can detect virtuals that are undectable by the fundamental group, quandle and Jones polynomial. Specific representations of the biquandle are discussed. These include a generalization of the Alexander module, and the group defined by Silver and Williams [24]. Silver and Williams and independently Sawollek [31] found the generalization of the Alexander polynomial described here in the context of invariants of virtuals. The generalization first appeared in [9] as an invariant of knots in thickened surfaces. In this section we show how the generalized Alexander module naturally arises in finding the simplest linear representation of the biquandle. This section also discusses informally the problems involved in making a fully general definition of the biquandle. It turns out the the axioms for a biquandle make demands that universal algebra in its usual context cannot meet. We explain how a generalization of the untyped lambda calculus of Church and Curry (originally devised for mathematical logic) can be used to formulate an appropiate algebraic category for biquandles. These issues will be taken up in papers to follow the present one, including a joint paper with Roger Fenn and Mercedes Jordan [4]. In Section 8 we discuss the four dimensional application of virtual knots due to Shin Saton. Section 9 is a short epilogue.

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2 Defining Virtual Knots and Links

A classical knot [1] can be represented by a diagram. The diagram is a 4-regular plane graph with extra structure at its nodes. The extra structure is classically intended to indicate a way to embed a circle in three dimensional space. The shadow of a projection of this embedding is the given plane graph. Thus we are all familiar with the usual convention for illustrating a crossing by omitting a bit of arc at the node of the plane graph. The bit omitted is understood to pass underneath the uninterrupted arc. See Figure 1 .

From the point of view of a topologist, a knot diagram represents an "actual" knotted (possibly unknotted) loop embedded in three space. The crossing structure is an artifact of the projection to the plane.

I shall define a virtual knot (or link) diagram. The definition of a virtual diagram is just this: We allow a new sort of crossing, denoted as shown in Figure 1 as a 4-valent vertex with a small circle around it.

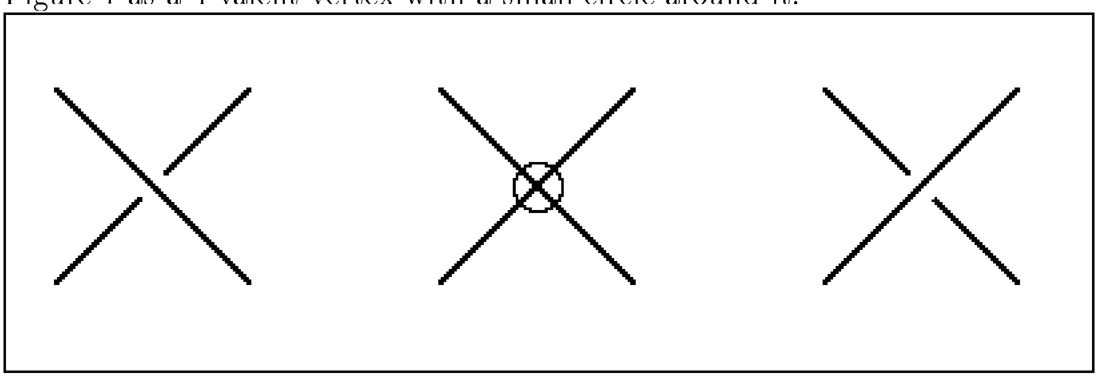

Figure 1 — Crossings and Virtual Crossings

This sort of crossing is called virtual. It comes in only one flavor. You cannot switch over and under in a virtual crossing. However the idea is not that a virtual crossing is just an ordinary graphical vertex. Rather, the idea is that the virtual crossing is not really there.

If I draw a non-planar graph in the plane it necessarily acquires virtual crossings. These crossings are not part of the structure of the graph itself. They are artifacts of the drawing of the graph in the plane. The graph theorist often gets rid of a crossing in the plane by making it into a knot theorist's crossing, thereby indicating a particular embedding of the graph in three dimensional space. This is just what we do not do with our virtual knot crossings, for then they would be indistinct from classical crossings. The virtual crossings are not there. We shall make sense of that property by the following axioms generalising classical Reidemeister moves. See Figure 2.

The moves fall into three types: (A) Classical Reidemeister moves relating classical crossings; (B) Shadowed versions of Reidemeister moves relating only virtual crossings; (C) A triangle move that relates two virtual crossings and one classical crossing.

The last move (type C) is the embodiment of our principle that the virtual crossings are not really there. Suppose that an arc is free of classical crossings. Then that arc can be arbitrarily moved (holding its endpoints fixed) to any new location. The new location will reveal a new set of virtual crossings if the arc that is moved is placed transversally to the remaining part of the diagram. See Figure 4 for illustrations of this process and for an example of unknotting of a virtual diagram.

The theory of virtual knots is constructed on this combinatorial basis - in terms of the generalised Reidemeister moves. We will make invariants of virtual knots by finding functions well-defined on virtual diagrams that are unchanged under the application of the virtual moves. The remaining sections of this paper study many instances of such invariants.

Remark. In Figure 3 we have illustrated the allowed analog for virtuals of the third Reidemeister move, and two suggestive relatives of that move that we call the *Forbidden Moves* of type $F1$ and $F2$. These moves are not part of the equivalence relation for virtual knots, but it is actually interesting

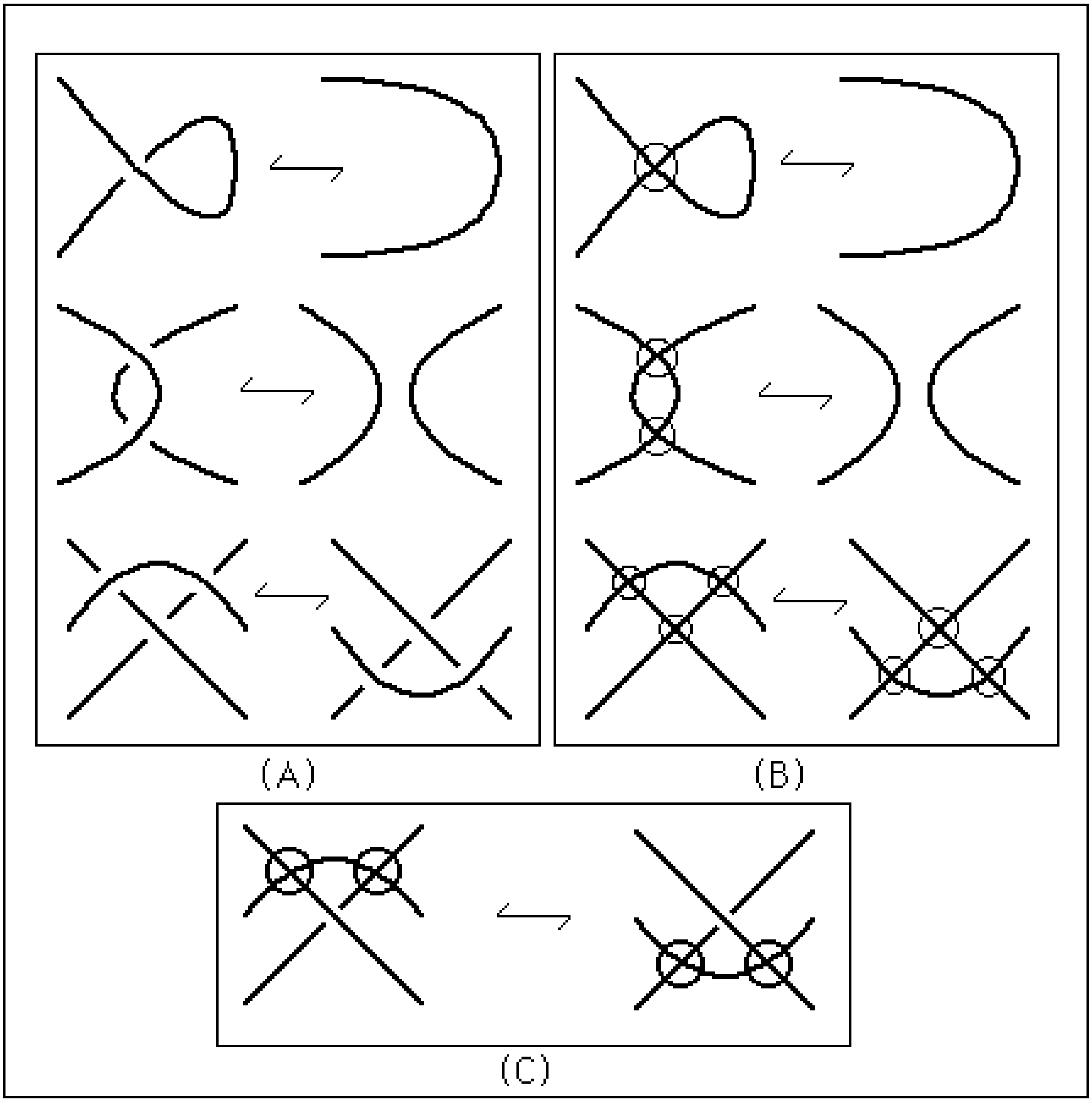

Figure 2 — Generalised Reidemeister Moves for Virtual Knot Theory

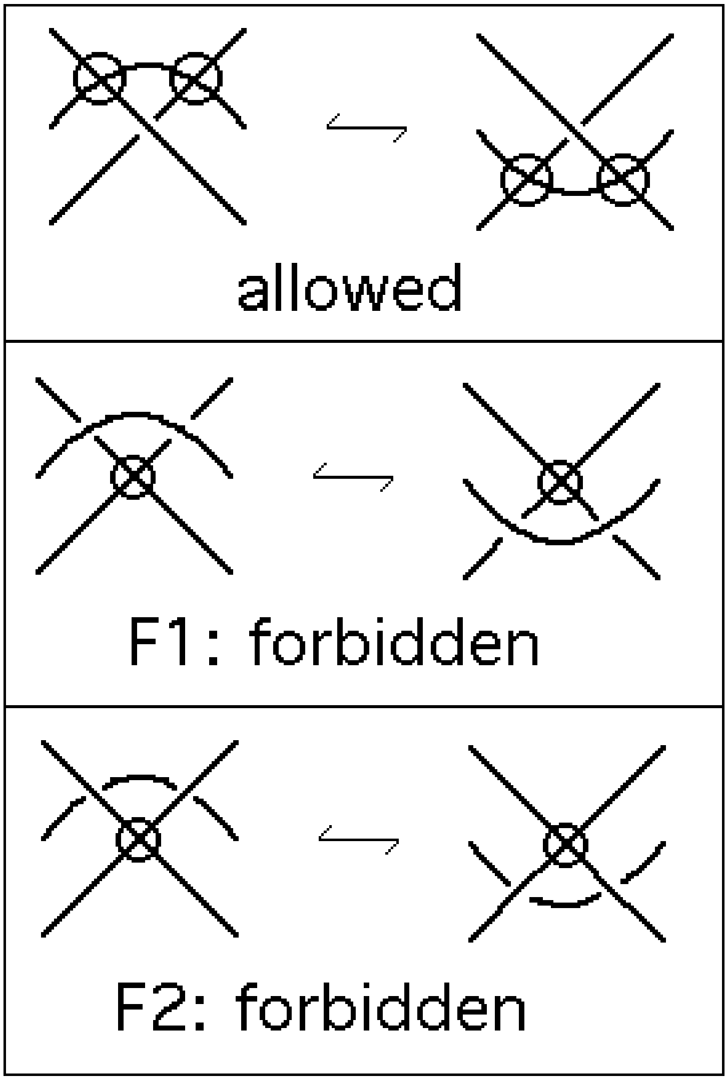

Figure 3 — The Forbidden Moves

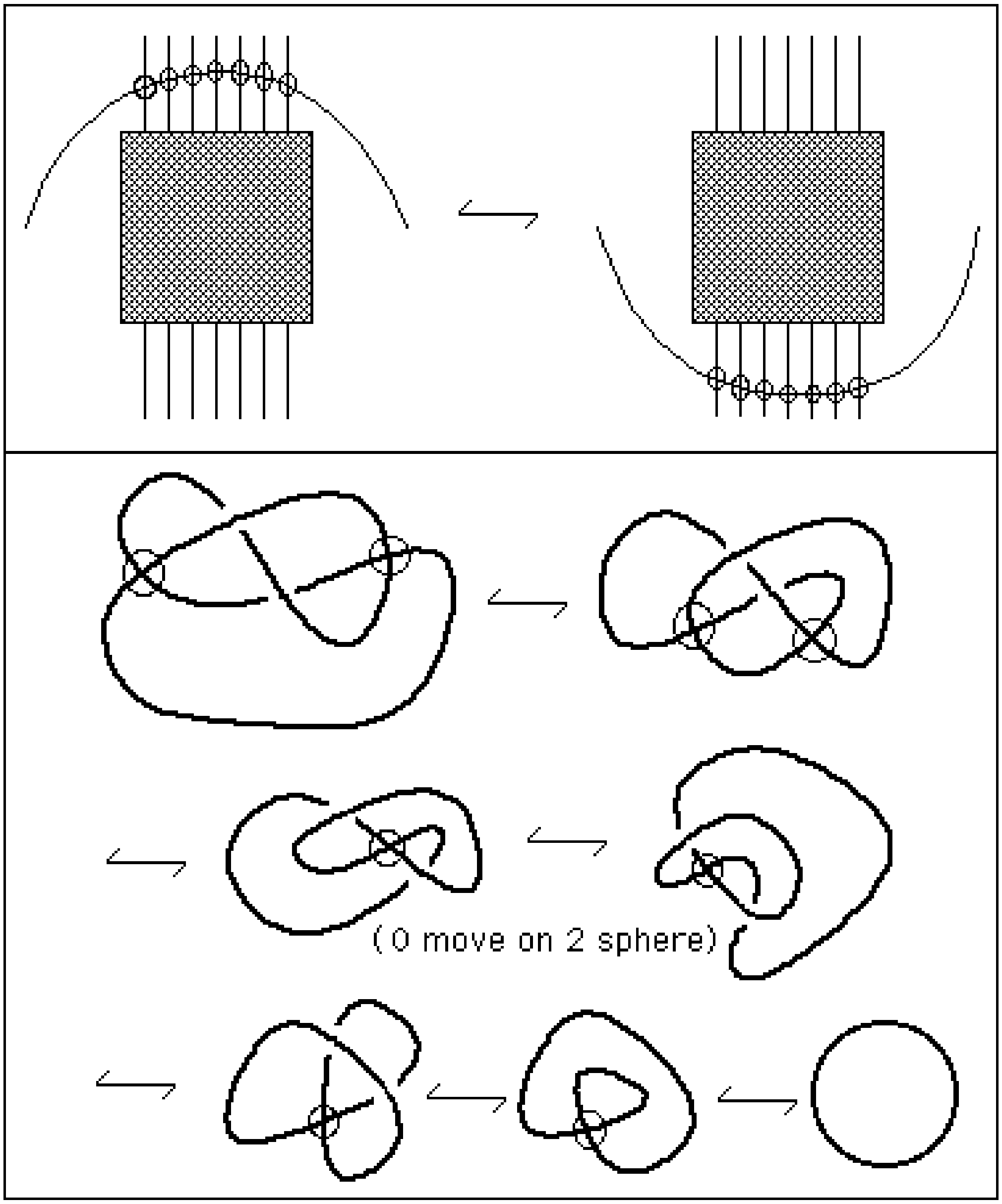

Figure 4 — Virtual Moves

to include one or both of them to obtain a quotient theory of the theory of virtual knots and links. We shall call the theory obtained by adding the move $F1$ the theory of *Welded Knots and Links* in analogy to the *welded braids* of Rourke and Fenn [30]. It turns out that if both of the forbidden moves are added to the theory, then it is possible to unknot any knot [19], [16]. The theory with both forbidden moves added will be called the theory of *Fused Links* and will be the subject of a separate paper. In regard to these forbidden moves, we point out that the allowed analog of the third Reidemeister move is actually an exemplar of a single generalized type of virtual move, namely if in a virtual diagram one encounters an arc with a consequtive sequence of virtual crossings, then one can excise this arc and reconnect the endpoints by a new arc placed anywhere transversal to the diagram with virtual crossings. We call this generalized move a *virtual detour*. See Figure 4. It is easy to see that the equivalence relation for virtual knots and links is generated by the classical Reidemeister moves plus the virtual detour. In this sense the virtual knots are diagrammatic analogues of the trajectories of a particle that is moving in three dimesional space, but occasionally "tunnels" from one location to another distant location. The virtual detours in the diagram are connections to indicate these tunnels.

3 Motivations

While it is clear that one can make a formal generalisation of knot theory in the manner so far described, it may not be yet clear why one should generalize in this particular way. This section explains two sources of motivation. The first is the study of knots in thickened surfaces of higher genus (classical knot theory is actually the theory of knots in a thickened two-sphere). The second is the extension of knot theory to the purely combinatorial domain of Gauss codes and Gauss diagrams. It is in this second domain that the full force of the virtual theory comes into play.

3.1 Surfaces

Consider the two examples of virtual knots in Figure 5. We shall see later in this paper that these are both non-trivial knots in the virtual category. In Figure 5 we have also illustrated how these two diagrams can be drawn (as knot diagrams) on the surface of a torus. The virtual crossings are then seen as artifacts of the projection of the torus to the plane.

The knots drawn on the toral surface represent knots in the three manifold $T \times I$ where I is the unit interval and T is the torus. If S_g is a surface of genus g, then the knot theory in $S_g \times I$ is represented by diagrams drawn on S_g taken up to the usual Reidemeister moves transferred to diagrams on this surface.

As we shall see in the next section, abstract invariants of virtual knots can be interpreted as invariants for knots that are specifically embedded in $S_g \times I$ for some genus g. The virtual knot theory does not demand the use of a particular surface embedding, but it does apply to such embeddings. This constitutes one of the motivations.

In fact, *we can formulate the theory of virtual knots in terms of knots embedded in thickened surfaces.* To do this we take a standard interpretation of each virtual crossing by regarding it as a representative of one arc passing through a handle that has been attached to the two dimensional sphere on which the original diagram is drawn. See Figure 6. Two representations are said to be equivalent if one can be obtained from the other by Reidemeister moves on the surface combined with the addition and subtraction of *empty 1-handles* from the surface. (Each handle is restricted to have no more than one arc passing through it.) It is easy to see that the addition and subtraction of the empty handles allows exactly the detour properties of the diagrams that we have emphasized. Thus virtual knots are knots in thickened surfaces up to stabilization by empty handles. The author is indebted to Dror Bar Natan for pointing out this formulation in terms of empty handle stabilization. See also [13].

3.2 Gauss Codes

A second motivation comes from the use of so-called *Gauss codes* to represent knots and links. The Gauss code is a sequence of labels for the crossings with each label repeated twice to indicate a walk along the diagram from a given starting point and returning to that point. In the case of multiple link

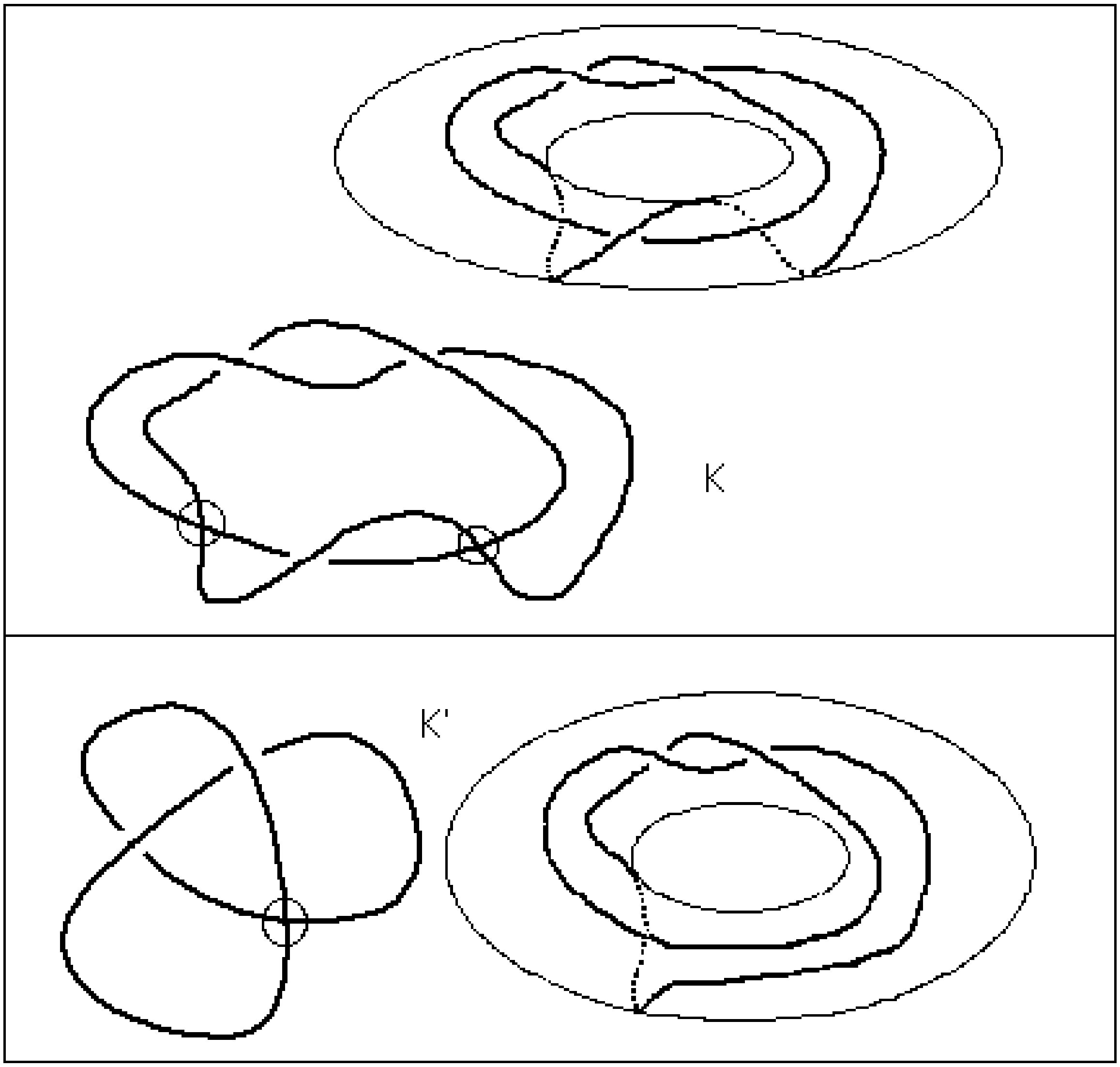

Figure 5 — Two Virtual Knots

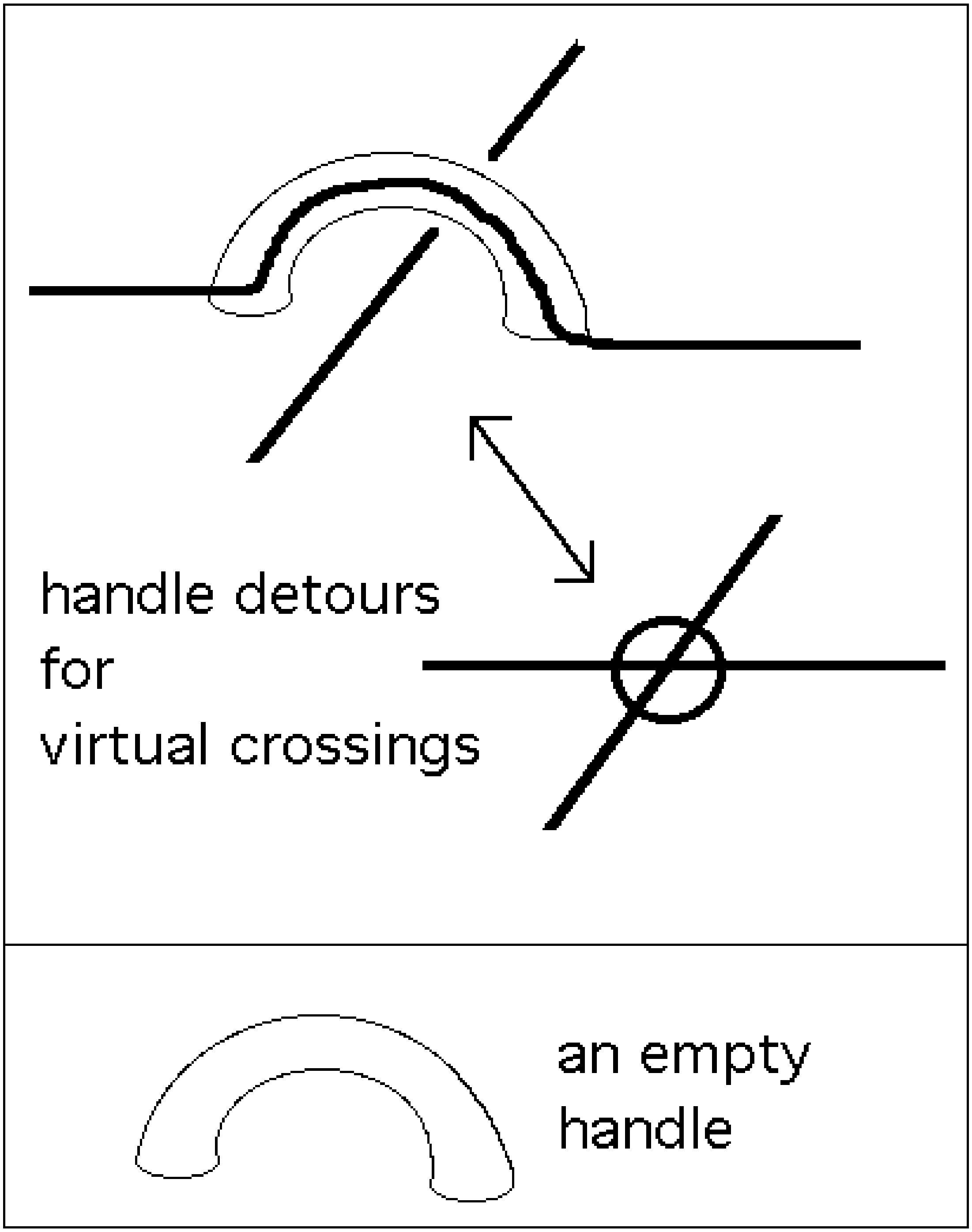

Figure 6 — Handle Detours

components we mean a sequence labels, each repeated twice and intersticed by partition symbols "/" to indicate the component circuits for the code.

A *shadow* is the projection of a knot or link on the plane with transverse self-crossings and no information about whether the crossings are overcrossings or undercrossings. In other words, a shadow is a 4-regular plane graph. On such a graph we can count circuits that always cross (i.e., they never use two adjacent edges in succession at a given vertex) at each crossing that they touch. Such circuits will be called the *components* of the shadow since they correspond to the components of a link that projects to the shadow.

A single component shadow has a Gauss code that consists in a sequence of crossing labels, each repeated twice. Thus the trefoil shadow has code 123123. A multi-component shadow has as many sequences as there are components. For example 12/12 is the code for the Hopf link shadow.

Along with the labels for the crossings one can add the symbols O and U to indicate that the passage through the crossing was an overcrossing (O) or an undercrossing (U). Thus

$$123123$$

is a Gauss code for the shadow of a trefoil knot and

$$O1U2O3U1O2U3$$

is a Gauss code for the trefoil knot.

The Hopf link itself has the code $O1U2/U1O2$. See Figure 7.

Suppose that g is such a sequence of labels and that g is free of any partition labels. Every label in g appears twice. The first necessary criterion for the planarity of g is given by the following definition and Lemma.

Definition. A single component Gauss code g is said to be *evenly intersticed* if there is an *even* number of labels in between the two appearances of any label.

Lemma 1. If g is a single component planar Gauss code, then g is evenly intersticed.

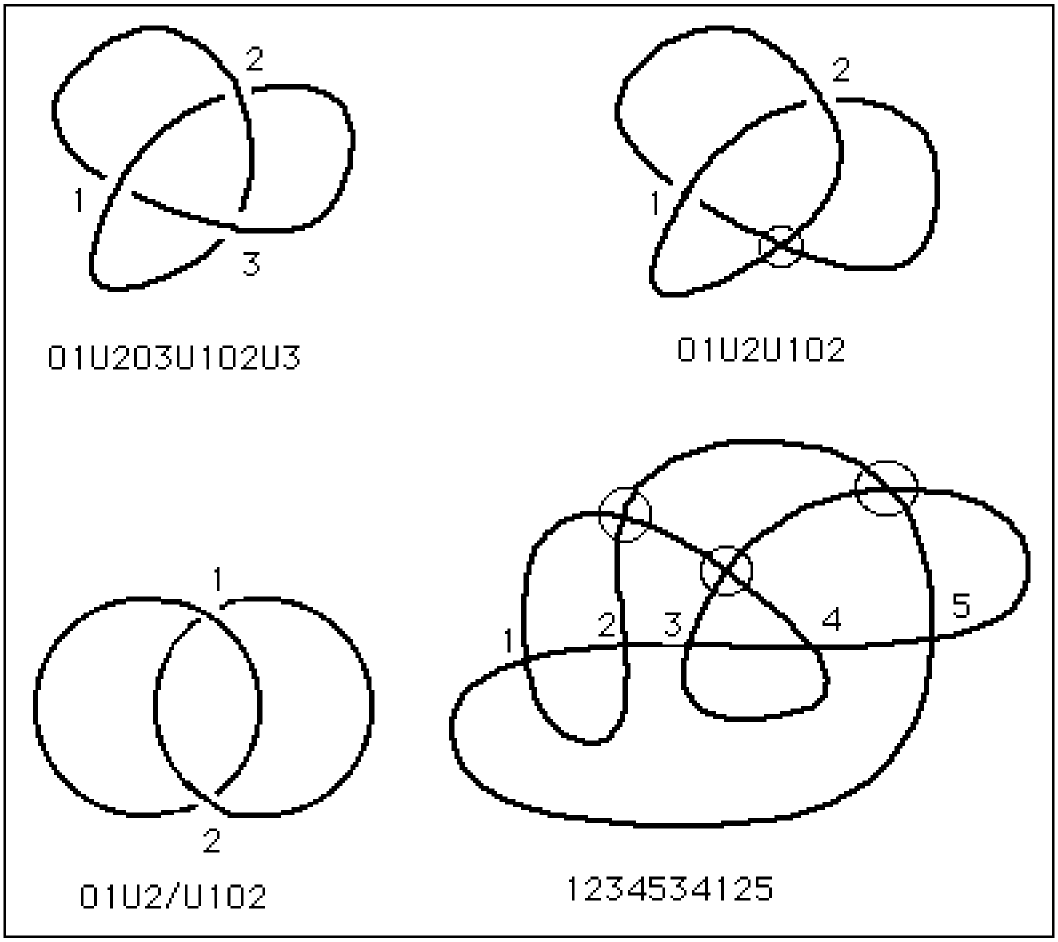

Figure 7 — Planar and Nonplanar Codes

Proof. This follows directly from the Jordan curve theorem in the plane.//

Example. The necessary condition for planarity in this Lemma is not sufficient. The code $g = 1234534125$ is evenly intersticed but not planar as is evident from Figure 7.

Non-planar Gauss codes give rise to an infinite collection of virtual knots.

Local orientations at the crossings give rise to another phenomenon: virtual knots whose Gauss codes have planar realisations with different local orientations from their classical counterparts.

By orienting the knot, one can give orientation signs to each crossing relative to the starting point of the code—using the convention shown in Figure 8. This convention designates each oriented crossing with a *sign* of $+1$ or -1. We say that the crossing has positive sign if the overcrossing line can be turned through the smaller angle (of the two vertical angles at the crossing) to coincide with the direction of the undercrossing line. The signed code for the standard trefoil is

$$t = O1 + U2 + O3 + U1 + O2 + U3+,$$

while the signed code for a figure eight knot is

$$f = O1 + U2 + O3 - U4 - O2 + U1 + O4 - U3 - .$$

Here we have appended the signs to the corresponding labels in the code. Thus, crossing number 1 is positive in the figure eight knot, while crossing number 4 is negative. See Figure 8 for an illustration corresponding to these codes.

Now consider the effect of changing these signs. For example let

$$g = O1 + U2 + O3 - U1 + O2 + U3 - .$$

Then g is a signed Gauss code and as Figure 8 illustrates, the corresponding diagram is forced to have virtual crossings in order to accommodate the change in orientation. The codes t and g have the same underlying (unsigned)

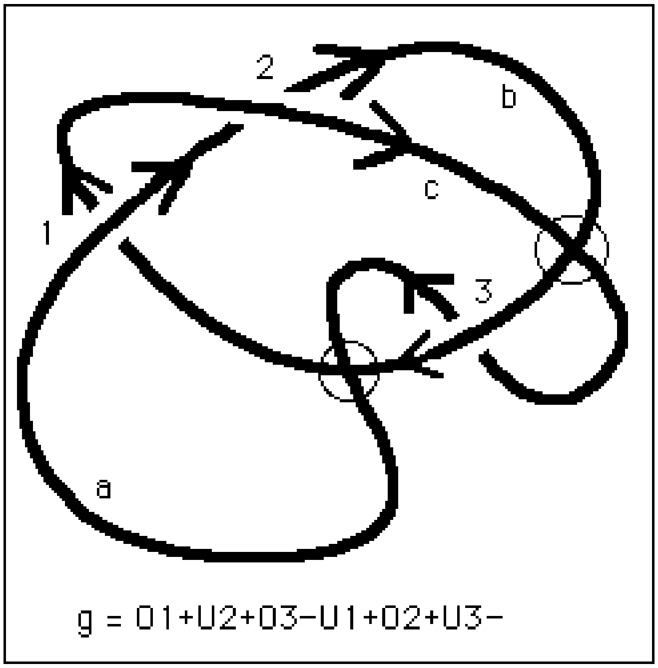

Figure 8 — Signed Gauss Codes

Gauss code $O1U2O3U1O2U3$, but g corresponds to a virtual knot while t represents the classical trefoil.

Definition of Virtuality. Carrying this approach further, we *define* a virtual knot as an equivalence class of oriented Gauss codes under abstractly defined Reidemeister moves for these codes—with no mention of virtual crossings. (We omit here the enumeration of the abstract moves on the Gauss codes. They are easily obtained by translation from the diagrams of the oriented Reidemeister moves.) The virtual crossings become artifacts of a planar representation of the virtual knot. The move sets of type B and C for virtuals are diagrammatic rules that make sure that this representation of the oriented Gauss codes is faithful. Note, in particular, that the move of type C does not alter the Gauss code. With this point of view we see that the signed codes are knot theoretic analogues of the set of all graphs, and that the classical knot (diagrams) are the analogues of the planar graphs. This is the fundamental combinatorial motivation for our definitions of virtual knots and their equivalences.

4 Fundamental Group, Crystals, Racks and Quandles

The fundamental group of the complement of a classical knot can be described by generators and relations, with one generator for each arc in the diagram and one relation for each crossing. The relation at a crossing depends upon the type of the crossing and is either of the form $c = b^{-1}ab$ or $c = bab^{-1}$ as shown in Figure 9.

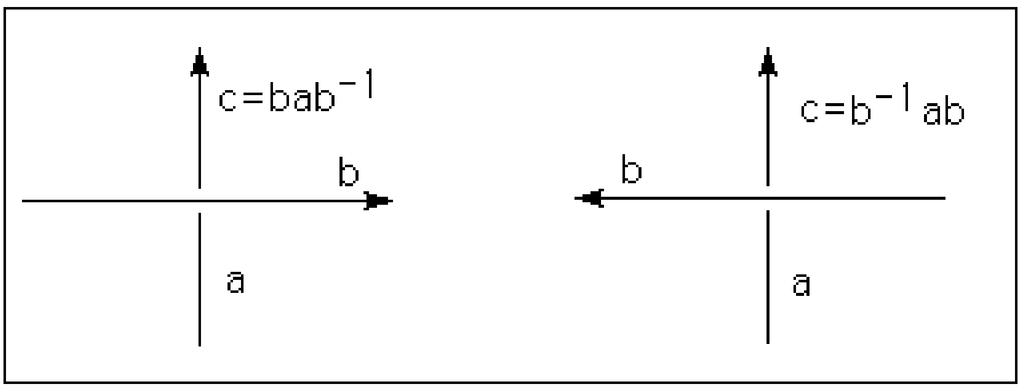

Figure 9 — Generators and Relations for the Fundamental Group

We define the group $G(K)$ of an oriented virtual knot or link by this same scheme of generators and relations. An *arc* of a virtual diagram proceeds from one classical undercrossing to another (possibly the same) classical undercrossing. Thus no new generators or relations are added at a virtual crossing. It is easy to see that $G(K)$ is invariant under all the moves for virtuals and hence is an invariant of virtual knots.

There are virtual knots that are non-trivial but have trivial fundamental group. (We say that the fundamental group of a knot is trivial if it is isomorphic to the infinite cyclic group.) The virtual K' in Figure 5 is such an example.. We shall show that K' is a non-trivial virtual in the next section by using a generalisation of the bracket polynomial.

A generalization of the fundamental group called the quandle, rack or crystal (depending on notations and history) also assigns relations (in a different algebra) to each crossing. The quandle generalises to the virtual category. We first discuss the involutory quandle, $IQ(K)$, for a (virtual) knot or link K. The $IQ(K)$ does not depend upon the local orientations of the diagram and it assigns to each crossing the relation $c = a * b$ as in Figure 10.

The operation $a*b$ is a non-associative binary operation on the underlying set of the quandle, and it satisfies the following axioms:

1. $a * a = a$ for all a.
2. $(a * b) * b = a$ for all a and b.
3. $(a * b) * c = (a * c) * (b * c)$ for all a, b, c.

The algebra under these axioms with generators and relations as defined above is called the involutory quandle, $IQ(K)$. It is easy to see that the $IQ(K)$ is well-defined for K virtual.

An important special case of $IQ(K)$ is the operation $a*b = 2b-a$ where a and b are elements of a cyclic group Z/nZ for some modulus n. In the case of a classical knot K, there is a natural choice of modulus $D(K) = Det(M(K))$ where $M(K)$ is a minor of the matrix of relations associated with the set of equations $c = 2b - a$. This is called the determinant of the knot, in the classical case. More generally, one can define the determinant for any finite group presentation whose abelianization is the integers (as is the case with virtual knot groups). In our case one takes the greatest common divisor of the set of $(n-1) \times (n-1)$ minors of the $n \times n$ coloring relation matrix associated with the knot. In this way we can define the determinant $|D(K)|$ of any virtual knot. If K is virtual then $|D(K)|$ is an invariant of K. The virtual knot labelled K in Figure 5 has determinant equal to 3. The non-triviality of the determinant shows that this knot is knotted and in fact that it has non-trivial fundamental group. For more information on virtual knot groups see [23].

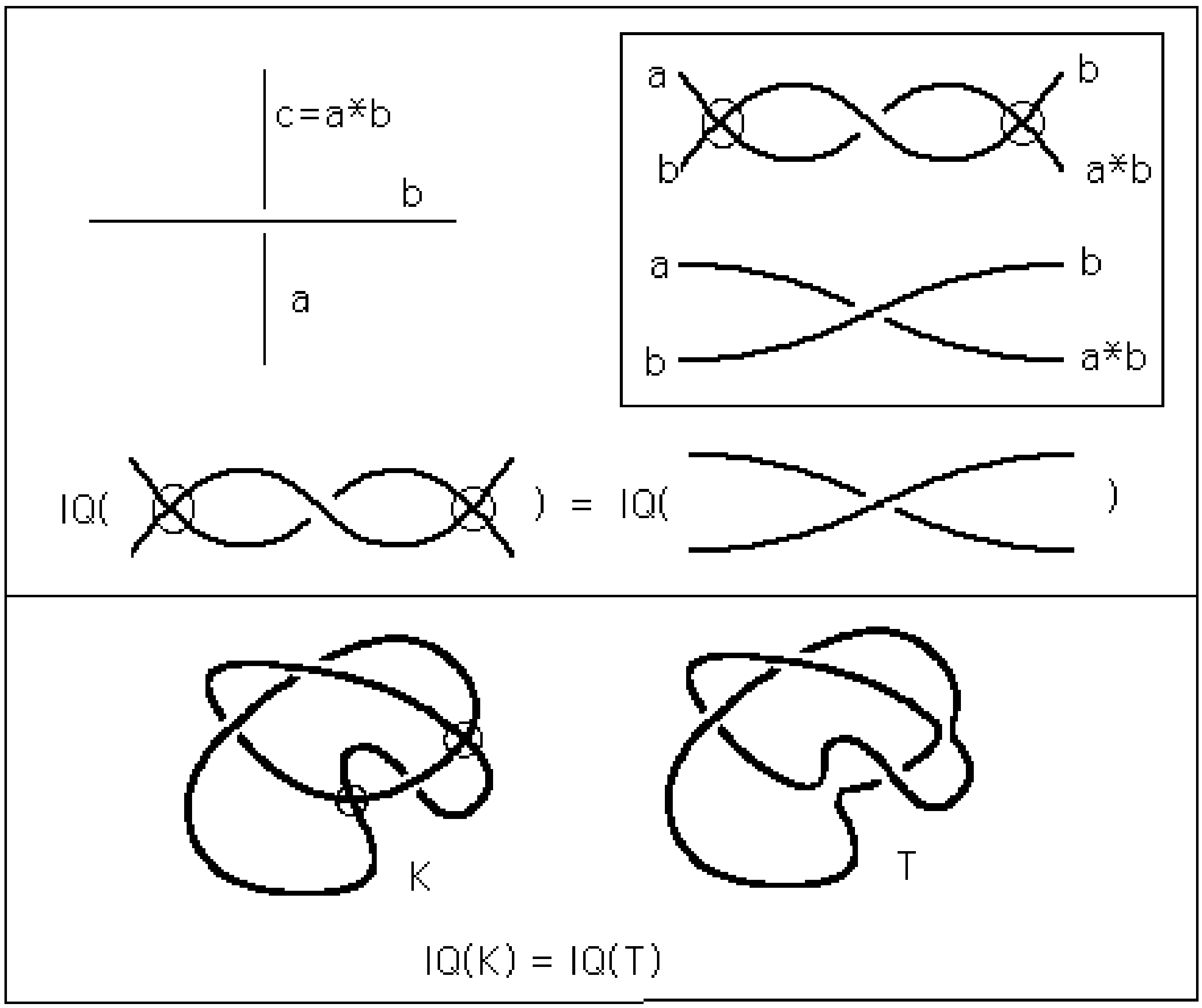

Figure 10 — The Involutory Quandle

Another example of an involutory quandle is the operation $a * b = ba^{-1}b$. In classical knot theory this yields the fundamental group of the two-fold branched covering along the knot.

Here is a useful lemma about the IQ for virtuals.

Lemma 5.

$$IQ(K_{vxv}) = IQ(K_{\overline{x}})$$

where x denotes a crossing in the diagram K, vxv denotes that x is flanked by virtuals, and $K_{\overline{x}}$ denotes the diagram obtained by smoothing the flanking virtuals, and switching the intermediate crossing.

In other words the IQ for a classical crossing flanked by two virtual crossings is the same as the IQ of the diagram where the two virtual crossings are smoothed and the classical crossing is switched.

Proof. See Figure 10. //

Remark. In Figure 10 we illustrate that $IQ(K) = IQ(T)$ where K is the virtual knot also shown in Figure 8 and T is the trefoil knot.

Finally we discuss the full quandle of a knot and its generalization to virtuals. For this discussion the exponential notation of Fenn and Rourke [11] is convenient. Instead of $a * b$ we write a^b and assume that there is an operation of order two

$$a \longrightarrow \overline{a}$$

so that

$$\overline{\overline{a}} = a,$$

and for all a and b

$$\overline{a^b} = \overline{a}^b.$$

This operation is well-defined for all a in the underlying set Q of the quandle.

By definition

$$a^{bc} = (a^b)^c$$

for all a,b and c in Q.
The operation of exponentiation satisfies the axioms

1. $a^a = a$
2. $a^{b\bar{b}} = a$
3. $a^{(b^c)} = a^{\bar{c}bc}$

It follows that the set of the quandle acts on itself by automorphisms

$$x \longrightarrow x^a.$$

This group of automorphisms is a representation of the fundamental group of the knot. Note that if we define a^b by the formula

$$a^b = bab^{-1}$$

and

$$\bar{b} = b^{-1},$$

then we get the fundamental group itself as an example of a quandle. The *rack* [11] or *crystal* [15] is obtained by eliminating the first axiom. This makes the rack/crystal an invariant of framed knots and links. The three axioms correspond to invariance under the three Reidemeister moves.

If we now compare Lemma 5 with its possible counterpart for the full fundamental group or the quandle, we see that it no longer holds. Figure 11 shows the new relations in the quandle that are obtained after smoothing the two virtual crossings and switching the classical crossing. While the quandle of the simplified diagram is no longer isomorphic to the original quandle, the fact that we can articulate the change is often useful in computations.

Example. Consider the virtual knot K of Figure 8. We have seen that K has the same IQ as the trefoil knot. However, the quandle and fundamental group of K are distinct from those of the trefoil knot, and K is not equivalent

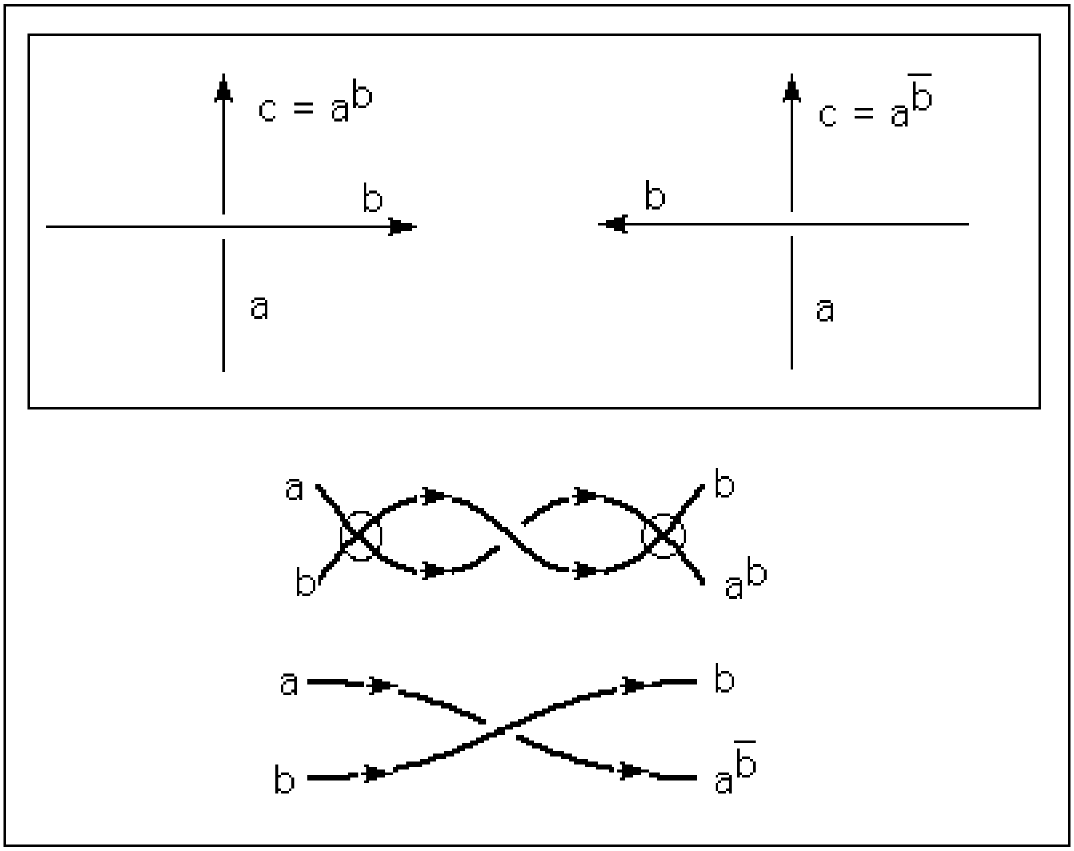

Figure 11 — Change of Relations for the Full Quandle

to any classical knot. To see this consider the *Alexander quandle* [15] defined by the equations

$$a^b = ta + (1-t)b$$

and

$$a^{\bar{b}} = t^{-1}a + (1-t^{-1})b.$$

This quandle describes a module (the Alexander module) M over $Z[t,t^{-1}]$. In the case of the virtual knot K in Figure 8, we have the generating quandle relations $a^c = b$, $b^a = c$, $c^{\bar{b}} = a$. This results in the Alexander module relations $b = ta + (1-t)c$, $c = tb + (1-t)a$, $a = t^{-1}c + (1-t^{-1})b$. From this it is easy to calculate that the module $M(K) = \{0, m, 2m\}$ for a non-zero element m with $3m = 0$ and $tm = 2m$. Thus the Alexander module for K is cyclic of order three. Since no classical knot has a finite cyclic Alexander module, this proves that K is not isotopic through virtuals to a classical knot.

It should be remarked that the Alexander polynomial of a knot can be regarded as a generalization of the determinant of a knot. The colors are taken to be elements of the Alexander moddule (See [15] and [22]). Just as with the determinant of a knot, the Alexander polynomial is defined to be a greatest common divisor the the set of $(n-1)\times(n-1)$ minors of the $n \times n$ relation matrix for the Alexander module. Since, as we shall see below, there are actually *two* fundamental quandles associated with a virtual knot (The second,in our language, is the quandle of the mirror image of the original knot.), there are two Alexander polynomials associated with a virtual. There are many questions related to the Alexander polynomials of virtual knots. We shall treat these matters elsewhere.

Finally, it should be remarked that the full quandle $Q(K)$ classifies a classical prime unoriented knot K up to mirror images. By keeping track of a *longitude* for the knot, one gets a complete classification. In the context of the quandle, the longitude can be described as the automorphism

$$\lambda : Q(K) \longrightarrow Q(K)$$

defined by the formula

$$\lambda(x) = x^{a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_k^{\epsilon_k}}$$

where $\{a_1, a_2, ..., a_k\}$ is an ordered list of quandle generators encountered (as one crosses underneath) as overcrossing arcs as one takes a trip around the diagram. The ϵ denotes whether the generator is encountered with positive or negative orientation, and x^ϵ denotes x if $\epsilon = 1$ and $\overline{x}$ if $\epsilon = -1$. For a given diagram the longitude is well-defined up to cyclic re-ordering of this list of encounters. Exactly the same definition applies to virtual knots. It is no longer true that the quandle plus longitude classifies a virtual knot, as our examples of knotted virtuals with trivial fundamental group show.

On the other hand, we can use the quandle to prove the following result. This proof is due to Goussarov, Polyak and Viro [29].

Theorem 6. If K and K' are classical knot diagrams such that K and K' are equivalent under extended virtual Reidemeister moves, then K and K' are equivalent under classical Reidemeister moves.

Proof. Note that longitudes are preserved under virtual moves (adding virtual crossings to the diagram does not change the expression for a longitude). Thus an isomorphism from $Q(K)$ to $Q(K')$ induced by extended moves preserves longitudes. Since the isomorphism class of the quandle plus longitudes classifies classical knots, we conclude that K and K' are classically equivalent. This completes the proof. //

Remark. We would like to see a purely combinatorial proof of Theorem 6.

5 Bracket Polynomial and Jones Polynomial

The bracket polynomial [27] extends to virtual knots and links by relying on the usual formula for the state sum of the bracket, but allowing the closed loops in the state to have virtual intersections. Each loop is still valued at $d = -A^2 - A^{-2}$ and the expansion formula

$$< K >= A < K_a > + A^{-1} < K_b >$$

still holds where K_a and K_b denote the result of replacing a single crossing in K by smoothings of type a and type b as illustrated in Figure 12.

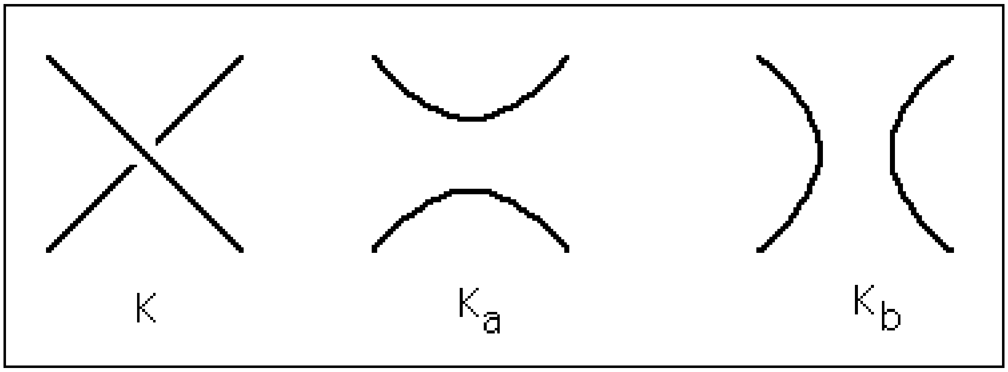

Figure 12 — Bracket Smoothings

We must check that this version of the bracket polynomial is invariant under all but the first Reidemeister move (See the moves shown in Figure 2). Certainly the usual arguments apply to the moves of type (A). Moves of type (B) do not disturb the loop counts and so leave bracket invariant. Finally the move of type (C) receives the verification illustrated in Figure 13. This completes the proof of the invariance of the generalised bracket polynomial under move (C).

We define the writhe $w(K)$ for an oriented virtual to be the sum of the crossing signs—just as in the classical case.

The f-polynomial is defined by the formula

$$f_K(A) = (-A^3)^{-w(K)} < K > (A).$$

The Laurent polynomial, $f_K(A)$ is invariant under all the virtual moves including the classical move of type I.

Remark. It is worth noting that f_K can be given a state summation of its

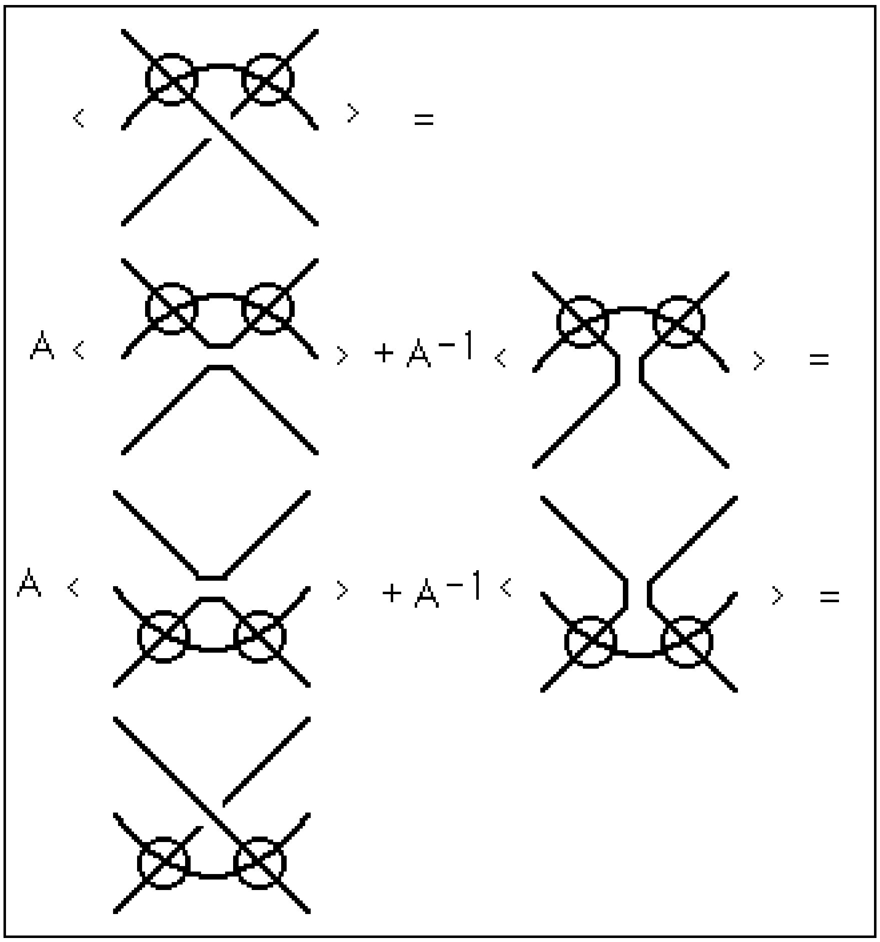

Figure 13 — Type (C) Invariance of the Bracket

own. Here we modify the vertex weights of the bracket state sum to include a factor of $-A^{-3}$ for each crossing of positive sign, and a factor of A^{+3} for each factor of negative sign. It is then easy to see that

$$f_{K_+} = -A^{-2} f_{K_0} - A^{-4} f_{K_\infty}$$
$$f_{K_-} = -A^{+2} f_{K_0} - A^{+4} f_{K_\infty}$$

where K_+ denotes K with a selected positive crossing, K_- denotes the result of switching only this crossing, K_0 denotes the result of making the oriented smoothing of this crossing, and K_∞ denotes the result of making an unoriented smoothing at this crossing. The states in this oriented state sum acquire sites with unoriented smoothings, but the procedure for evaluation is the same as before. For each state we take the product of the vertex weights multiplied by $d^{||S||-1}$ where $d = -A^2 - A^{-2}$ and $||S||$ denotes the number of loops in the state. Then f_K is the sum of these products, one for each state.

The following Lemma makes virtual calculations easier.

Lemma 7. $< K_{vxv} >=< K_x >$ where x denotes a crossing in the diagram K, vxv denotes that x is flanked by virtuals, and K_x denotes the diagram obtained by smoothing the flanking virtuals, and leaving the crossing the same.

Proof. The proof is shown in Figure 14.//

Note that this result has a different form from our corresponding Lemma about the involutory quandle $IQ(K)$. As a result we get an example of a virtual knot that is non-trivial (via the IQ) but has $f_K = 1$. *Hence we have a virtual knot K with Jones polynomial equal to 1.* The example is shown in Figure 15. Note that in Figure 10 we illustrated that this K has the same involutory quandle as the trefoil knot. We will see in Section 6 that K is not equivalent to a classical knot.

We now compute the bracket polynomial for our previous example with trivial fundamental group and we find that $< K' >= A^2 + 1 - A^{-4}$ and $f_{K'} = (-A^3)^{-2} < K' >= A^{-4} + A^{-6} - A^{-10}$. Thus K' has a non-trivial Jones polynomial. See Figure 16.

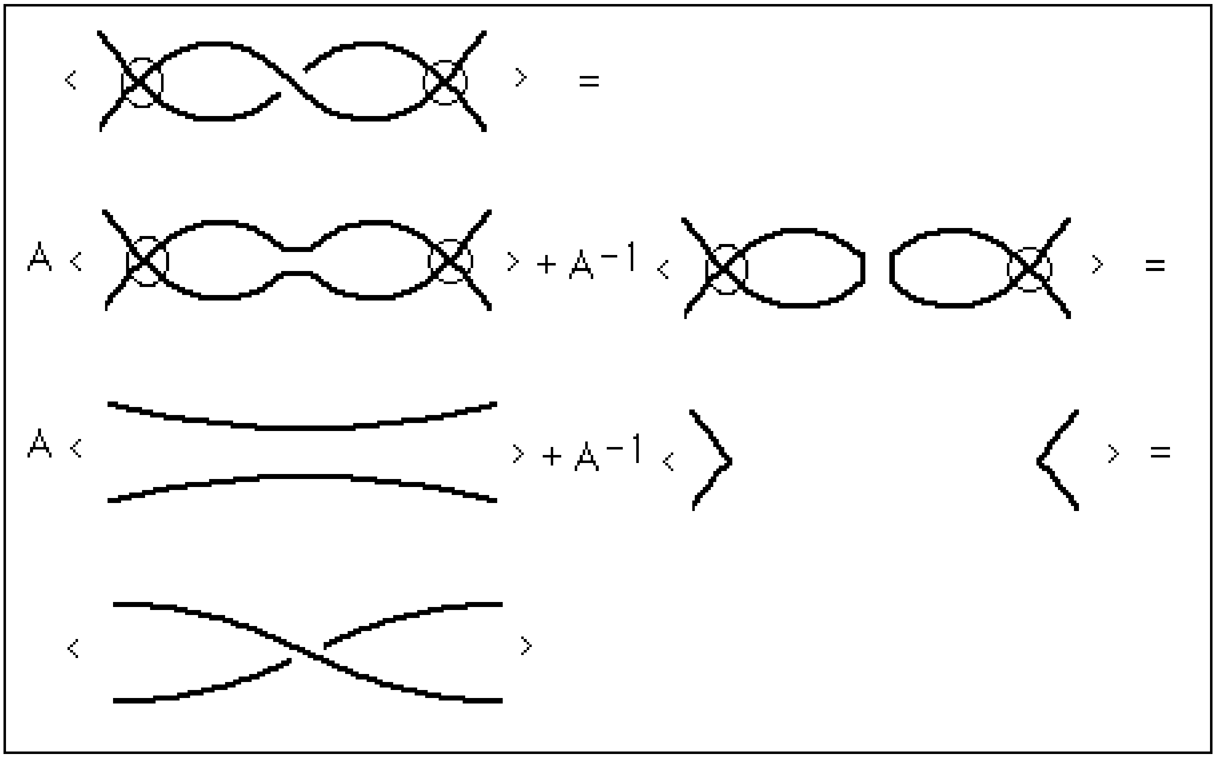

Figure 14 — Removal of Flanking Virtual Crossings

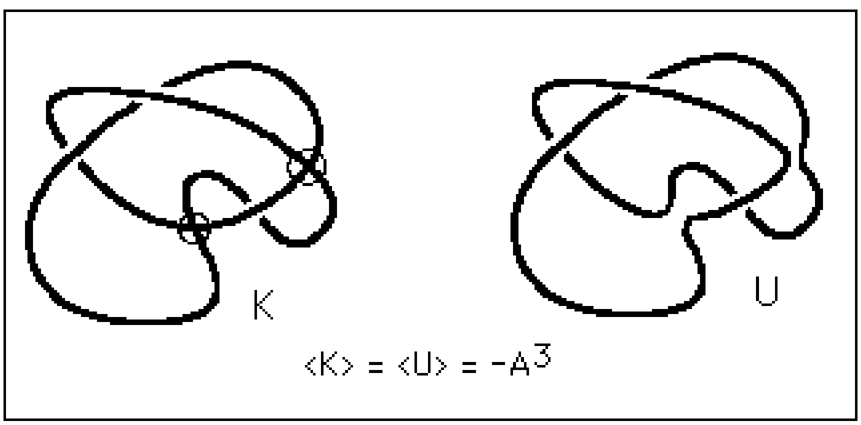

Figure 15 — A Knotted Virtual with Trivial Jones Polynomial

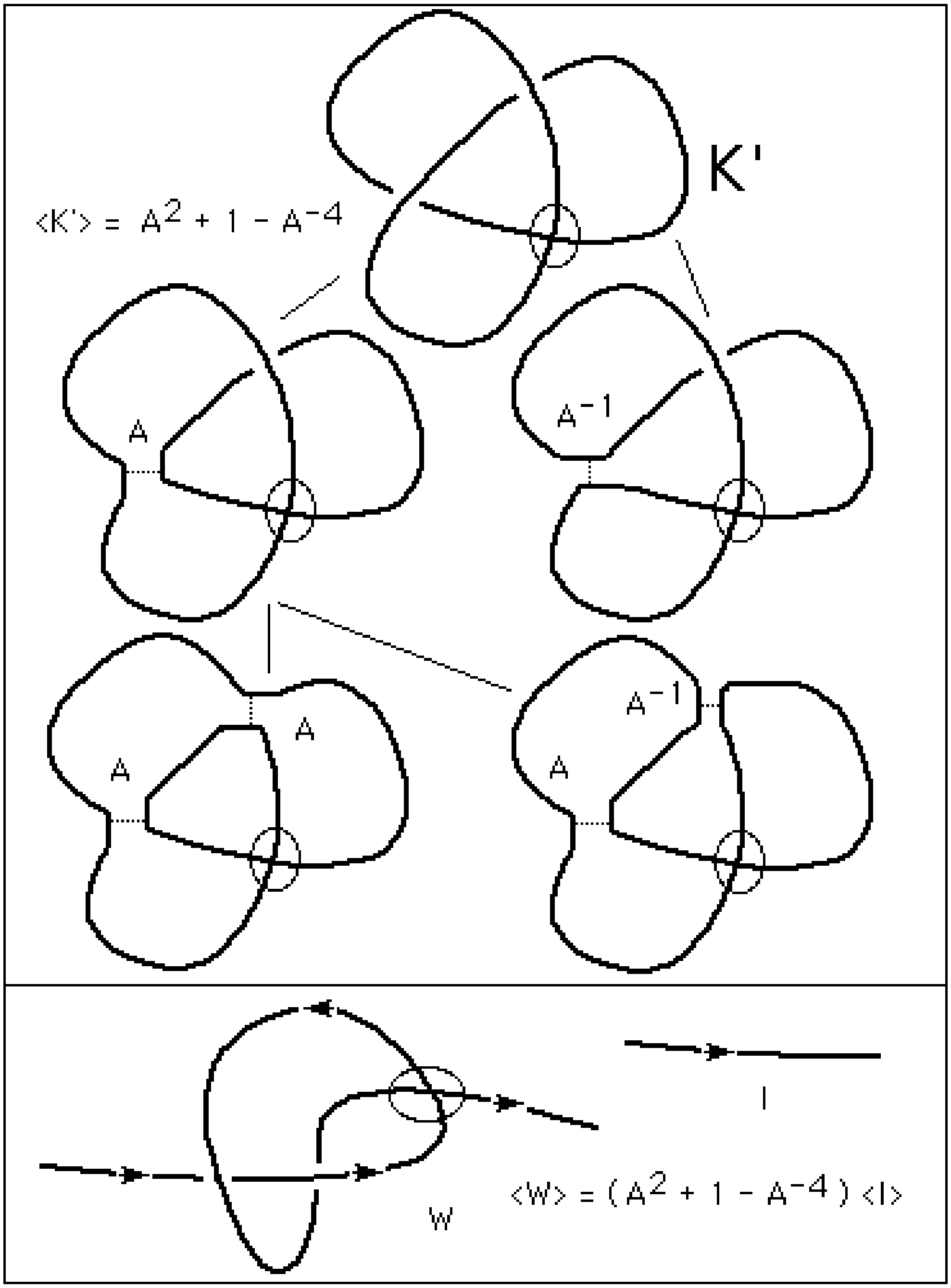

Figure 16 — Calculation of $< K' >$

In Figure 16 we also indicate the result of placing a tangle W into another knot or link. Since this is the same as taking a connected sum with K' it has the effect of multiplying the bracket polynomial by $A^2 + 1 - A^{-4}$. Thus if L is any knot or link and $K' + L$ denotes the connected sum of K' along some component of L then $< K' + L >= (A^2 + 1 - A^{-4}) < L >$ while $Q(K' + L) = Q(L)$ (as we verified in the last section). Thus for any knot L, successive connected sums with K' produces an infinite family of distinct virtual knots, all having the same quandle (hence same fundamental group).

Finally, we note that if the knot is given as embedded in $S_g \times I$ for a surface of genus g, and if its virtual knot diagram K is obtained by projecting the diagram on S_g into the plane, then $< K >$ computes the value of the extension of the bracket to the knots in $S_g \times I$ where all the loops have the same value $d = -A^2 - A^{-2}$. This is the first order bracket for link diagrams on a surface.

In Figure 5 we illustrated the non-trivial knot K with trivial Jones polynomial as embedded in $S_1 \times I$. This knot in $S_1 \times I$ is actually not trivial as can be seen from the higher Jones polynomials that discriminate loops in different isotopy classes on the surface.

In Figure 17 is another example of a virtual knot E and a corresponding embedding in $S_1 \times I$. In this case, E is a trivial virtual knot (as is shown in Figure 4), but the embedding of E in $S_1 \times I$ is non-trivial (even though it has trivial fundamental group and trivial bracket polynomial). The non-triviality of this embedding is seen by simply observing that it carries a non-trivial first homology class in the thickened torus. In fact, if you expand the state sum for the bracket polynomial and keep track of the isotopy classes of the curves in the states, then the bracket calculation also shows this non-triviality by exhibiting as its value a single state with a non-contractible curve.

Virtual knot theory provides a convenient calculus for working with knots in $S_g \times I$. The virtuals carry many properties of knots in $S_g \times I$ that are independent of the choice of embedding and genus. This completes our quick survey of the properties of the bracket polynomial and Jones polynomial for virtual knots and links. Just as uncolorable graphs appear when one goes beyond the plane (for planar graph coloring problems) so knots of unit Jones

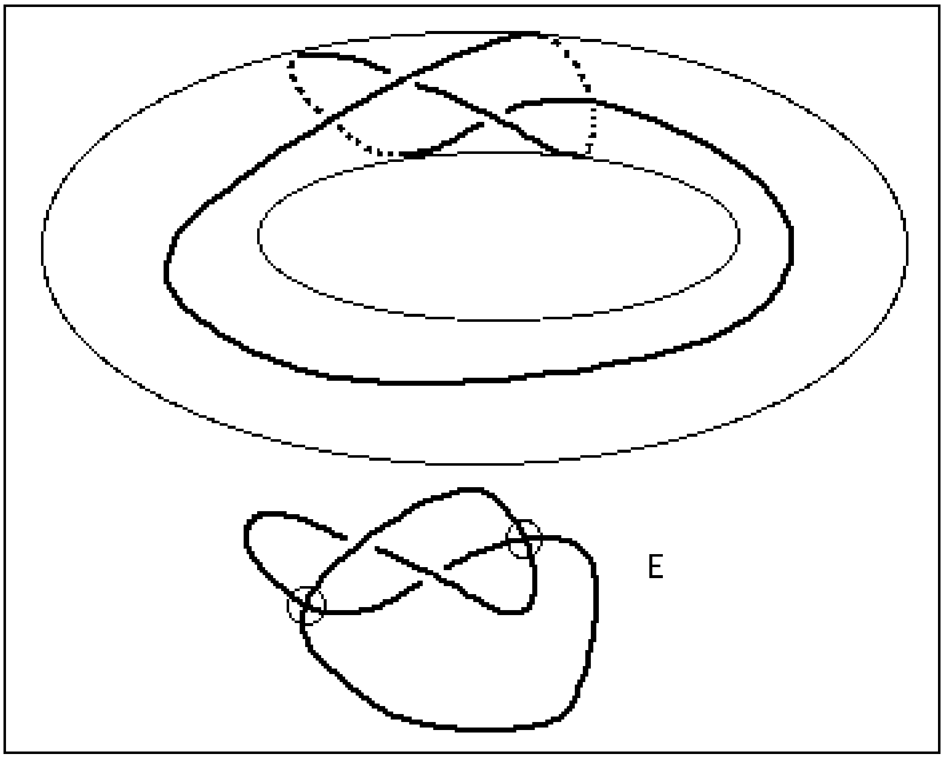

Figure 17 — A Knot in $S_1 \times I$ with Trivial Jones Polynomial

polynomial appear as we leave the diagrammatic plane into the realm of the Gauss codes.

6 Making Infinitely Many Virtuals with Unit Jones Polynomial

The purpose of this section is to give a generalization of the examples in the previous section that gave non-trivial virtual knots with unit Jones polynomial. Consider the following operation on a crossing in a knot or link diagram: Switch the crossing and place two virtual crossings on either side of it as shown in Figure 18 (Compare Figures 10 and 14) Call this operation on a crossing in a diagram K a *virtual switch* of the crossing.

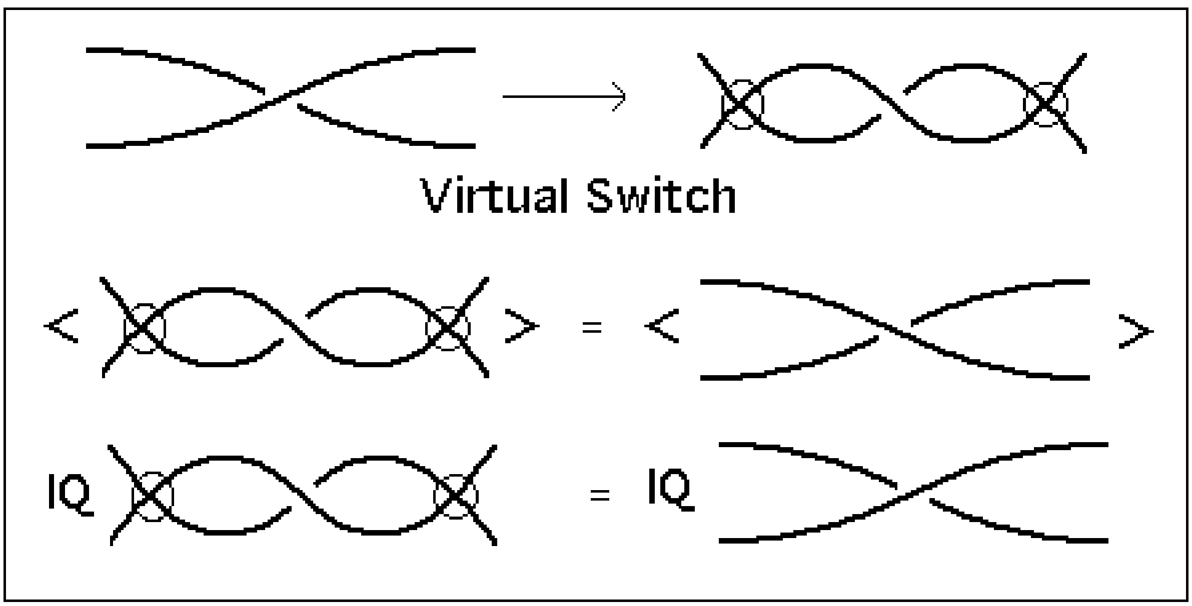

Figure 18 - Virtual Crossing Switch

Lemma. Let K be a (virtual) link diagram. Let K' be obtained from the diagram K by a single virtual switch of a crossing i in K. Then $IQ(K) = IQ(K')$ and $V_{K'} = V_{SK}$ where SK is the diagram obtained from K by switching the crossing i without and virtual replacement.

Proof. This Lemma follows directly from the results in the previous two sections where we saw that the bracket polynomial was unchanged by the addition or removal of flanking virtuals at a crossing, while the involutory quandle of a virtual diagram with virtual crossings flanking a real crossing is isomorphic to the diagram obtained by removing the virtuals and switching the crossing. //

Theorem. Let K be any classical non-trivial knot diagram. Let $S = \{i_1, ..., i_n\}$ be a subset of the crossings of K such that the knot K' obtained by switching these crossings is unknotted. Let K^v be the virtual knot diagram obtained from K by performing a virtual switch at each of the crossings in the set S. Then K^v is a non-trivial virtual knot with unit Jones polynomial.

Proof. By the Lemma we know that $IQ(K^v = IQ(K)$. Winker [32] has shown that K is a non-trivial classical knot if and only if $IQ(K)$ is non-trivial. Thus, we know that $IQ(K^v$ is non-trivial and hence that K^v is non-trivial. On the other hand, the Jones polynomial of K^v is equal, by the Lemma, to the Jones polynomial of the knot K' obtained from K by switching all the crossings in the set S. Since K' is unknotted, it follows that K^v has unit Jones polynomial.//

This Theorem shows that there are infinitely many non-trivial virtual knots with unit Jones polynomial. Are any of these knots (constructed as in the Theorem) equivalent to classical knots? If any one of the virtual knots K^v *is* equivalent to a classical knot, then this will be an example of a classical non-trivial knot with unit Jones polynomial. It may be that no knot K^v is equivalent to a classical knot. An answer to this question in either direction will be exceedingly interesting. See Figure 19 an example of a pair K, K^v where K is classical and knotted and K^v is virtual, knotted and with unit Jones polynomial.

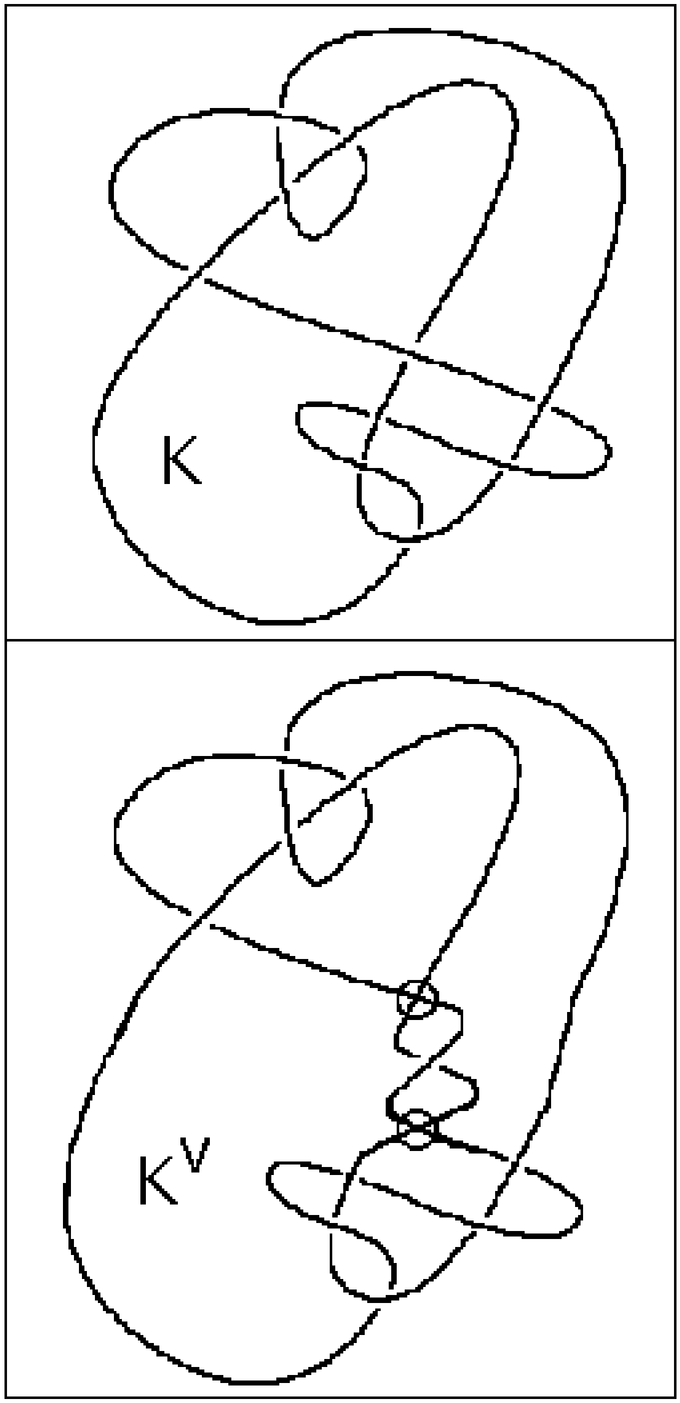

Figure 19 - A Diabolical Pair

7 Biquandles and the Silver-Williams Group

In [23] Dan Silver and Susan Williams define a group associated with a virtual knot or link that can detect virtual knottedness unseen by the fundamental group or by the rack or quandle. The Silver-Williams group is an example of a *biquandle*, a generalization of the quandle that involves both the upper and the lower arcs at a crossing as operators. We will begin this section with a discussion of the notion of a biquandle. We thank Roger Fenn for helpful conversations about the potential of the *birack* and biquandle for studying virtual knots and links. A joint paper on biracks and biquandles with Roger Fenn and Mercedes Jordan is in preparation [4].

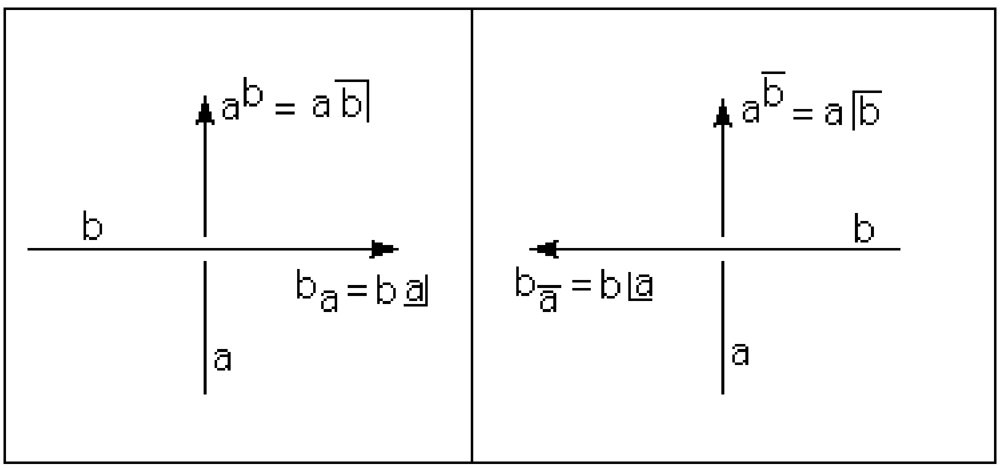

Figure 20 - Biquandle Relations at a Crossing

View Figure 20. In this Figure we have repeated the format for the operations in a quandle, but now the overcrossing arcs have two labels, one on each side of the crossing. In a biquandle there is an algebra label on each *edge* of the diagram. An edge of the diagram corresponds to an edge of the underlying plane graph of that diagram. Let the edges oriented toward a crossing be called the *input* edges for the crossing, and the edges oriented away from the crossing be called the *output* edges for the crossing. Let a and b be the input edges for a positive crossing, with a the label of the undercrossing input and b the label on the overcrossing input. Then in the biquandle, we label the undercrossing output by

$$c = a^b$$

just as in the case of the quandle, but the overcorossing output is labelled

$$d = b_a.$$

We usually read a^b as – the undercrossing line a is acted upon by the overcrossing line b to produce the output $c = a^b$. In the same way, we can read b_a as – the overcossing line b is operated on by the undercrossing line a to produce the output $d = b_a$. The biquandle labels for a negative crossing are similar but with an overline (denoting an operation of order two) placed on the letters just as we did in case of the quandle. Thus in the case of the negative crossing, we would write

$$c = a^{\overline{b}}$$

and

$$d = b_{\overline{a}}.$$

To form the biquandle, $BQ(K)$, we take one generator for each edge of the diagram and two relations at each crossing (as described above). This system of generators and relations is then regarded as encoding an algebra that is generated freely by the biquandle operations as concatenations of these symbols and subject to the biquandle algebra axioms. These axioms are a transcription in the biquandle language of the requirement that this algebra be invariant under Reidemeister moves on the diagram.

Another way to write this formalism for the biquandle is as follows

$$a^b = a\,\overline{\quad \mathrm{b}}|$$
$$a_b = a\,\underline{\quad \mathrm{b}}|$$
$$a^{\overline{b}} = a|\overline{\mathrm{b}\quad}$$
$$a_{\overline{b}} = a|\underline{\mathrm{b}\quad}.$$

We call this the *operator formalism* for the biquandle. The operator formalism has advantages when one is performing calculations, since it it possible to maintain the formulas on a line, rather than extending them up and down the page as in the exponential notation. On the other hand the exponential notation has intuitive familiarity and is good for displaying certain results. The axioms for the biquandle, are exactly the rules needed for invariance of this structure under the Reidemeister moves. Note that in analyzing invariance under Reidemeister moves, we visualize representative parts of link diagrams with biquandle labels on their edges. The primary labelling occurs at a crossing. At a positive crossing with over input b and under input a, the under output is labelled $a\overline{\ b|}$ and the over output is labelled $b\underline{\ a|}$. At a negative crossing with over input b and under input a, the under output is labelled $a\overline{|b\ }$ and the over output is labelled $b\underline{|a\ }$. At a virtual crossing there is no change in the labellings of the lines that cross one another. In Figure 21 we illustrate the effect of these conventions and how it leads to the following algebraic transcription of the directly oriented second Reidemeister move.

$$a = a\overline{\ b|}\overline{|\,b\underline{\ a|}\ }$$

$$b = b\underline{\ a|}\underline{|\,a\overline{\ b|}\ }.$$

The reverse oriented second Reidemeister move gives a different sort of identity, as shown in Figure 22. For the reverse oriented move, we must assert that given elements a and b in the biquandle, then there exists an element x such that

$$x = a\overline{\ b\underline{|x\ }\ |}$$

and that

$$a = x\overline{|b\ }$$

$$b = b\underline{|x\underline{\ a|}}.$$

See Figure 22. The assertion about the existence of x can be viewed as asserting the existence of a fixed point for a certain operator. In this case the operator is $F(x) = a\overline{\ b\underline{|x\ }\ |}$. It is characteristic of certain axioms in the biquandle that they demand the existence of such fixed points. Another

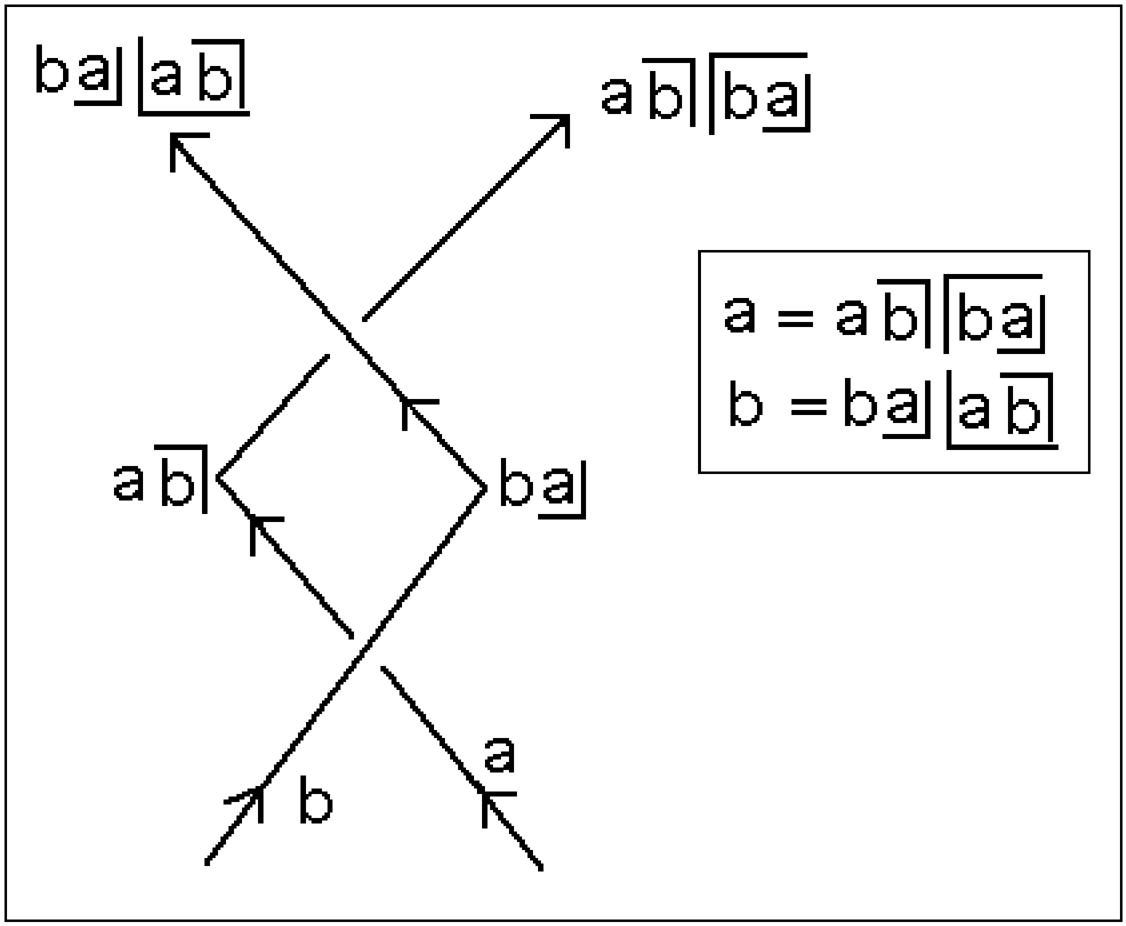

Figure 21 — Direct Two Move

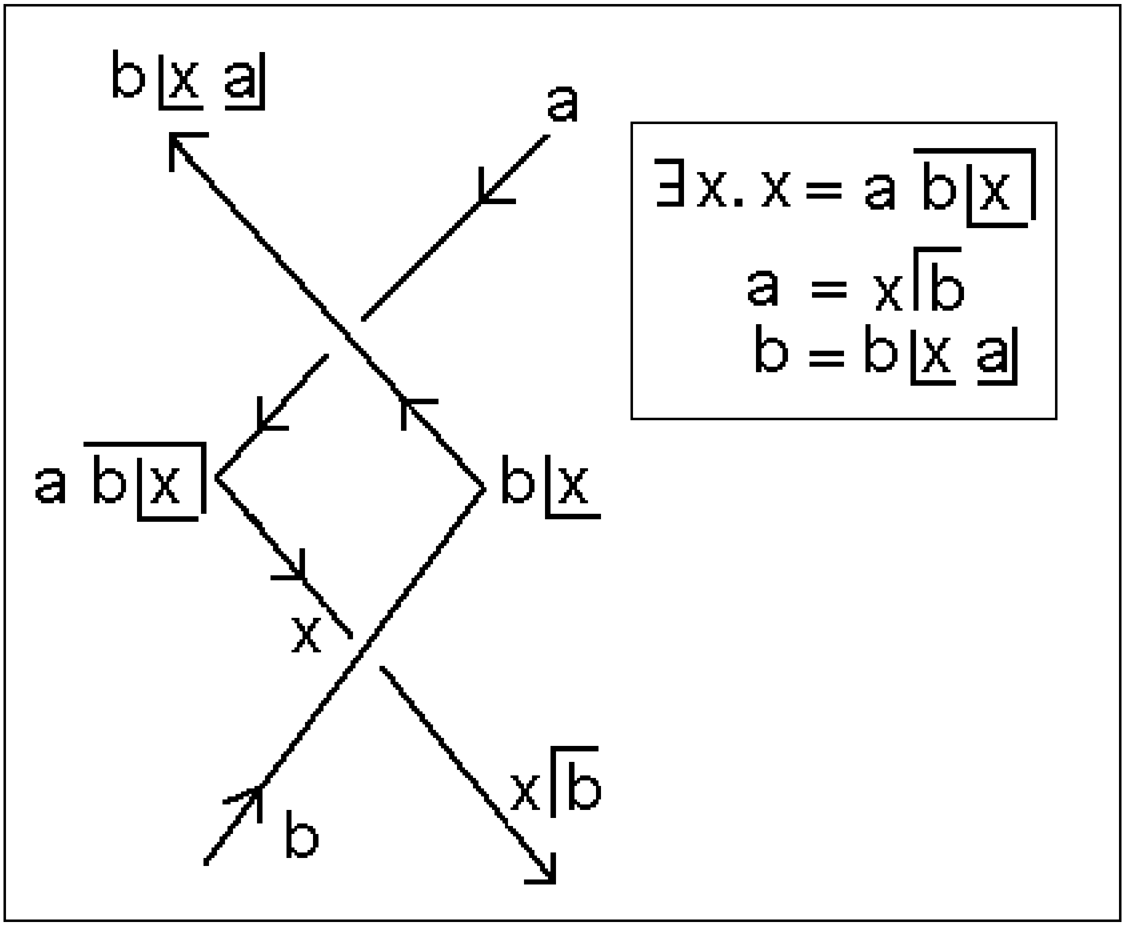

Figure 22 — Reverse Two Move

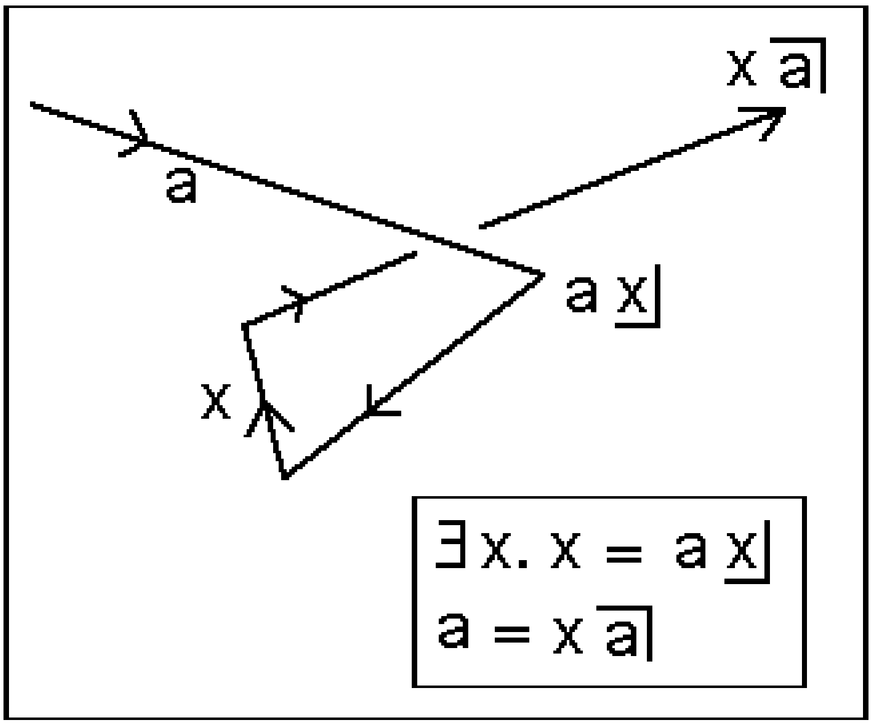

Figure 23 — First Move

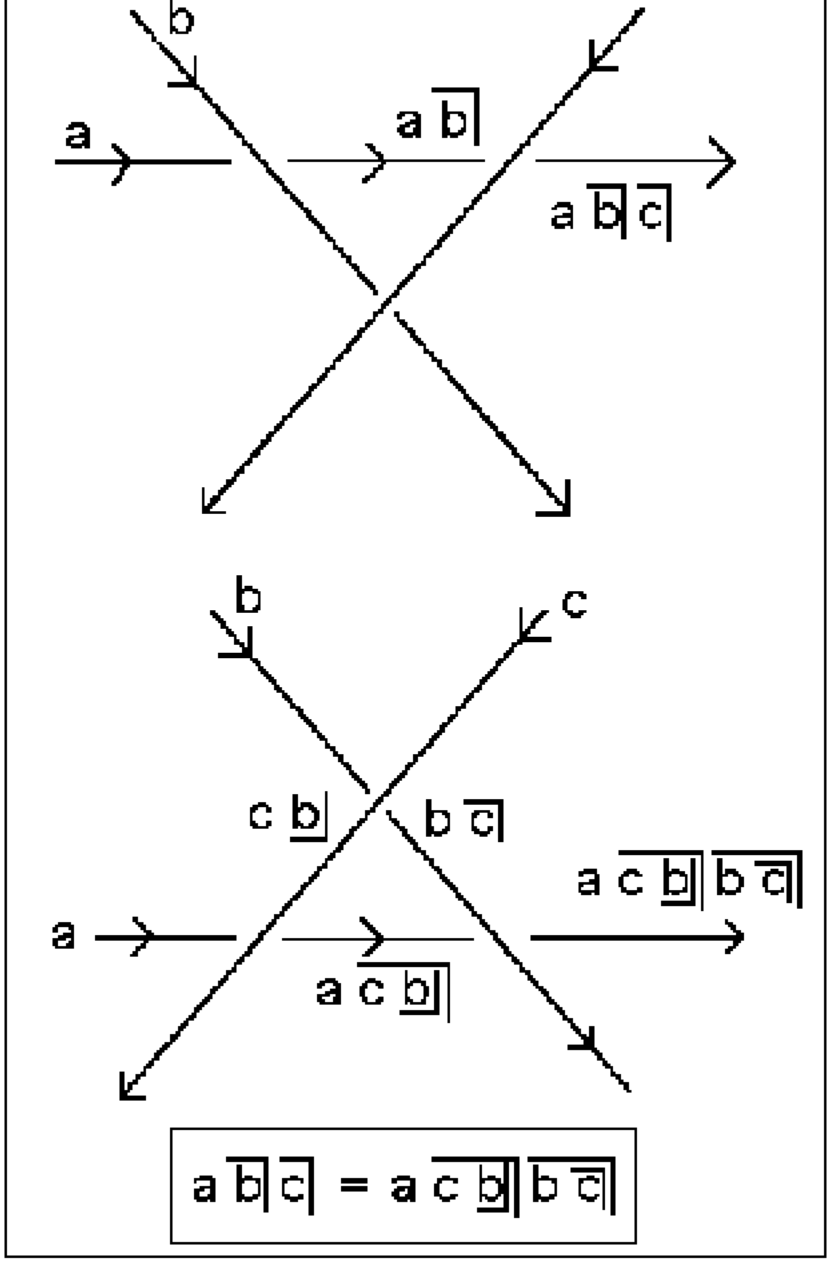

Figure 24 — Third Move

example is the axiom corresponding to the first Reidemeister move (one of them) as illustrated in Figure 23. This axiom states that given an element a in the biquandle, then there exists an x in the biquandle such that $x = a\,\underline{\mathrm{x}|}$ and that $x\,\overline{\mathrm{a}|} = a$. In this case the operator is $G(x) = a\,\underline{\mathrm{x}|}$. It is unusual that an algebra would have axioms asserting the existence of fixed points with respect to operations involving its own elements. We plan to take up the study of this aspect of biquandles in a separate publication. Here, we shall not write a complete list of biquandle axioms. However, another one of them (invariance under a third Reidemeister move, generalizing the distributive law in the quandle) is illustrated in Figure 24. The complete axioms will be published elsewhere. The version of the third Reidemeister move shown in this figure has algebraic form

$$a\,\overline{\mathrm{b}|}\,\overline{\mathrm{c}|} = a\,\overline{\mathrm{c}\,\underline{\mathrm{b}|}\,|}\,\overline{\mathrm{b}\,\overline{\mathrm{c}|}\,|}.$$

At the end of this section we discuss that nature of a general algebraic apparatus that can handle existence of fixed points required for the definition of the general biquandle. For now, we look at a specific biquandle structure.

In order to uncover a specific example of a biquandle structure, suppose that

$$a\,\overline{\mathrm{b}|} = ta + sb$$

$$a\,\underline{\mathrm{b}|} = va + ub$$

where a,b,c are members of a module M over a ring R and t,s,v and u are in R. We use invariance under the Reidemeister moves to determine relations among these coefficients.

As a first example, suppose that we want to implement the axiom for the first Reidemeister move, as explained above. Then, for any element a we require the existence of an element x such that

$$x = a\,\underline{\mathrm{x}|}$$

Here this means

$$x = va + ux$$

thus

$$(1-u)x = va.$$

We see that we can solve for the fixed point via

$$x = va/(1-u)$$

when (1-u) is invertible in the base ring for the module. A formal solution is

$$x = va(1 + u + u^2 + u^3 + ...)$$

and this formal power series corresponds to the use of infinitary words in the algebra. This example illustrates the issues related to fixed points, but in fact in this case we find that a more persipciuous analysis of the whole situation occurs by starting with an analysis of invariance under the third Reidemeister move.

Taking the equation for the third Reidemeister move discussed above, we have

$$a\,\overline{\mathrm{b}|}\,\overline{\mathrm{c}|} = t(ta+sb)+sc = t^2a + tsb + sc$$

$$\overline{a\;\mathrm{c}\;\underline{\mathrm{b}|}|}\;\overline{\mathrm{b}\;\overline{\mathrm{c}|}|} = t(ta + s(vc+ub)) + s(tb+sc)$$

$$= t^2a + ts(u+1)b + s(tv+s)c.$$

From this we see that we have a solution to the equation for the third Reidemeister move if $u = 0$ and $s = 1 - tv$. Assuming that t and v are invertible, it is not hard to see that the following equations not only solve this single Reideimeister move, but they give a biquandle structure, satisfying all the moves.

$$a\,\overline{\mathrm{b}|} = ta + (1-tv)b$$

$$a\,\underline{\mathrm{b}|} = va$$

$$a|\overline{\mathrm{b}} = t^{-1}a + (1 - t^{-1}v^{-1})b$$

$$a|\underline{\mathrm{b}} = v^{-1}b.$$

Thus we have a simple generalization of the Alexander quandle and we shall refer to this structure, with the equations given above, as the *Alexander Biquandle.*

Just as one can define the Alexander Module of a classical knot, we have the Alexamder Biquandle of a virtual knot or link, obtained by taking one generator for each *edge* of the knot diagram and taking the relations in the above linear form. Let $ABQ(K)$ denote this module structure for an oriented link K.

For example, consider the virtual knot in Figure 15. This knot gives rise to a biquandle with generators a,b,c,d and relations

$$d = c\,\overline{\mathrm{a}|}$$

$$b = a\,\underline{\mathrm{c}|}$$

$$c = b\,\overline{\mathrm{d}|}$$

$$a = d\,\underline{\mathrm{b}|}.$$

writing these out in $ABQ(K)$, we have

$$d = tc + (1 - vt)a$$

$$b = va$$

$$c = tb + (1 - vt)d$$

$$a = vd.$$

eliminating b and d and rewriting, we find

$$v^{-1}a = tc + (1 - vt)a$$

$$c = tva + (1 - vt)v^{-1}a.$$

Eliminating c, we are left with a module generated by a with relation

$$[(t^{-1} - t) + (v^{-1} - v) + (vt - v^{-1}t^{-1})]a = 0.$$

The polynomial that annihilates a can be regarded as a two variable generalization of the Alexander polynomial. The non-triviality of the module $ABQ(K)$ shows that K is a non-trivial virtual knot. We had previously checked this fact using the Jones polynomial. The classical Alexander polynomial (via the fundamental group) is trivial since the fundamental group of K is isomorphic to the integers.

Here is another example of the use of this polynomial. Let D denote the diagram in Figure 25. It is not hard to see that this virtual knot has unit Jones polynomial, and that the fundamental group is isomorphic to the integers. The biquandle does detect the knottedness of D. The relations are

$$a\,\overline{\;\mathrm{d}\;}\rceil = b,\; d\,\underline{\;\mathrm{a}\;}\rfloor = e$$

$$c\,\overline{\;\mathrm{e}\;}\rceil = d,\; e\,\underline{\;\mathrm{c}\;}\rfloor = f$$

$$f\lceil\overline{\;\mathrm{b}\;} = a,\; b\lfloor\underline{\;\mathrm{f}\;} = c$$

from which we obtain the relations (eliminating c, e and f)

$$b = ta + (1 - tv)d$$

$$d = tv^{-1}b + (1 - tv)vd$$

$$a = t^{-1}v^2d + (1 - t^{-1}v^{-1})b.$$

The determinant of this system is the generalized Alexander polynomial for D:

$$t^2(v^2 - 1) + t(v^{-1} + 1 - v - v^2) + (v - v^2).$$

This proves that D is a non-trivial virtual knot.

In fact the polynomial that we have computed is the same as the polynomial invariant of virtuals of Sawollek [31] and defined by an alternative method by Silver and Williams [24]. Sawollek defines a module structure essentially the same as our Alexander Biquandle. Silver and Williams first define a group. The Alexander Biquandle proceeds from taking the abelianization of the Silver-Williams group. We will explain this, beginning with the Silver-Williams group. This generalization of the Alexander polynomial and Alexander module first appeared in [9] as an invariant of knots in thickened surfaces. The idea was also explored by Oleg Viro in unpublished work [26] and by Scott Carter and Masahico Saito in unpublished work.

Let G be a group written multiplicatively. Assume that G admits an action of $Z \times Z$, the direct product of the integers with itself. We view Z as an infinite cyclic group, and take t and v to be *multiplicative* generators

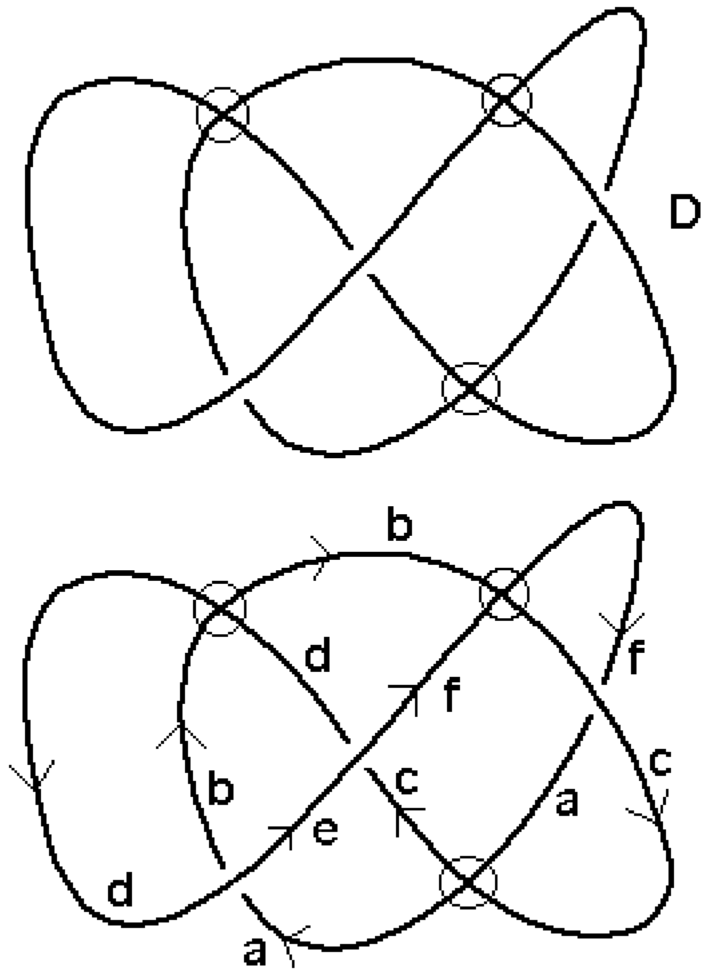

Figure 25 – Unit Jones, Integer Fundamental Group

of $Z \times Z$. For g in G, we write g^t and g^v for the actions of t and v on the element g. We now define the following operations:

$$a\ \overline{\ \mathrm{b}}| = (b^{tv})^{-1} a^t b$$

$$a\ \underline{\ \mathrm{b}}| = a^v$$

$$a|\overline{\mathrm{b}\ } = b a^{\bar{t}} (b^{\overline{tv}})^{-1}$$

$$a|\underline{\mathrm{b}\ } = a^{\bar{v}}$$

where $\overline{x} = x^{-1}$ for elements in the operator group $Z \times Z$. It is easy to verify that these operations define a biquandle. Given a group G with an operation of $Z \times Z$, we let $SW(G)$ denote the biquandle structure on G defined by these equations. The Silver-Williams group is the group associated to a link diagram that is obtained by taking one generator for each edge of the diagram and the two relations at each crossing corresponding to the above equations. Here we have used the operators t and v. Silver and Williams construct a more general group where there is one operator t_i for each link component, plus the extra operator v. They note that if v is set equal to the identity,then the resulting group is isomorphic to commutator subgroup of the fundamental group of the link complement. Note that if we abelianize the equations for the Silver-Williams biquandle, writing the abelianized group additively (keeping the operators t and v in multiplicative form), then the resulting equations are exactly the equations for the Alexander Biquandle described above. This puts all these structures in the same framework.

7.1 Fixed Points and the Lambda Calculus of Church and Curry

First lets recall the need for fixed points in the definition of a Biquandle. The axiom for the invariance under the first Reidemeister move reads: Given an element a in a biquandle BQ, there exists an x in BQ satisfying the equation $x = x\ \underline{\ \mathrm{a}}| = a_x$ and $a = x\ \overline{\ \mathrm{a}}| = x^a$. In this definition the second equation is a normal relation involving the elements x and a, but the first equation is an existence statement about x satisfying the equation $x = a_x$. Consider the simplest instance of this situation. Let $BQ = (a|)$ denote the "free biquandle" generated by the single element a. In the usual case of universal algebras the "free object" on a single generator is constructed by taking all finite algebraic expressions involving the formalism of the algebra and the generating element, subject to the natural equivalence relations that ensue for this algebra (this depends on the axioms which usually are expressed as relations, not as existence statements). But here we are asked to know that there is an x satisfying the equation above. One solution for x is the infinitary expression

$$x = a_{a_{a_{a_{\dots}}}}$$

since this formally satisfies the equation $x = a_x$. It is not clear to this author how to add infinitary expressions in a controlled way to obtain an adequate definition of a free biquandle. The alternative that we choose here is a construction based on the Chruch-Curry untyped lambda calculus [2]. In order to explain this construction we will first describe the standard untyped lambda calculus, and then we explain how it is modified to fit the biquandle. For another point of view about the relationship of lambda calculus with knot theory, see the author's paper "Knot Logic" [8].

Suppose that A is a non-associative algebra with one binary operation, denoted by justaposition ab for elements a and b of A. An *algebraic expression F(x) of one variable in* A is an appropriately parenthesized expression using elements of A and appearances of the variable x. Such an expression can be regarded as a mapping $F : A \longrightarrow A$ and we shall call such mappings *algebraic mappings from* A *to* A. A is said to be a *lambda algebra* if for every algebraic mapping F from A to A there is a corresponding element F in A (we denote them by the same letter) such that for all x in A

$$Fx = F(x).$$

For example, if we define $F(x) = xx$, then there is an element F in the lambda algebra A such that $Fa = aa$ for every a in A. Every algebraic mapping of a lambda algebra is represented by an element of the algebra. Now we prove the basic fixed point theorem for lambda algebras [2].

Theorem. Let A be a lambda algebra and let F be any element in A. Then there exists an element J in A sucn that $F(J) = FJ = J$. Thus every element of A has a fixed point, and consequently, every algebraic mapping of A to itself has a fixed point.

Proof. Define a new algebraic mapping G by the formula

$$GX = F(XX).$$

Then by the axiom for lambda algebra, G is an element of A and so we can apply G to itself, obtaining

$$GG = F(GG).$$

This shows that $J = GG$ satisfies the equation $F(J) = J$. This completes the proof. //

A lambda algebra can be approximated from a given non-associative algebra A_0 by an infinite tower of adjoinings of algebraic mappings of successive algebras. That is, given an algebra A, we let A^* denote the algebraic mappings of A to itself, and we let A' denote the new algebra obtained from A by adjoining A^* to A so that for F in A^*, $FX = F(X)$ for all X in A. No relations are imposed on XF for X in A unless X itself has a definition as an algebraic mapping, in which case $XF = X(F)$. It is important to note that X might be defined as an algebraic mapping "after" the definition of F. We then define a sequence of algebras $A_{n+1} = A'_n$ and continue this sequence through infinite ordinals via unions of the algebras in the case of limit ordinals. Let $A(\omega)$ denote the algebra constructed to the level of the ordinal ω. Each of these algebras approximates the concept of a lambda algebra, and each will have algebraic mappings that are not yet realized inside the algebra. Since there is no final ordinal, this bare process of approximation will not give a set theoretic model of a lambda algebra. In order to produce such models one needs to put further restrictions on the structure of the mapping set, as in the topological models of Dana Scott [20].

Another point of view on lambda algebra is that it is analogous to a computer programming language in which one can always make new definitions of functions that are to be included in the language (such as $GX = F(XX)$). Let L denote this language. Then the syntax of L is specified beforehand, and along with this is a capacity to define new symbols in terms of the original alpbabet of the language. The underlying set of defined symbols that represent elements of the algebra can evolve in time. Two users of the language may start with the same formal structure and evolve quite different patterns of user-defined structures. It can be arranged that two such structures can be merged into a larger structure that encompasses them both. Let us call a structure of this sort a *lambda language* L. It is understood that in a lambda language there is a well-defined notion of algebraic mapping of the language to itself. An algebaic mapping is any well-formed expression E in L that has one free variable

Now lets turn to the biquandle structure. Note that up until now, we do not have any binary operation in the biquandle that corresponds to (non-associative) juxtaposition of symbols unless these symbols are of the form

$$a\,\overline{\;\mathrm{b}\,}\rceil,\ a\,\underline{\;\mathrm{b}\,}\rfloor,\ a\lceil\overline{\mathrm{b}\;},\ a\lfloor\underline{\mathrm{b}\;}.$$

In other words, we use the operator formalism to delineate the different binary operations in the biquandle. For the purpose of using a lambda calculus formalism in the biquandle it is useful to add to the operator formalism a non-associative binary operation denoted ab for elements a and b of the biquandle. This operation can be interpreted in terms of diagrams by extending the diagrams for the virtual knots and links to include trivalent graphical vertices. This will be the subject of another paper. For our purposes the binary operation is simply given without any further relations except the usual rules for cancelling parentheses as in $((ab)(c)) = (ab)c$. With this in mind, we can posit that the biquandle is a lambda algebra with respect to this binary operation. Then the desired fixed points exist just as in the abstract discusssion above. For example, if we need an x such that $x = a_x$ then we consider first the function defined by

$$Gx = a_{xx},$$

using the juxtaposition operation to define the element xx. Then, regarding G as an element of the lambda algebra, we have

$$GG = a_{GG},$$

and so GG is the desired fixed point.

By taking every (universal) biquandle to be a lambda algebra in this way, we obtain the needed existences of fixed points and can proceed in the usual way of universal algebra to take the equivalence classes generated by the axioms for the biquandle plus the consequences of the specific relations.

This completes our sketch of the general construction of biquandles and their relation to the lambda calculus of Church and Curry.

8 Embeddings in Four-Space, Welded Braids and Welded Knots

In this section we describe a beautiful application of virtual knot theory that is due to Shin Satoh [21]. Satoh discovered that virtual knot diagrams can be interpreted as representing special embeddings of a torus in four dimensional space. That is, each component of a virtual link corresponds to an embedded torus. This correspondence is obtained as follows: First view Figure 26.

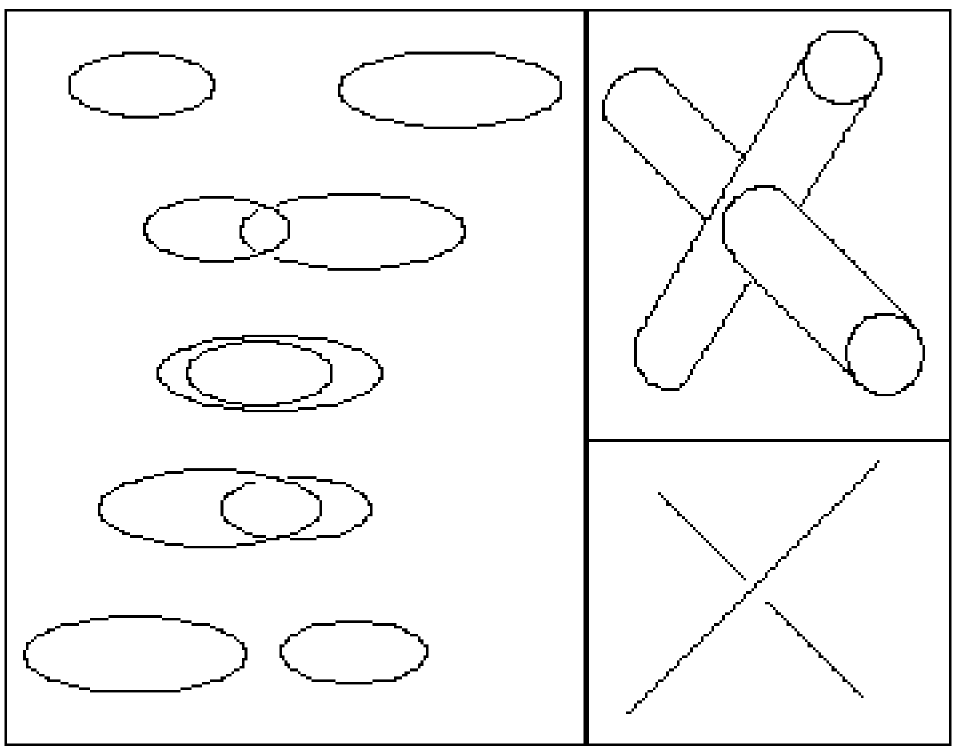

Figure 26 – Linking Tubes in Four Space

Here we have illustrated a "movie" of two tubes embedded in four dimensions so that in each three dimesional hyperspace there are two embedded circles. Each circle traces out one of the tubes in four-space. We denote the fourth spatial coordinate by t and refer to it as "time". At time zero the two circles are separated from one another. As time proceeds, the left circle and the right circle approach one another, and the left circle slides through the right circle and continues on the right while the right circle continues on to the left. In this process, the left circle is seen to first slide under the right circle and then slide out over it. The result is two tubes in four-space that are non-trivially braided with one another. If we were to make an immersion diagram of these tubes in three-space (by projection from four-space), one tube would appear to enter the side of the other and then leave it by piercing it once more. We indicate this immersion in Figure 26. Figure 26 also illustrates Satoh's ingenious coding of this tubular braiding in four dimensions: He uses the *real crossing* to indicate the four dimensional braiding of the tubes as shown in Figure 26. The assymetry of the real crossing allows us to identify the undercrossing arc with the tube whose circle goes under and through the circle of the other tube. Satoh identifies the *virtual* crossing with the situation where the circles just go around each other. In a three space immersion, this can project as simply two tubes that do not touch and *it does not matter whether one tube is depicted as overpassing or underpassing the other tube in three space* as these give the same embedding in four space. Thus Satoh obtains a mapping

$$S : VK \longrightarrow RT$$

where VK denotes virtual knot diagrams depicted in the plane, and RT denotes ribbon tori in four space. A ribbon torus embedding is exactly what we have described, namely an embedding of a torus in four space that has an immersion diagram whose only singularities correspond to braidings of the type represented by a crossing as described above.

The Satoh mapping S is not an isomorphism. Many different virtual diagrams can represent the same torus embedding in four space. In fact the first ($F1$) of our forbidden moves of type III is an isotopy of ribbon torus. See Figure 3 for an illustration of the two forbidden moves (types $F1$ and $F2$). Thus we can define WK, the *Welded Knots* to be the quotient of VK by the equivalence relation generated by forbidden move A. We then have a factorization of the Satoh map

$$P : VK \longrightarrow WK$$

$$S' : WK \longrightarrow RK$$

so that

$$S' \circ P = S$$

where S' is the Satoh mapping from welded knots to ribbon torus embeddings, and P is the projection of virtual knots to welded knots.

Of course this description raises at once the question of invariants of welded knots. The most direct observation that we can make is that *the quandle of a virtual knot is invariant under the forbidden move $F1$*. And Satoh proved that the quandle of a virtual knot K is isomorphic to the quandle of the corresponding ribbon torus embedding in four space. Thus the invariants related to the fundamental group carry through in this pattern. On the other hand, we do not know how to easily extend the Jones polynomial and other quantum invariants to welded knots. Welded knots are clearly worth study, but present new challenges for their classification.

The classification of welded knots is also distinct from the classification of ribbon torus embeddings in four space. Such embeddings have to be studied on their own grounds via analogs of the Reidemeister moves in four-space. In this way Satoh's construction raises a host of new problems and ideas.

The reader should note that it follows from our description that the fundamental groups of virtual knot diagrams are isomorphic to the fundamental groups of certain ribbon torus embeddings in four space. This fact was first observed by Silver and Williams [23] by an algebraic argument. These constructions give a direct topological interpretation of this fact.

Finally, we note that the term em welded knots has been assigned because the *welded braids* of Rourke, Fenn and Rimanyi [30] are precisely the braids that correspond to the theory of welded knots. The Rourke-Fenn moves consist in the rules for virtual braids plus the moves on braids that correspond to forbidden move $F1$. The interpretation we have given of welded braiding in terms of embedded tubes in four space shows that the Rourke-Fenn welded braids map to the braid group whose configuration space is the set of disjoint unlinked unknotted circles in three space. This is sometimes called the *motion group* of circles in three space [3]. In fact, it follows from this interpretation and calculations of Tom Imbo [25] that the welded braids on n strands are isomorphic to the motion group of n unknotted unlinked circles in three space. This gives a beautiful and unexpected interpretation to the welded braids of Rourke and Fenn.

Remark. It is worthwhile pointing out that the virtual braid group is an extension of the classical braid group by the symmetric group. If V_n denotes the n–strand virtual braid group, then V_n is generated by braid generators σ_1, ..., σ_{n-1} and virtual gnerators c_1,..., c_n where each virtual generator c_i has the form of the braid generator σ_i with the crossing replaced by a virtual crossing. Among themselves the braid generators satisfy the usual braiding relations. Among themselves, the virtual generators are a presentation for the symmetric group S_n. The relations that relate virtual generators and braiding geneerators are as follows:

$$\sigma_i^{\pm} c_{i+1} c_i = c_{i+1} c_i \sigma_{i+1}^{\pm},$$
$$c_i c_{i+1} \sigma_i^{\pm} = \sigma_{i+1}^{\pm} c_i c_{i+1},$$
$$c_i \sigma_{i+1}^{\pm} c_i = c_{i+1} \sigma_i^{\pm} c_{i+1}.$$

It is easy to see from this description of the virtual braid groups that all the braiding generators can be expressed in terms of the first braiding generator σ_1 (and its inverse) and the virtual generators. One can also see that Alexander's Theorem generalizes to virtuals: Every virtual knot is equivalent to a virtual braid. In [14] a Markov Theorem is proven for virtual braids.

9 Discussion

This paper has been a survey of aspects of virtual knot theory. Two problems that arose here should be underlined. The first is the construction of Section 6 of infinitely many virtual knots with unit Jones polynomial. It is an open question whether any of these are isotopic to classical knots. This gives a new twist to the problem of classical knot detection by the Jones polynomial. The second is the construction and use of the biquandle as an invariant of virtuals.

This structure presents conceptual problems in its very definition due to the need for the existence of certain fixed points in the algebra. On the other hand there are concrete examples of biquandles that are quite well-defined and useful. More exploration is needed in this domain.

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