QUANDLES, KNOT INVARIANTS AND THE N-FOLD BRANCHED COVER

by Steven K. Winker

1 Introduction

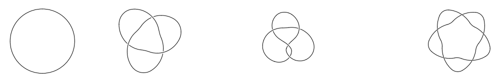
This paper focuses on the class of algebraic objects called *involutory quandles* and, in particular, on the connection between involutory quandles and topological knots and links. The paper is intended to be a self-contained treatment of the topic and should be accessible to readers with various backgrounds. Section 2 surveys the available material concerning involutory quandles and includes several important examples. Section 3 covers an exposition of a more general class of algebraic objects called *quandles* and their connection with knots and links, and the results discussed are those of [Joyce1982a]. As will be shown, quandles associated to a knot are analogous in a number of ways to the fundamental group of the knot complement. In particular, the involutory quandle is obtained from a given quandle as the image of a particular homomorphism.

In Section 4, we develop a new type of a diagram for an arbitrary quandle. This diagram encodes the multiplication table in an efficient and notationally convenient form. Such a diagram often contains repeated geometric patterns that provide intuitive clues concerning the algebraic structure of the associated quandle. (This type of a diagram is compared and contrasted with another type of diagram [Joyce1979] at the end of Section 4.3.) Sections 4.5 - 4.11 develop a method to construct the new diagram directly from a quandle presentation thereby bypassing calculation of the multiplication table. This method is applied in Section 4.8, where we construct diagrams of involutory quandles for various knots and links. In Section 4.12, we distinguish the 4-quandles of the square and granny knots with the aid of a quandle diagram.

Our main theorem—any tame knot with a trivial involutory quandle or a trivial n-quandle must be trivial—is proven in Section 5.2. The proof involves certain types of groups obtained from quandles: the conjugate group of the knot quandle (Joyce's Adconj); the involutory conjugate group; and its even subgroup. Our discussion begins with Joyce's result that the conjugate group of the quandle associated to a knot or link is isomorphic to the knot (or link) group. As a result, we obtain the statement that the even subgroup of an involutory conjugate group is isomorphic to the fundamental group $\pi_1(M^{(2)})$ of the two-fold branched cover of S^3 along a knot or link. The main theorem is proved using the Smith Conjecture.

2 Brief Survey and Examples

A knot is an embedding of a circle S^1 into the 3-sphere S^3 (Figure 2.1). What we actually draw is its projection onto the plane.



Trivial Trefoil Figure-eight $(5, 2)$ Torus

Figure 2.1. *Knots*

A link is an embedding of a disjoint union of two or more circles into S^3 (Figure 2.2).

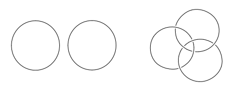
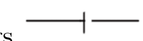


Figure 2.2. *Links*

We will consider only *tame* knots and links, those knots and links with a finite number of crossings  in the projection. The *knot group,* an algebraic object associated with a knot or link, is an *invariant* of the knot. That is, the knot group remains the same (up to an isomorphism) no matter how the knot is deformed through ambient isotopy of S^3 (Section 3.1). For example, while using the knot group, the trefoil knot cannot be deformed into the figure-eight knot (Figure 2.1) by showing the corresponding knot groups are not isomorphic.

The knot group is not the only algebraic invariant associated to a knot or a link. We shall also consider another invariant called an involutory quandle of the knot or link.

Definition 2.0.1 *An* involutory quandle *is a set Q with a binary operation $\triangleright$ called* product*, written $x \triangleright y$, which satisfies the following three axioms.*

1. $x \triangleright x = x$ *for all* $x \in Q$.

2. $(x \triangleright y) \triangleright y = x$ *for all* $x, y \in Q$.

3. $(x \triangleright z) \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z$ *for all* $x, y, z \in Q$.

We construct an involutory quandle for the trefoil knot in this section and describe the general procedure for constructing involutary quandles for knots and links in Section 4. We shall see close connections between quandles and groups and, in particular, between the involutory quandle of a knot or link and the knot group.

The involutory quandle has advantages over the knot group for certain purposes. First, the involutory quandle of a tame knot or link is either finite or "not too infinite". For example, we shall see the involutory quandle of the trefoil knot has three elements. (The knot group for any nontrivial knot is "very infinite," and the corresponding Cayley diagram is impractical due to its complexity.)

Second, involutory quandles distinguishes certain pairs of links which cannot be distinguished using their groups. For example, the two links in Figure 2.3 have the same knot groups but have finite involutory quandles of differing cardinality. The involutory quandle for the first link of Figure 2.3 has 8 elements and 24 elements for the second link. (See Section 4.8, Example 4.8.4 for details.) Note these links can be distinguished by examining their corresponding two-fold branched covering spaces. This observation suggests a connection between the involutory quandle and the two-fold branched cover. This connection will be closely examined in Section 5.2. (Note knots with different knot groups can have the same associated involutory quandle [e.g. the figure-eight knot and the $(5, 2)$ torus knot in Figure 2.1], and, as conjectured by J. Simon, the knot group determines the involutory quandle for knots, as opposed to links).

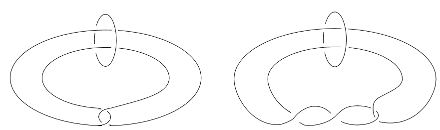
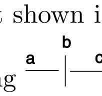


Figure 2.3. *Two links with homeomorphic complements in* S^3 *but non-isomorphic involutory quandles. The involutory quandle of the link at left has 8 elements,while the involutory quandle of the link at right has 24 elements.*

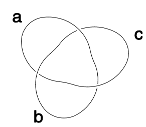
A third advantage of the involutory quandle is the involutory quandle can be studied conveniently by means of a diagram analogous to the Cayley diagrams (Section 4). With the advantages stated, we now construct the diagram an involutory quandle.

We begin with a diagram of the involutory quandle of the trefoil knot shown in Figure 2.4. Label arcs of the knot projection by a, b, c (Figure 2.4, upper left). For each crossing 

in the projection, define the relation

$$a \triangleright b = c.$$

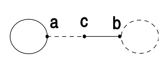
The relations are listed in Figure 2.4, upper right. (The relation $c \triangleright b = a$ obtained by reading the projection in the opposite direction is redundant.) These relations, together with the axioms of an involutory quandle (see Definition 2.0.1), yield a multiplication table for the involutory quandle of the knot (see Figure 2.4, lower right). The multiplication table is obtained from the relations and the axioms by a purely algebraic process. Note while a one-to-one correspondence between the elements of the involutory quandle and the labeled arcs of the original knot occurs in this particular case, such a correspondence does not occur in general. Usually the labeled arcs of a knot correspond to the set of generators of the associated involutory quandle. (An involutory quandle usually has more elements than generators). The correspondence between arcs and distinct generators of an involutory quandle is also not necessarily one-to-one as two different arcs of the knot projection may correspond to the same element in the involutory quandle (see Figure 2.5).



$$a \rhd b = c$$

$$b \rhd c = a$$

$$c \rhd a = b$$



$\rhd$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Figure 2.4. *Trefoil knot, relations, and multiplication table for the involutory quandle ([Joyce1982a]). Our diagram of the involutory quandle appears at lower left.*

We define the diagram of an involutory quandle in terms of the multiplication table (Figure 2.4, lower left). The vertices a, b, c of the diagram represent the elements of the involutory quandle. The solid arcs represent right multiplication by a, and the dashed arcs represent right multiplication by b.

Thus, the solid arc from c to b implies $c \rhd a = b$; the solid arc from a back to a indicates $a \rhd a = a$; and the dashed arc from a to c indicates $a \rhd b = c$. In Section 4.5, the diagram of an involutory quandle encodes all information present in its multiplication table. In the example, the information concerning the right multiplication by a and b allows us to find right multiplication by c as well. We shall ultimately derive the diagram of an involutory quandle directly from the relations (Sections 4.5 - 4.7) without intermediate computation of the multiplication table.

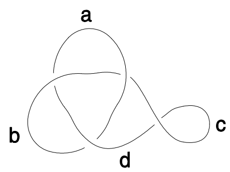


Figure 2.5. *Two arcs of knot projection may correspond to same element of the associated involutory quandle. In the projection above $c \rhd c = d$ (for right-most crossing), but $c \rhd c = c$ as well (Axiom* 1 *in Definition 2.0.1).*

Using Figure 2.6, we suggest the intuitive possibilities of quandle diagrams. We provide several knots of increasing complexity and provide the corresponding involutory quandle diagrams to the right. Since the corresponding involutory quandle diagrams reveal an increase in complexity, we ask how the complexity of the quandle diagram is related to the complexity of the knot We have no general answer yet. However, we show any nontrivial knot must have a nontrivial involutory quandle (see proof in Section 5.2).

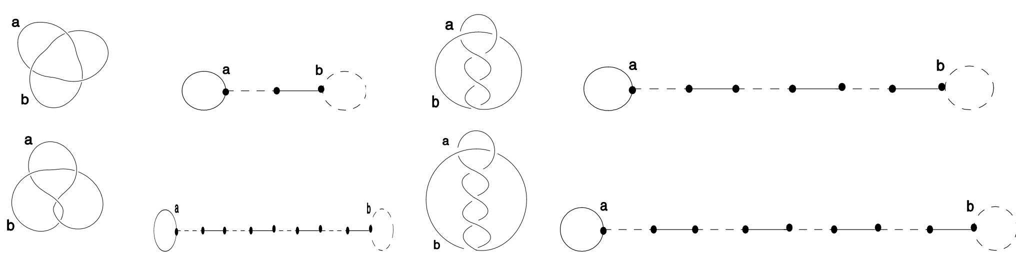


Figure 2.6. *Increasing complexity in knots and in their involutory quandle diagrams*

We conclude this section with an example of the involutory quandle for the Borromean rings (Figure 2.7). The quandle diagram of this link has three components, one component for each component of the link. We show only the component which contains element a in Figure 2.7. The quandle diagram can be drawn on a conical surface and is discussed further in Section 4.7. The quandle diagram in Figure 2.7 shows the involutory quandle of the prime link is infinite. If the diagram is collapsed by letting

$$x \rhd c = x$$

for any x, and the c component of the link is deleted, then the resulting quandle diagram is the same as for the trivial link of 2 components.

In Section 4.12, quandles are used to show the square knot and the granny knot are distinct. Quandles are related to the n-fold branched covering spaces in Section 5.

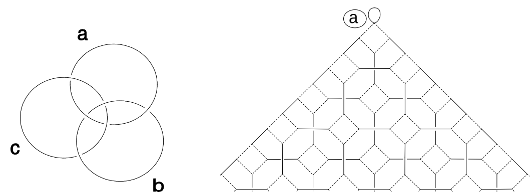


Figure 2.7. Borromean rings and its involutory quandle's diagrams (*one of three components*)

3 The Quandle of a Knot

In this section, we define a quandle of a knot [Joyce1982a] both topologically and algebraically. A combinatorial review of elementary knot theory and the knot group introduces the approach. (For further reading see [Crowell and Fox1977] and [Rolfsen 1976]) An algebraic structure of a quandle naturally arises by considering a mapping called a disk with a path–a meridian path spanned by a disk. Homotopy equivalence classes and product operations are used to define an algebraic structure of a quandle. A particular homomorphic image of this quandle is called an involutory quandle. We describe a more direct geometric application of the involutory quandle in Section 5.2.

3.1 Combinatorial Knot Theory

We introduce basic notions of knot theory from a combinatorial point of view. A *knot* is an embedding of a circle S^1 within the three dimensional sphere S^3 (see Figure 3.1.1).

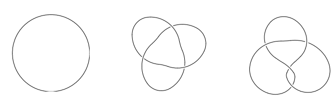


Figure 3.1.1. *Trivial knot, trefoil, and figure-eight knot*

We represent a knot by drawing a *projection* of the knot onto the plane. A *link* is an embedding of disjoint union of two or more circles into S^3 (see Figure 3.1.2). Each embedded circle is called a *component* of the link; a knot is a link with one component.

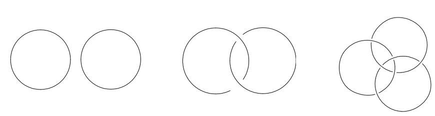


Figure 3.1.2. *Trivial link* (*unlink*)*, simplest nontrivial link, and Borromean rings*

The following terminology will be used in describing presentations of knot groups and quandles. An *arc* is an unbroken curve in a knot projection. We label them by letters a, b, c... (Figure 3.1.3).

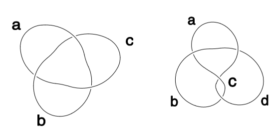
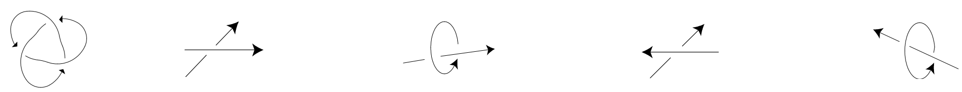


Figure 3.1.3. *Labeling of arcs*

At each *crossing*, three arcs meet–an overcrossing arc, a, and two undercrossing arcs, b and c. We consider only *tame* projections, those with a finite number of crossings, and tame knots and links, those having a tame projection. An *oriented* knot or link has a preferred direction of travel along each component. Orientation leads to *positive* and *negative* crossing and linking as illustrated in Figure 3.1.4.



Positive Crossing Positive Linking Negative Crossing Negative Linking

Figure 3.1.4. *Orientation of knots and links*

Deformation of knots in S^3 is formalized using the following definition of *ambient isotopy*, (see Figure 3.1.5).

Definition 3.1.1 *An* ambient isotopy *between two knots K_1 and K_2 is a continuous mapping $h : S^1 \times I \to S^3$ such that*

$$h(S^1 \times \{0\}) = K_1 \text{ and } h(S^1 \times \{1\}) = K_2,$$

and for all $x \in I$, the mapping h restricted to $S^1 \times \{x\}$ is a knot.

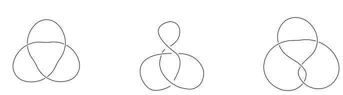


Figure 3.1.5. *Ambient isotopy*

An ambient isotopy defines an equivalence relation between knots in S^3, and a *knot type* is defined as an equivalence class of the ambient isotopy relation. Analogously, we define ambient isotopy of links. The following three *elementary knot moves* shown in Figure 3.1.6 describe an ambient isotopy using knot projection.

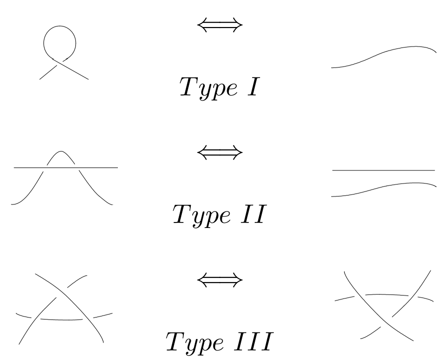


Figure 3.1.6. *The three elementary knot moves*

For example, in Figure 3.1.5 moves **I** and **III** were applied to pass from the left-most figure to the right-most. The elementary moves on knot projections yield all ambient isotopies between tame projections as shown in [Alexander and Briggs1976]. Therefore, to define an invariant of a knot type it suffices to show the invariance under the three above moves.

3.2 The Knot Group

In this section, we define the knot group both topologically and combinatorially. Given a tame oriented knot or link $K \subset S^3$ we choose a *basepoint* $P \in S^3 - K$. A *path* is a continuous map $\alpha : I \to S^3 - K$ such that $\alpha(0) = \alpha(1) = P$, as shown in Figure 3.2.1.

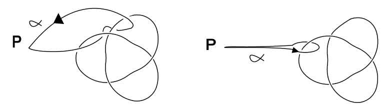


Figure 3.2.1. *Paths* α *with the basepoint* P

A *meridian* is a path singly and positively linked with a single arc (see Figure 3.2.1, right) of the knot projection. Meridians will play a special role. Deformation of paths is formalized as the equivalence relation of *path homotopy* (Figure 3.2.2).

Definition 3.2.1 *A* homotopy *of paths* α_1 *and* α_2 *(path homotopy) is a continuous map* $h : I \times I \to S^3 - K$ *such that*

$$h(I \times \{0\}) = \alpha_1 \text{ and } h(I \times \{1\}) = \alpha_2,$$

and for all $x \in I$ *the map* h *restricted to* $I \times \{x\}$ *is a path.*

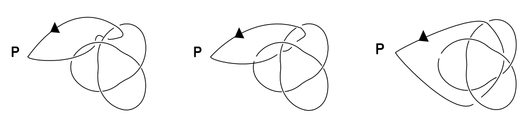


Figure 3.2.2.*Path homotopy*

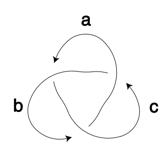
A homotopy between paths is an equivalence relation described in terms of the three elementary knot moves and an additional move passing the path through itself. Multiplication of paths is defined by concatenation of paths.

Definition 3.2.2 *The* product *of two paths* α_1 *and* α_2 *is the path* $\alpha : I \to S^3 - K$ *defined as follows*

$$\alpha(x) = \begin{cases} \alpha_1(2x), & when \quad 0 \leq x \leq \frac{1}{2} \\ \alpha_2(2x-1), & when \quad \frac{1}{2} \leq x \leq 1 \end{cases}.$$

With this product, the set of equivalence classes of path homotopy relation in $S^3 - K$ with the fixed basepoint P forms a group–the fundamental group of a knot complement or the knot group $\pi_1(S^3 - K)$. This group depends only on the complement of K in S^3 and not on the connection of K to its complement. By contrast, the quandle and the involutory quandle, defined in Sections 3.3 and 3.4, will utilize this connection.

A presentation of $\pi_1(S^3 - K)$ is obtained as follows (see Figure 3.2.3). In the projection of an oriented knot (or link) K, label arcs by a, b, c, ... These labels will serve as the generators of $\pi_1(S^3 - K)$. A relation in the presentation of the knot group is obtained for every crossing of the knot projection as follows. Whenever an arc a crosses under b to become c, the relation $b^{-1}ab = c$ is read if the crossing is positive and $bab^{-1} = c$ if it is negative.



$$\pi_1\left(S^3 - K\right) = \langle a, b, c : b^{-1}ab = c,\ c^{-1}bc = a,\ a^{-1}ca = b\rangle$$

Figure 3.2.3. *Wirtinger presentation of knot group*

Such a presentation has the following topological interpretation. Each generator a represents a meridian linking the arc a positively and passing under no other arc of the knot projection. The relations follow from the path homotopy at the crossings. Using Van Kampen theorem, these relations are sufficient to define $\pi_1(S^3 - K)$ up to the isomorphism. Consequently, invariance of the group thus presented is proved combinatorially by examining the effect of the three knot moves, as in Figure 3.2.4.

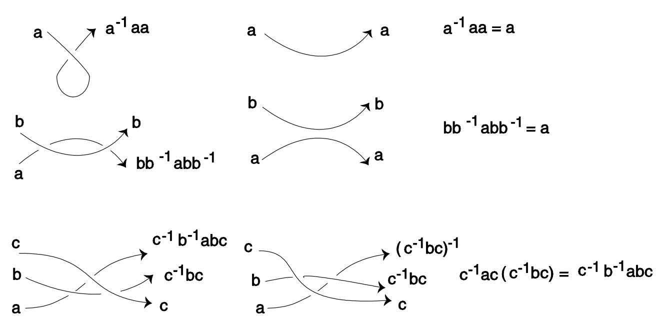


Figure 3.2.4. *Invariance of the knot group under elementary moves*

Meridians will now be discussed in another connection. Knowledge of which paths are meridians is useful in distinguishing knots. For instance, the two non-equivalent knots shown in Figure 3.2.5 have isomorphic knot groups, but the isomorphism fails to map meridians. Similarly, consider the links in Figure 3.2.5. The complements are homeomorphic as demonstrated by cutting along the shaded disk, twisting 2π, and re-pasting. Therefore, the knot groups are isomorphic. However, the isomorphism does not preserve meridians.

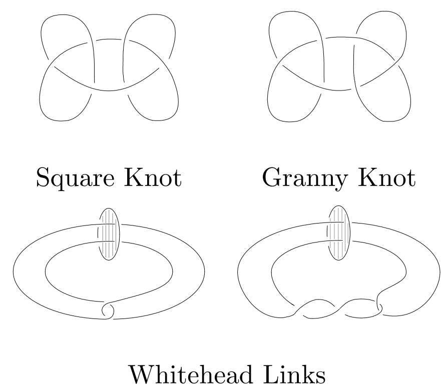


Figure 3.2.5. *Distinct knots and links with isomorphic knot groups*

The quandle is introduced in order to incorporate the special role of meridians into an algebraic structure.

3.3 Definition of the Quandle of a Knot

For a knot or a link K, a meridian $m \in \pi_1 \left(S^3 - K\right)$ can be spanned by a disk to become a *disk with a path,* an element of an algebraic object called the *knot quandle* $Q(K)$. Formally, a disk with path is a continuous map to S^3 from the following topological object.

Definition 3.3.1 *A* disk with an interval *is a topological object illustrated in Figure* 3.3.1 *below.*

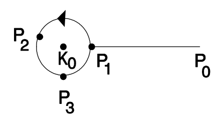


Figure 3.3.1. *Disk with an interval*

Note the preferred direction of travel P_1 P_2 P_3 P_1 around the circumference of the disk.

Definition 3.3.2 *A* disk with a path *is a continuous map* $\delta : D \to \left(S^3,\ K,\ P\right)$ *from a disk with interval* D *to an oriented knot or link* K *in* S^3 *with the basepoint* $P \in S^3 - K$*, which satisfies the following conditions.*

1. δ *maps* P_0 *to* P *and* K_0 *into* K*.*

2. *No point in* D*, other than* K_0*, is mapped to* K*.*

3. *The path* α *obtained by tracing the image of the circumference* P_0 P_1 P_2 P_3 P_1 P_0 *of* D *is singly and positively linked with* K*. The path* α *is called the* meridian *of* δ*.*

A disk with a path and its meridian are illustrated in Figure 3.3.2. Deformation of a disk with a path is formalized as the equivalence relation of homotopy.

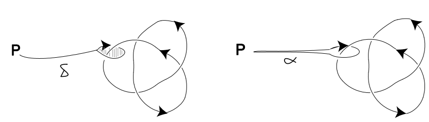


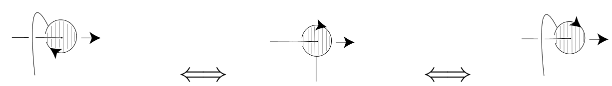
Figure 3.3.2. *Disk with path* δ *and its meridian* α

Definition 3.3.3 *A* homotopy *of disks with paths* δ_1 *and* δ_2 *is a continuous map* $h : D \times I \to \left(S^3,\ K,\ P\right)$ *such that*

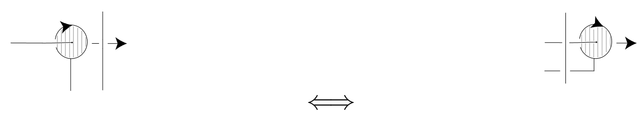
$$h(D \times \{0\}) = \delta_1 \text{ and } h(D \times \{1\}) = \delta_2,$$

and the restriction of h *to* $D \times \{x\}$ *is a disk with a path for all* $x \in I$.

Homotopy of disks with path is exemplified by homotopy of paths with the addition of the two move types of Figure 3.3.3. The homotopy classes of disks with paths are the elements of the quandle $Q(K)$ associated to K. Note two non-homotopic disks with paths may have homotopic meridians (see Figure 3.3.4). In Section 4.6, these two non-homotopic disks with paths are shown in a diagram of the involutory quandle associated to this link.



Rotation



Translation

Figure 3.3.3. *Homotopy of disks with path. These moves are in addition to the elementary moves of Figures* 3.1.6 *and* 3.2.2.

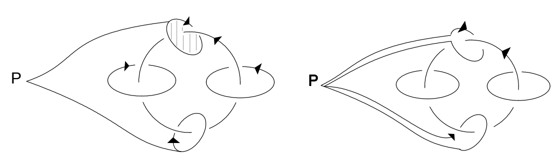


Figure 3.3.4. *Non-homotopic disks with path, which nevertheless have homotopic meridians*

The product $a \triangleright b$ of two disks with paths is now defined as illustrated in Figure 3.3.5. The path component of a is extended by appending it by the meridian of b to yield the disk with path $a \triangleright b$. The inverse product $a \triangleright^{-1} b$ is obtained by instead appending a by the inverse of the meridian of b.

Definition 3.3.4 *A* quandle $Q(K)$ *of a knot (or link)* $K \subset S^3$ *with the basepoint* $P \in S^3 - K$ *is an algebraic structure whose elements are the homotopy equivalence classes of disks with path* $\delta : D \to \left(S^3,\ K,\ P\right)$ *and whose product operations are* $\triangleright$, $\triangleright^{-1}$*as defined above.*

The quandle satisfies the following equalities which can be verified by use of homotopy. For all x, y, $z \in Q(K)$ we have:

1. $x \triangleright x = x$

2. $(x \triangleright y) \triangleright^{-1} y = x = \left(x \triangleright^{-1} y\right) \triangleright y$

3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

The above identities are used to define the algebraic notion of a quandle in Section 4.1.

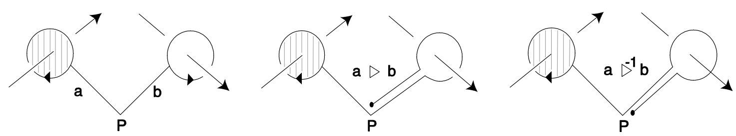


Figure 3.3.5. Product of disks with paths

We now discuss the *presentation* for quandle $Q(K)$. Briefly, a presentation consists of the set of *generators* $a, b, c, ...$ for the quandle and the set of *relations* (equalities) involving those generators. Therefore, an algebraic structure of a quandle given in terms of generators and relations is the set of all formal products of generators, modulo the equivalence relation generated by the axioms and the relations (see Sections 4.1 and 4.2).

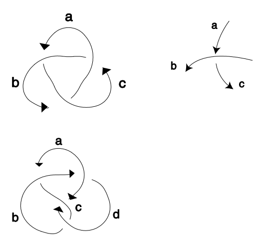
A presentation of $Q(K)$ is obtained as follows (Figure 3.3.6). In a tame projection of K, label the arcs $a, b, c, ...$. These labels are generators of the quandle. At each crossing, a relation is obtained; whenever an arc a crosses under b to become c, the relation

$$a \triangleright b = c$$

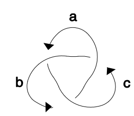
is read if the crossing is positive and

$$a \triangleright^{-1} b = c$$

if the crossing is negative. Presentations for the trefoil and the figure-eight knot are given in Figure 3.3.6. The procedure for finding the multiplication table of $Q(K)$ using the presentation of $Q(K)$ is discussed in Section 4.



$a \rhd b = c$



$Q\,(K) = \langle a, b, c : a \rhd b = c,\ b \rhd c = a,\ c \rhd a = b$

$$Q\,(K) = \langle a, b, c, d : a \rhd^{-1} b = c,\ c \rhd d = b,\ b \rhd^{-1} a = d,\ d \rhd c = a$$

Figure 3.3.6. *Presentations of knot quandles*

That the algebraic structure $Q(K)$ is a knot invariant is intuitively clear and is proven in Figure 3.3.7 by showing $Q(K)$ is preserved by the 3 elementary knot moves. In the Figure 3.3.7, only certain cases of the possible orientations were considered. In each of these cases the invariance under the knot moves is a consequence of the axiom given in the definition of a quandle. However, most of the remaining cases are consequences of more involved identities derived in Section 4.1 and in Lemma 4.4.7 from the axioms given in Definition 2.0.1.

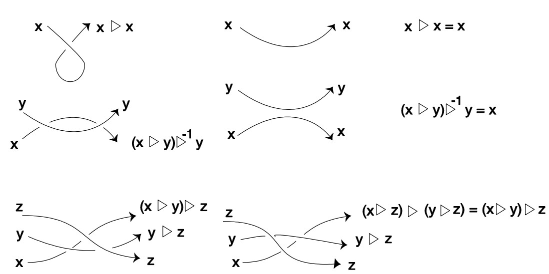


Figure 3.3.7. *Invariance of the knot quandle under the elementary knot moves*

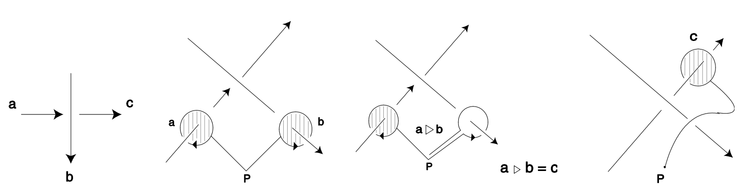


Figure 3.3.8. *Relation* $a \rhd b = c$ *at a crossing verified by homotopy*

A presentation of $Q\,(K)$ as in Figure 3.3.6 has the following topological interpretation. Each generator (arc label) a represents a homotopy class of a disk with a path linked singly and positively around the arc a. The path of the disk crosses over, but not under, the arcs of the knot projection. The relations read for each crossing are a consequence of the homotopy of a disk with a path relation (see Figure. 3.3.8). The algebraic structure $Q(K)$ defined in terms of generators and relations is, in fact, the same algebraic structure defined in terms of disks with paths and the homotopy relation [Joyce1982a]. The theorem for quandles used as proof for this statement is analogous to the Van Kampen theorem for the fundamental group.

3.4 Definition of the Involutory Quandle of a Knot

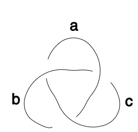
We now turn our focus to the *involutory quandle*, a homomorphic image of the quandle associated to a knot. The involutory quandle $IQ\,(K)$ of a knot or link K is obtained from the quandle $Q\,(K)$ by identifying the operations $\triangleright$ with $\triangleright^{-1}$. In other words, by adding relations $x \triangleright y = x \triangleright^{-1} y$ for all $x,\, y \in Q(K)$ to the presentation of $Q(K)$. Equivalently, the presentation of $IQ\,(K)$ is defined using the following identities for all x, y:

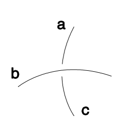
1. $x \triangleright x = x$

2. $(x \triangleright y) \triangleright y = x$

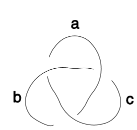
3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

and the generators and relations are obtained as in Figure 3.4.1. In a tame projection of K, label the arcs using $a, b, c, \ldots.$ These labels serve as the generators of $IQ\,(K)$. Each crossing yields a relation obtained as follows. Whenever an undercrossing arc labeled by a crosses under the overcrossing arc b to become c, we have the relation $a \triangleright b = c$. Presentations for the trefoil and the figure eight knot are shown in Figure 3.4.1.

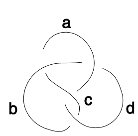




$a \triangleright b = c$



$IQ\,(K) = I\langle a, b, c : c,\; b \triangleright c = a,\; c \triangleright a = b$



$$IQ\,(K) = I\langle a, b, c, d : a \triangleright b = c,\; c \triangleright d = b,\; b \triangleright a = d,\; d \triangleright c = a$$

Figure 3.4.1. *Presentation of an involutory quandles of knots*

Note the orientation of K can be omitted as it is redundant for defining a presentation of an involutory quandle. The omission of the orientation implies the relation for each crossing can be either $a \triangleright b = c$ or $c \triangleright b = a$. In fact, these two relations can be shown to be equivalent assuming axioms of the involutory quandle (see Lemma 4.1.6).

As presented, an involutory quandle is a knot invariant. This statement is proven by verifying $IQ(K)$ is preserved by examining three elementary knot moves (see Section 3.3) or recognizing $Q\,(K)$ is a knot invariant and $IQ\,(K)$ is defined as a quotient of $Q(K)$.

$IQ(K)$ is indirectly connected with the knot K through the topological interpretation of $Q(K)$. A more direct topological interpretation of $IQ(K)$ is established in Section 5.2.

4 Diagramming of Quandles

In this section, we produce a *diagram* (see Definitions 4.3.1, 4.3.9) for an arbitrary quandle $Q(K)$, which encodes properties of the algebraic structure in an often concise way. In order to produce a diagram of $Q(K)$, preliminary information is necessary. This information includes basic definitions and lemmas given in Section 4.1; a discussion of a presentation for quandle described in Section 4.2; a definition and simple examples of diagrams discussed in Section 4.3; and the canonical left association introduced in Section 4.4. Thus, in Sections 4.5-4.7, we develop a method of constructing a diagram of a quandle from its presentation and focus on the definition of an involutory quandle (Definition 4.1.2). Diagrams of involutory quandles associated to various knots and links will be constructed in Section 4.8. Correctness of the method for constructing diagrams of involutory quandles will be discussed in Sections 4.9-4.10. Finally, in Section 4.11, the method for constructing diagrams will be extended to arbitrary (not necessarily involutory) quandles and then used to distinguish 4-quandles of the square and granny knots in Section 4.12.

4.1 Algebraic Definition of Quandles

In this section, we give basic definitions, results, and examples concerning quandles. The discussion is based entirely on [Joyce1982b].

Definition 4.1.1 *A quandle is a set of elements with two binary operations* $\triangleright$, $\triangleright^{-1}$*satisfying the following axioms.*

Axiom 1 (*idempotency*) $x \triangleright x = x$

Axiom 2 (*right cancellation*) $(x \triangleright y) \triangleright^{-1} y = x = \left(x \triangleright^{-1} y\right) \triangleright y$

Axiom 3 (*distributivity*) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

Idempotency for the operation $\triangleright^{-1}$, $x \triangleright^{-1} x = x$, is obtained from axioms **1** and 2**.**

Definition 4.1.2 *An involutory quandle* Q *is a quandle in which* $x \triangleright y = x \triangleright^{-1} y$ *for all elements* x, y*. Equivalently, an involutory quandle is a set of elements with one binary operation* $\triangleright$ *satisfying the following axioms.*

Axiom 1 (*idempotency*) $x \triangleright x = x$

Axiom 2 (*right cancellation*) $(x \triangleright y) \triangleright y = x$

Axiom 3 (*distributivity*) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

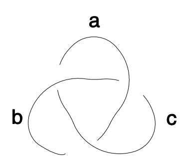
From this point on, the omission of parentheses in the product denotes left association, e.g. $x \triangleright y \triangleright z$ denotes $x \triangleright y \triangleright z = (x \triangleright y) \triangleright z$. Note quandles are non-associative in general as shown in the following examples. The multiplication tables given in Figures 4.1.1 and 4.1.2 clearly satisfy the axioms of an involutory quandle.

Example 4.1.3 *The involutory quandle in Figure* 4.1.1 *is generated by the elements* a, b, c, *and satisfies the following relations*:

$$\begin{aligned} a \triangleright b &= c, \\ b \triangleright c &= a, \\ c \triangleright a &= b. \end{aligned}$$

Lemma 4.2.7 supports this is in fact an involutory quandle for the trefoil knot,

$$IQ = \langle a,\ b,\ c : a \triangleright b = c,\ b \triangleright c = a,\ c \triangleright a = b\rangle.$$



$\triangleright$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Figure 4.1.1. Involutory quandle for the trefoil knot

Note this quandle is non-associative. For instance,

$$(a \triangleright b) \triangleright c = c \triangleright c = c$$

but

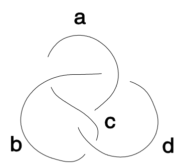
$$a \triangleright (b \triangleright c) = a \triangleright a = a.$$

Example 4.1.4 *The involutory quandle of Figure* 4.1.2 *is generated by* a, b, c, d *and satisfies the following relations.*

$$\begin{aligned} a \triangleright b &= c, \\ c \triangleright d &= b, \\ b \triangleright a &= d, \\ d \triangleright c &= a. \end{aligned}$$

Lemma 4.2.7 demonstrates the quandle in Figure 4.1.2 *is an involutory quandle for the figure-eight knot and has the following presentation.*

$$IQ = \langle a,\ b,\ c,\ d \mid a \triangleright b = c,\ c \triangleright d = b,\ b \triangleright a = d,\ d \triangleright c = a \rangle.$$



$\triangleright$	a	b	c	d	e
a	a	c	d	e	b
b	d	b	e	c	a
c	e	a	c	b	d
d	b	e	a	d	c
e	c	d	b	a	e

Figure 4.1.2. Involutory quandle of the figure-eight knot

As in Example 4.1.3,

$$(a \triangleright b) \triangleright c \neq a \triangleright (b \triangleright c)$$

and

$$a \triangleright b \neq b \triangleright a,$$

Thus, the quandle in Figure 4.1.2 *is non-associative and non-commutative.*

The following class of quandles will be discussed in Section 5.

Definition 4.1.5 *An* n-quandle Q *is a quandle satisfying* $x \triangleright^n y = x$ *for all* x, y, *where the relation* $x \triangleright^n y = x$ *is defined inductively as follows*:

$$x \triangleright^1 y = x \triangleright y$$

and

$$x \triangleright^{m+1} y = (x \triangleright^m y) \triangleright y.$$

In particular, a quandle Q *is an involutory quandle if and only if* Q *is a* 2*-quandle.*

The following lemma aids in an algorithm for finding the multiplication table for a quandle.

Lemma 4.1.6 *For any involutory quandle,*

$$x \rhd y = z \Longleftrightarrow z \rhd y = x.$$

For any quandle,

$$x \rhd y = z \Longleftrightarrow z \rhd^{-1} y = x.$$

Proof. For an involutory quandle, if $x \rhd y = z$, then we obtain $z \rhd y = (x \rhd y) \rhd y = x$ from axiom 2. Analogous argument applies in the other direction. Similar arguments prove an arbitrary quandle. ■

Further discussion of properties of quandles, including the existence of the canonical form for the left association and the fundamental asymmetry between right and left multiplication, is discussed in Section 4.4. Connections between quandles and groups are discussed in Section 5. This section ends with the following standard definitions for homomorphism, isomorphism, and automorphism of quandles.

Definition 4.1.7 *A* homomorphism *of quandles* Q, Q' *is a map*

$h : Q \to Q'$ such that $h(x \rhd y) = h(x) \rhd h(y)$ for all $x, y \in Q$.

Definition 4.1.8 *An* isomorphism *of quandles is a homomorphism that is one-to-one and onto.*

Definition 4.1.9 *An* automorphism *of a quandle* Q *is an isomorphism from* Q *to itself.*

Remark 4.1.10 *For a homomorphism* h *of quandles,*

$$h\left(x \rhd^{-1} y\right) = h(x) \rhd^{-1} h(y)$$

and

$$\begin{aligned} h\left(x \rhd^{-1} y\right) &= h\left(x \rhd^{-1} y\right) \rhd h(y) \rhd^{-1} h(y) \;\; (\textit{by axiom } \mathbf{2}) \\ &= h\left(x \rhd^{-1} y \rhd y\right) \rhd^{-1} h(y) \;\; (\textit{Definition 4.1.7}) \\ &= h(x) \rhd^{-1} h(y) \;\; (\textit{by axiom } \mathbf{2}). \end{aligned}$$

Definition 4.1.11 *An* inner automorphism *of a quandle Q is an automorphism given by $\varphi_q(x) = x \triangleright q$ for some $q \in Q$, a product of such mappings and their inverses.*

Using axiom **3** of a quandle, φ_q is indeed an automorphism,

$$\varphi_q (x \triangleright y) = (x \triangleright y) \triangleright q = (x \triangleright q) \triangleright (y \triangleright q) = \varphi_q (x) \triangleright \varphi_q (y) .$$

The set of inner automorphisms $\left\{\varphi_q\right\}_{q \in Q}$ generates a group of *inner automorphism* denoted by $\mathfrak{Inn}\,(Q)$. The group $\mathfrak{Inn}\,(Q)$ is a subgroup of the group $\mathfrak{Aut}\,(Q)$ of all automorphism of quandle Q. Both cases, $\mathfrak{Inn}\,(Q) = \mathfrak{Aut}\,(Q)$ and $\mathfrak{Inn}\,(Q) \underset{\neq}{\subsetneq} \mathfrak{Aut}\,(Q)$, can occur. For example, equality holds when Q is the involutory quandle of the trefoil knot, and the strict inclusion holds when Q is the involutory quandle of the figure-eight knot.

Definition 4.1.12 *The* algebraic components *of a quandle Q are the orbits under the inner automorphism group Inn Q..*

That is, q, $r \in Q$ are in the same component of Q iff there are $q_i \in Q$ and $e_i = \pm 1$, $i = 1, 2, ..., n$ such that

$$q \triangleright^{e_1} q_1 \triangleright^{e_2} ... \triangleright^{e_{n-1}} q_{n-1} \triangleright^{e_n} q_n = r$$

A quandle associated with a knot has exactly one algebraic component, and a quandle associated to a link has as many algebraic components as the link (Lemma 4.6.8).

4.2 Presentation of Groups and Quandles

Presentations are a convenient way of describing algebraic structures and, as mentioned in Section 3, arise repeatedly in an algebraic approach to knot theory.

Example 4.2.1 *The presentation of the group of the trefoil knot*

$$\langle a,\ b,\ c : b^{-1}ab = c,\ c^{-1}bc = a,\ a^{-1}ca = b\rangle.$$

The symbols a, b, c before the colon are the generators. *The equalities following the colon are the* relations.

Example 4.2.2 *The presentation of a quandle of a trefoil knot*

$$Q = \langle a,\ b,\ c : a \rhd b = c,\ b \rhd c = a,\ c \rhd a = b\rangle.$$

In this section, we discuss a presentation of an algebraic structure from the perspective of the universal algebra [Graetzer and Taylor]. To construct an algebra defined by a presentation, begin with a set called the *universe of words* on the generators. The universe of words on a set of generators is the set of all formal expressions (words) which are obtained by concatenation of the generators using the operations appropriate to the algebra studied (product and inverse for groups; $\rhd$, $\rhd^{-1}$for quandles; $\rhd$ for involutory quandles). Thus, in a group with generators a, b, c such words include a, b, c, ab, ca, a^{-1}, $a\left(b^{-1}\right)$, $(ab)^{-1}$, $(ab)\,c$, $(ab)\,(ba)$, etc. In a quandle with generators a, b, c, the set of words include $a \rhd b$, $c \rhd a$, $a \rhd^{-1} b$, $(a \rhd b) \rhd^{-1} a$, $\left(a \rhd^{-1} b\right) \rhd^{-1} \left(b \rhd^{-1} c\right)$, etc. We record the following for the later reference.

Definition 4.2.3 *The* universe of words $U(S,\ \rhd)$ *on the generating set S and the operation $\rhd$ consists of the elements of S and the word $v \rhd w$ for all pairs of words v, $w \in U(S,\ \rhd)$.*

Definition 4.2.4 *The* universe of words $U\left(S,\ \rhd,\ \rhd^{-1}\right)$ *on the generating set S and the operations $\rhd$, $\rhd^{-1}$consist of the elements of S and the words $v \rhd w$ and $v \rhd^{-1} w$ for each pair of words $v, w \in U\left(S,\ \rhd,\ \rhd^{-1}\right)$.*

These universes are used to present involutory quandles and arbitrary quandles respectively.

Algebraic operations on words in the given universe is defined in a purely formal way. For instance, the product of $(ab)^{-1}$ and c^{-1} is $(ab)^{-1}\,c^{-1}$; the inverse of a^{-1} is $(a^{-1})^{-1}$; the $\rhd$ product of $a \rhd b$ and $a \rhd^{-1} b$ is

$(a \triangleright b) \triangleright \left(a \triangleright^{-1} b\right)$. Evidently, formal operations do not satisfy the axioms for the desired algebra nor the relations given in its presentation. Therefore, *equivalence classes* of words are used. The set of all words of the universe is partitioned into equivalence classes. Each equivalence class is one element of the algebra according to the *equivalence relation* $=$ defined as follows.

1. For any words u, v in the universe, $u = v$ if and only if the identity $u = v$ is either an instance of one of the axioms for the algebra or it is one of the relations given in the presentation.

2. For any words u, v, w in the universe:

   - $u = u$ (reflexivity);
   - if $u = v$ then $v = u$ (symmetry);
   - if $u = v$ and $v = w$ then $u = w$ (transitivity).

3. Well-definedess for the unary operations. If the algebra in question is a group and $u = v$, then $u^{-1} = v^{-1}$.

4. Presentation of an involutory quandle for binary operations. If $u = v$, then for any word w of the universe,

$$uw = vw \text{ and } wu = wv$$

   in the case of a group, and

$$\begin{aligned} u \triangleright w &= v \triangleright w, \\ w \triangleright u &= w \triangleright v, \\ u \triangleright^{-1} w &= v \triangleright^{-1} w, \text{ and} \\ w \triangleright^{-1} u &= w \triangleright^{-1} v \end{aligned}$$

   in the case of a quandle. (The last two equalities are omitted in the case of an involutory quandle.)

The condition 1 ensures the axioms and relations hold for the multiplication of equivalence classes. Condition 2 assures $=$ is in fact an equivalence relation. Conditions 3 and 4 assure the operations on the equivalence classes are well-defined. This means the product xy appears in the same equivalence class no matter which x, y are chosen within their respective equivalence classes.

Remark 4.2.5 *The above definition of a presentation of an algebraic structure is not generally practical for the actual computation. Among the many difficulties, the following are most significant. First, the universe of words is always infinite, even when the resulting presentation is finite. Similarly, all the equivalence classes are infinite sets, as they are the sets of equalities defining the equivalence classes. Second, one needs to have some means for naming the individual equivalence classes while computing. Common practices include assigning numbers to classes or referring to some element selected from the equivalence class. Except for the simplest objects, either of these practices leads to the irregular notation. Third, such methods may be criticized on the grounds they make no use of the special properties of quandles, such as their canonical forms (Section 4.4).*

Diagramming, a more practical computational method, is presented in Sections 4.5- 4.7. Various properties of quandles are incorporated intrinsically in such diagrams and in the method for producing them. Correctness of the practical method is proven in Sections 4.9 - 4.10 by demonstrating its equivalence to the abstract definition given in this section.

Canonicalization, or the reduction of all words to some standard canonical form, aids efficient computation. For instance, in groups we have

$$(xy)^{-1} = y^{-1}x^{-1} \text{ and } x\,(yz) = (xy)\,z.$$

Therefore, any word can be written in the canonical form of a left-associated product of the generators and their inverses. In Section 4.4, we discuss displaying quandles and involutory quandles in canonical form.

The common notion of free object is defined as follows.

Definition 4.2.6 *A* free object *(a group, a quandle, or an involutory quandle) is a presentation of an algebraic structure with no relations.*

Thus, any algebraic structure given via a presentation may be viewed as a free structure on the given set of generators, modulo the equivalence relation generated by the set of relations present in the presentation.

The following lemma can be useful in obtaining the multiplication table corresponding to a given presentation.

Lemma 4.2.7 *Let Q be a quandle (respectively, involutory quandle) defined by a presentation consisting of the generating set S and the set of relations R. Let Q' be a quandle (respectively, involutory quandle) which satisfies the following properties.*

1. *Every element of S is an element of Q'.*
2. *The elements of Q' form a subset of $U\left(S,\ \rhd,\ \rhd^{-1}\right)$ (respectively, $U(S,\ \rhd)$).*
3. *The relations R are satisfied in Q'.*
4. *Whenever $p \rhd q = r$ for $p,\ q,\ r \in Q'$, the equality $p \rhd q = r$ can be derived from the relations R and the axioms of quandles (respectively, involutory quandles) by rules 1 - 4 given above. Then $Q \cong Q'$.*

Proof. Q' is generated by S and satisfies the relations R. Thus, $Q^{'} = h(Q)$ for some homomorphism of quandles h. The homomorphism h is necessarily one-to-one by condition (4) of the lemma, and is onto by condition (2) ■.

Remark 4.2.8 *Lemma 4.2.7 can be generalized in an obvious way for the case of groups and, indeed, to any type of an universal algebra. Condition (4) must be asserted for each operation of the algebra. Note Lemma 4.1.6 automatically asserts condition (4) is for the operation $\rhd^{-1}$ in quandles.*

Remark 4.2.9 *There are particular presentations in which relations R force equality of two elements of S. In such cases, Lemma 4.2.7 holds in the following modified form. Condition (1) is restated, "Every element of S is either an element of Q' or is identified with an element of Q'." Therefore, the following sentence is added to condition (4). "The equalities mentioned in (1) are derived similarly."*

Applying Lemma 4.2.7, the reader may verify the multiplication tables given in Section 4.1 do correspond to the presentations stated there. As a proof, we notice the element e in the second example can be expressed as a word in the generators, $e = a \rhd d$. We shall develop more efficient computational techniques in the following sections.

We use the following notation for presentations. The notation $\langle S : R\rangle$ denotes a presentation for a group. $Q\langle S : R\rangle$ denotes a presentation for a quandle. $IQ\langle S : R\rangle$ denotes a presentation for an involutory quandle in which the axiom $x \rhd y \rhd y = x$ is assumed to hold. $Q_n\langle S : R\rangle$ denotes a presentation for an n-quandle, in which the identity $x \rhd^n y = x$ holds. Note $IQ\langle S : R\rangle = Q_2\langle S : R\rangle$. The two descriptions have the same meaning.

In order to examine homomorphisms of algebraic structures given by presentations (in Sections 4.12, 5.1 and 5.2), we conclude this section with the following Definition and Remark.

Definition 4.2.10 *Let S be the set of generators of quandle Q. Let $q = r$ be a quandle-theoretic (respectively, group-theoretic) relation on S, and let $h : S \to Q'$ be a map from S to a quandle $Q^{'}$ (respectively, group). Then the* image *of the relation $q = r$ via h is the relation $h(q) = h(r)$ obtained by replacing each occurrence of each generator $g \in S$ in $q = r$ by its image $h(g)$.*

For example, if

$$h(a) = a' \text{ and } h(c) = b' \rhd c',$$

then the image of the relation $a \rhd c \rhd a = c$ via h is the relation

$$a' \rhd (b' \rhd c') \rhd a' = (b' \rhd c').$$

Remark 4.2.11 *Let $Q = Q\langle S : R\rangle$ (respectively, $Q_n\langle S : R\rangle$, $\langle S : R\rangle$) be a presentation of quandle Q (respectively, presentation of an n-quandle, presentation of a group). Let $h : S \to Q'$ be any map from the generating set S to a quandle (respectively, n-quandle, group) Q'. Then h extends to a homomorphism $h : Q \to Q'$ of quandles if the image of each relation in R via h is a relation in Q'.*

Remark 4.2.11 holds not only for quandles, n-quandles, and groups, but for every algebraic variety in the universal algebra as well [Graetzer and Taylor]. It can be shown for every relation derivable in Q that the corresponding image via h is a relation derivable in Q'.

4.3 Diagrams of Quandles

In this section, we begin our discussion of diagrams for quandles. A diagram of a quandle Q provides a systematic way for counting elements of a quandle as well as a compact representation of the multiplication table. Intuitive clues are present in the regular geometric forms of many diagrams. Our goal is a method for constructing the diagram directly from the presentation of a quandle or involutory quandle. The method of constructing diagrams for involutory quandles is described in detail in Sections 4.5 - 4.7. In Section 4.8, we apply the method to a number of knots and classes of knots, and we give mathematically sound proof of its correctness in Sections 4.9 - 4.10. Before the full method can be described, we need a number of preliminary notions and results. Therefore, we simply construct a diagram of a quandle using its corresponding multiplication table in this section. We then use knowledge concerning the generating subset $S \subset Q$ to reduce the number of arcs in the diagram of a quandle. In Section 4.5, the reduced form of the diagram does, in fact, represent the entire multiplication table. The reader may also wish to refer to Remark 4.2.5, Remark 4.3.12, and Exercise 4.5.12 concerning alternatives to diagramming a quandle. We begin with the formal definition of a diagram.

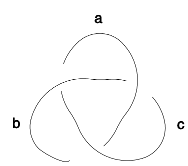
Definition 4.3.1 *The* diagram D *of an involutory quandle* Q *with the generating set* $S \subset Q$ *is the undirected graph whose vertices are labeled by the elements of* Q *using one-to-one correspondence and has an edge labeled by* $s \in S$ *connecting vertices* $p,\ q \in Q$ *whenever* $p \rhd s = q$*.*

Remark 4.3.2 *We use undirected graphs instead of directed graphs in Definition 4.3.1 because for an involutary quandle* $p \rhd s = q$ *iff* $q \rhd s = p$ *by Lemma 4.1.6. However, in the more general case of an arbitrary quandle (see Definition 4.3.9), a directed graph must be used.*

Remark 4.3.3 *We use the following conventions to construct a diagram for a quandle. We draw the arc labeled* a *using a solid arc; the arc labeled* b *by a dashed arc; and the arc labeled* c *by a dotted arc (see Figure* 4.3.1 *bottom center). This visual convention emphasizes the labeling and yields an uncluttered diagram for the quandle.*

Example 4.3.4 *Let* $Q = IQ\langle a,\ b,\ c\ :\ a \rhd b = c,\ b \rhd c = a,\ c \rhd a = b\rangle$ *be a presentation of the involutory quandle for the trefoil knot. Consider first the diagram obtained by taking* $S = \{a,\ b,\ c\} = Q$

(*Figure* 4.3.1, *bottom left*). *This diagram explicitly encodes the entire multiplication table for* Q. *For example, the solid arc from* c *to* b *indicates* $c \triangleright a = b$, *and the dashed arc from* b *back to* b *indicates* $b \triangleright b = b$. *This presentation of the multiplication table for* Q *is convenient since* Q *has only three elements. However, for quandles with more elements, the diagram is more readable if the number of arcs in the diagram can be reduced. We achieve this by reducing the size of the generating set* S, *so only arcs labeled with elements of* S *will remain in the diagram. In the present example, the set* S *can also be reduced to a set with only* 2 *elements* $\{a, b\}$. *Using the set* $\{a, b\}$, *we obtain the diagram at the bottom right of Figure* 4.3.1.



Relations:

$a \triangleright b = c$

$b \triangleright c = a$

$c \triangleright a = b$

$\triangleright$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

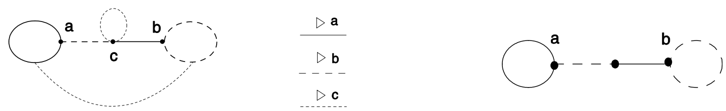


Figure 4.3.1. *Diagram of the involutory quandle for the trefoil knot*

In such a reduced diagram, the multiplication by c *is no longer explicitly indicated. However, the entire multiplication table remains encoded into the reduced diagram of* Q *and is retrieved in Section 4.4 - 4.5. In order to write a desired product as a canonical left-associated product of generators during this retrieving process, the following identity is applied*:

$$x \triangleright (y \triangleright z) = ((x \triangleright z) \triangleright y) \triangleright z.$$

This identity follows from axioms 2 *and* 3 *(see Lemma 4.4.1). Recall quandles are non-associative in general (see Example 4.1.3). We now illustrate computation possible by using diagrams. In order to multiply* $a \triangleright c$ *using a reduced diagram of* Q, *we first use the diagram to read an expression for* c *in terms of the generators* $a, b \in S$. *The expression is* $c = a \triangleright b$. *In order to find the product* $a \triangleright c$, *we first substitute* $c = a \triangleright b$. *Thus, we have*

$$a \triangleright c = a \triangleright (a \triangleright b)\,.$$

We apply the above identity to represent the result of the multiplication as a left-associated product:

$$a \triangleright (a \triangleright b) = ((a \triangleright b) \triangleright a) \triangleright b.$$

Therefore, we have

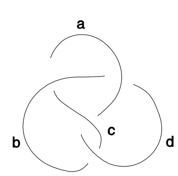
$$a \triangleright c = ((a \triangleright b) \triangleright a) \triangleright b.$$

To find the vertex of $((a \triangleright b) \triangleright a) \triangleright b$ which corresponds to the diagram of Q, trace from the vertex a along the dashed arc $\triangleright b$, the solid arc $\triangleright a$, and around the dashed arc $\triangleright b$ to the vertex b. Thus, we have

$$a \triangleright c = ((a \triangleright b) \triangleright a) \triangleright b = b.$$

The value of $b \triangleright c$ can be found similarly, and of course $c \triangleright c = c$.

Example 4.3.5 *Let $Q = I\langle a,\, b,\, c,\, d : a \triangleright b = c,\, c \triangleright d = b,\, b \triangleright a = d,\, d \triangleright c = a\rangle$, the involutory quandle of the figure-eight knot. Let $S = \{a,\, b\}$. We diagram Q in Figure 4.3.2. The reader may wish to construct the corresponding diagram in which $S = Q$ and observe the effect on readability of the diagram.*



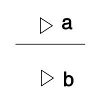
Relations:

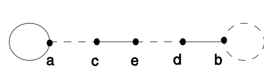
$a \triangleright b = c$

$b \triangleright a = d$

$c \triangleright d = b$

$d \triangleright c = a$





$\triangleright$	a	b	c	d	e
a	a	c	d	e	b
b	d	b	e	c	a
c	e	a	c	b	d
d	b	e	a	d	c
e	c	d	b	a	e

Figure 4.3.2. *Diagram of the involutory quandle of the figure-eight knot*

Remark 4.3.6 *The diagram of an involutory quandle Q generated by $S \subset Q$ has the following properties.*

1. *For every $q \in Q$ and $s \in S$, exactly one arc labeled by s meets the vertex labeled by q. (This arc may meet the vertex q at one or both ends.)*

2. *For every $s \in S$, there is a vertex labeled by s met by the arc labeled s at the both ends.*

In regard to these properties, Figure 4.3.3 illustrates examples of labeled graphs failing to be diagrams of involutory quandles.

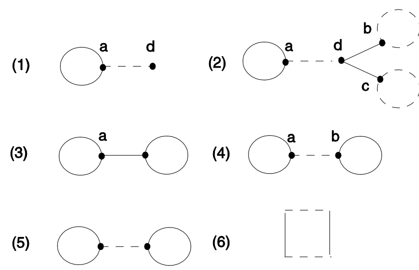


Figure 4.3.3. *Non-diagrams*

As before, we draw the arc labeled by a using a solid arc and the arc labeled by b using a dashed arc. Example (1) is not a diagram of an involutory quandle because no solid arc meets the right hand vertex labeled by d. In algebraic terms, no value is assigned to the product $d \triangleright a$. Example (2) is not a diagram of an involutory quandle because two different solid arcs meet the center vertex d. Algebraically, this means two distinct values are assigned to the product $d \triangleright a : b = d \triangleright a = e$. However, $b \neq e$. Similarly, in example (3), two different solid arcs meet the vertex a. Example (4) is not a diagram of an involutory quandle because no arc labeled by b meets the vertex b at both ends. Similarly, examples (5) and (6) cannot be diagrams of involutory quandles no matter how we label their vertices. In both examples, the presence of a dashed arc indicates the presence of the generator b, but no vertex can be labeled b because none of the vertices is met at the both ends by a dashed arc.

Remark 4.3.7 *Some diagrams of involutory quandles may not be connected graphs, as shown in Figure* 4.3.4. *Neither of the connected components taken by itself is a diagram of an involutory quandle. Non-connected diagrams are discussed in more details in Section 4.6.*

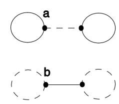


Figure. 4.3.4. A two-component diagram

Remark 4.3.8 *The conditions of Remark 4.3.6 do not suffice to ensure a labeled graph is the diagram of some involutory quandle. Consider the two-component counterexample of Figure* 4.3.5.

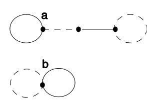


Figure 4.3.5. A cryptic non-diagram

From the diagram, $b \triangleright a = b$*. Hence, using algebraic reasoning,*

$$\begin{aligned} a \triangleright b &= a \triangleright (b \triangleright a) \ (\textit{equality substitution}) \\ &= (a \triangleright a) \triangleright (b \triangleright a) \ (\textit{using axiom } 1) \\ &= (a \triangleright b) \triangleright a \ (\textit{using axiom } 3). \end{aligned}$$

in contradiction to the diagram which indicates $a \triangleright b \neq (a \triangleright b) \triangleright a$.

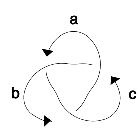
We generalize the notion of a diagram of an arbitrary quandle as follows.

Definition 4.3.9 *The diagram* D *of a non-involutory quandle* Q *generated by the set* $S \subset Q$ *is a directed graph whose vertices are labeled by the elements of* Q *using one-to-one correspondence between vertices and elements of* S *and has an arc directed from* p *to* q *labeled by* s *whenever* $p \triangleright s = q$*, where* $p, q \in Q$*,* $s \in S$.

Example 4.3.10 *The quandle for the trefoil knot,*

$$Q = \langle a,\ b,\ c : a \triangleright b = c,\ b \triangleright c = a,\ c \triangleright a = b \rangle,$$

has infinite cardinality and is inconvenient to diagram. In such a circumstances, it may be useful to consider certain homomorphic images of Q. For example, if we add relations $x \triangleright^3 y = x$ for all $x, y \in Q$ (Definition 4.1.5) to the presentation of Q, we obtain a 3-quandle which is also a knot invariant. In the case of the trefoil knot, such a homomorphic image is finite and has the multiplication table and the diagram shown in Figure 4.3.6. (The table for $\triangleright^{-1}$ can be obtained from the diagram or by applying Lemma 4.1.6).



$\triangleright$	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

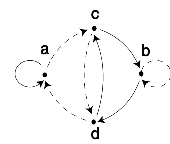
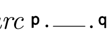
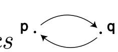
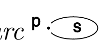
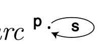


Figure 4.3.6. 3-*quandle for the trefoil knot*

Remark 4.3.11 *The diagram of an involutory quandle Q generated by S (Definition 4.3.1) can be converted to the directed diagram (Definition 4.3.9) as follows. Whenever an undirected arc connects two distinct vertices, we replace it by two directed arcs. Whenever an undirected arc*  *connects two distinct vertices, replace the undirected arc by two directed arcs* *. Whenever an undirected arc*  *connects a vertex to itself, replace the undirected arc with one directed arc* *.*

Remark 4.3.12 *In [Joyce1982a], a different type of a diagram is considered for various involutory quandles. Such a diagram is similar to those constructed in this section in that the diagram represents the multiplication tables, uses vertices to represent the elements, and uses arcs to represent multiplication. Such a diagram differs from the diagram in this section in the following ways.*

1. An arc of such a diagram may contain three or more vertices rather than just two.

2. The arcs of such a diagram are not labeled.

3. Such a diagram explicitly describes all entries of the multiplication table rather than just multiplication by generators.

4. Such diagrams could be difficult to draw for the large and infinite involutory quandles presented in Section 4.8.

5. On the other hand, such diagrams for small quandles have symmetries which may not be visible in the diagrams constructed in this section.

6. Such diagrams may not be easily applied to arbitrary non-involutory quandles.

7. Apparently, there is no simple procedure to construct such a diagram directly from a given presentation of a quandle. However, Sections 4.5 - 4.7 present relatively easy methods to construct the diagrams defined in this section.

Our next objective is to use the diagrams of involutory quandles constructed in this section to calculate products of any two elements–to find an algorithm for calculating the corresponding multiplication table of quandle Q. In order to do so, we will use the method of canonical left association as discussed in Section 4.4.

4.4 Canonical Forms for Products

In this section, we develop a method of representing products of elements in a quandle Q in the canonical left-associated form. For any free quandle or an involutory quandle Q, we obtain a one-to-one correspondence between the set of elements of Q and a subset of all canonical left-associated products. Representation of elements of a quandle Q in the canonical left-associated form yields an algorithm for computing products of elements of Q. Section 4.5 uses the canonical left-associated forms of elements of Q for the study of diagrams. Recall the convention that any product written without parentheses is assumed to be left-associated. For example, $a \rhd b \rhd c \rhd d$ means $((a \rhd b) \rhd c) \rhd d$.

Lemma 4.4.1 *In any involutory quandle Q, the following left association identity holds*:

$$x \rhd (y \rhd z) = ((x \rhd z) \rhd y) \rhd z$$

Proof. Using axiom 2,

$$x \rhd (y \rhd z) = ((x \rhd z) \rhd z) \rhd (y \rhd z)\,,$$

and using axiom 3,

$$((x \rhd z) \rhd z) \rhd (y \rhd z) = ((x \rhd z) \rhd y) \rhd z.$$

Thus, we have

$$x \rhd (y \rhd z) = ((x \rhd z) \rhd y) \rhd z.$$

■

Note an alternative set of axioms for involutory quandles is obtained by replacing axiom 3 by the left association identity just proved. Proof that axiom 3 (self-distributivity) is obtained from this alternate set of axioms is left to the reader.

Additionally, note Lemma 4.4.1 can be generalized to the product of any two elements of quandle Q expressed in the left-associated forms as follows.

Lemma 4.4.2 *For every involutory quandle Q the following identity holds*:

$$(a_0 \rhd a_1 \rhd ... \rhd a_m) \rhd (b_0 \rhd b_1 \rhd ... \rhd b_n) = a_0 \rhd a_1 \rhd ... \rhd a_m \rhd b_n \rhd ... \rhd b_1 \rhd b_0 \rhd b_1 \rhd ... \rhd b_n.$$

Proof. The above identity is obtained by repeated application of Lemma 4.4.1. For instance:

$$a \rhd (b_0 \rhd b_1 \rhd b_2) = a \rhd b_2 \rhd b_1 \rhd b_0 \rhd b_1 \rhd b_2.$$

Any product of elements of an involutory quandle can be expressed in the left-associated form by repeated use of Lemma 4.4.2. ■

Example 4.4.3 *What are the benefits of representing elements of an involutary quandle Q in the left-associated form? Consider the following exercise. Show the following identity holds in any involutory quandle*:

$$(x \rhd y) \rhd (y \rhd x) = (x \rhd (x \rhd y)) \rhd x.$$

This identity follows directly from the axioms. First, establish the following identity

$$((x \rhd y) \rhd x) \rhd y = x \rhd (x \rhd y).$$

Proof. We have

$$\begin{aligned} ((x \rhd y) \rhd x) \rhd y &= ((x \rhd y) \rhd y) \rhd (x \rhd y) \text{ (by axiom 3)} \\ &= x \rhd (x \rhd y) \quad \text{(by axiom 2)} \end{aligned}$$

Now we derive the desired equality.

$$\begin{aligned} (x \rhd y) \rhd (y \rhd x) &= (((x \rhd y) \rhd x) \rhd x) \rhd (y \rhd x) \quad \text{(by axiom 2)} \\ &= (((x \rhd y) \rhd x) \rhd y) \rhd x \text{ (by axiom 3)} \\ &= (x \rhd (x \rhd y)) \rhd x \text{ (by the identity just established).} \end{aligned}$$

Alternatively, show

$$(x \rhd y) \rhd (y \rhd x) = (x \rhd (x \rhd y)) \rhd x$$

using the left association identity. More precisely, the representation of the both sides of the identity as left-associated products yields the canonical form, $x \rhd y \rhd x \rhd y \rhd x$. ■

Theorem 4.4.4 *A free involutory quandle Q generated by the set S is an algebraic structure of which the elements are left-associated products of the form*

$$a_0 \triangleright a_1 \triangleright ... \triangleright a_n, \;\; n \geq 0,$$

where $a_i \in S$, $a_i \neq a_{i+1}$ for $0 \leq i \leq n-1$. Distinct left-associated products correspond to distinct elements of Q. The multiplication in Q is given as in Lemma 4.4.2 with the use of right cancellation and idempotency to remove adjacent occurrences of the same generator.

Note the idempotency property (axiom 1) of quandles allows for removal of only one occurrence of a generator, while the right cancellation property (axiom 2) allows for removal of two occurrences of a generator. In [Joyce1982b], the author describes free involutory quandles in terms of groups.

Proof. Let Q' denote an algebraic structure described in the statement of the theorem. Q' is an involutory quandle; Q' satisfies axiom 2. Let $a = a_0 \triangleright a_1 \triangleright ... \triangleright a_m$ and $b = b_0 \triangleright b_1 \triangleright ... \triangleright b_n$ then we have

$$(a \triangleright b) \triangleright b = a_0 \triangleright a_1 \triangleright ... \triangleright a_m \triangleright b_n \triangleright ... \triangleright b_1 \triangleright b_0 \triangleright b_1 \triangleright ... \triangleright b_n \triangleright b_n \triangleright ... \triangleright b_1 \triangleright b_0 \triangleright b_1 \triangleright ... \triangleright b_n$$

(by Lemma 4.4.2)

$$= a_0 \triangleright a_1 \triangleright ... \triangleright a_m$$

(by repeated right cancellation)

$$= a.$$

Analogous computations verify axioms 1 and 3. Q' evidently satisfies conditions (1) - (4)of Lemma 4.2.7. Hence, $Q = Q'$. ■

Corollary 4.4.5 *The result of representing a product in the canonical left-associated form using the left association and axioms* 1 *and* 2 *is independent of the order in which sub-expressions are represented in the canonical form.*

Proof. Suppose different orders of representing sub-expressions of a given product in the canonical left-associated form yield different final canonical forms. Distinct left-associated products representing different elements of a free involutary quandle by Theorem 4.4.4 contradicts the fact both canonical forms represent the same product (an element of a free quandle). ■

Remark 4.4.6 *The expression resulting from representing an element of a free quandle in the canonical left-associated form is unique even when axiom* 1 *is not included in the set of rules for representing quandles in canonical form.*

Now we generalize the above results to the case of an arbitrary quandle (not necessarily an involutary quandle).

Lemma 4.4.7 *Let Q be a quandle. Then, for all x, y, $z \in Q$, the following identities are equivalent.*

1. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (*axiom* 3 *for quandles*)
2. $x \triangleright (y \triangleright z) = \left(\left(x \triangleright^{-1} z\right) \triangleright y\right) \triangleright z$
3. $(x \triangleright y) \triangleright^{-1} z = \left(x \triangleright^{-1} z\right) \triangleright \left(y \triangleright^{-1} z\right)$
4. $x \triangleright \left(y \triangleright^{-1} z\right) = ((x \triangleright z) \triangleright y) \triangleright^{-1} z$
5. $\left(x \triangleright^{-1} y\right) \triangleright^{-1} z = \left(x \triangleright^{-1} z\right) \triangleright^{-1} \left(y \triangleright^{-1} z\right)$
6. $x \triangleright^{-1} \left(y \triangleright^{-1} z\right) = \left((x \triangleright z) \triangleright^{-1} y\right) \triangleright^{-1} z$
7. $\left(x \triangleright^{-1} y\right) \triangleright z = (x \triangleright z) \triangleright^{-1} (y \triangleright z)$
8. $x \triangleright^{-1} (y \triangleright z) = \left(\left(x \triangleright^{-1} z\right) \triangleright^{-1} y\right) \triangleright z$

We note alternate axiom sets for quandles can be obtained by replacing axiom 3 with any one of the identities (2) - (8).

Proof. In our proof we use axiom 2.

(1) $\Longrightarrow$ (2) : We have

$$x \triangleright (y \triangleright z) = \left(\left(x \triangleright^{-1} z\right) \triangleright z\right) \triangleright (y \triangleright z) = \left(\left(x \triangleright^{-1} z\right) \triangleright y\right) \triangleright z \text{ by (1).}$$

(2) $\Longrightarrow$ (3) : We have

$$\begin{aligned} (x \triangleright y) \triangleright^{-1} z &= \left(x \triangleright \left(\left(y \triangleright^{-1} z\right) \triangleright z\right)\right) \triangleright^{-1} z = \left(\left(x \triangleright^{-1} z\right) \triangleright \left(y \triangleright^{-1} z\right)\right) \triangleright z \triangleright^{-1} z \text{ by (2)} \\ &= \left(x \triangleright^{-1} z\right) \triangleright \left(y \triangleright^{-1} z\right). \end{aligned}$$

$(3) \Longrightarrow (4)$: We use a similar argument as in the proof $(1) \Longrightarrow (2)$.

$(4) \Longrightarrow (5)$: We have

$$\begin{aligned}
\left(x \rhd^{-1} y\right) \rhd^{-1} z &= \left[\left(\left(x \rhd^{-1} y\right) \rhd^{-1} z\right) \rhd \left(y \rhd^{-1} z\right)\right] \rhd^{-1} \left(y \rhd^{-1} z\right) \\
&= \left[x \rhd^{-1} y \rhd^{-1} z \rhd z \rhd y^{-1} z\right] \rhd^{-1} \left(y \rhd^{-1} z\right) \text{ by (4)} \\
&= \left(x \rhd^{-1} z\right) \rhd^{-1} \left(y \rhd^{-1} z\right).
\end{aligned}$$

$(5) \Longrightarrow (6) \Longrightarrow (7) \Longrightarrow (8) \Longrightarrow (1)$: Reverse the roles of $\rhd$, $\rhd^{-1}$ in $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5)$.

■

The result of Lemma 4.4.7 generalizes the remark $(1) \Longrightarrow (3), (5), (7)$ mentioned in [Joyce1979].

With the aid of a modified notation we can capture identities (2), (4), (6), (8) as one. Let $a \rhd^e b$ denote $a \rhd b$ when $e = +1$ and $a \rhd^{-1} b$ when $e = -1$. Then the left association identity

$$x \rhd^d (y \rhd^e z) = \left(\left(x \rhd^{-e} z\right) \rhd^d y\right) \rhd^e z$$

captures all four identities (2), (4), (6), (8). The following result corresponds to Lemma 4.4.2.

Lemma 4.4.8 *In any quandle Q,*

$$\begin{aligned}
&\left(a_0 \rhd^{d_1} a_1 \rhd^{d_2} ... \rhd^{d_m} a_m\right) \rhd^{e_0} (b_0 \rhd^{e_1} b_1 \rhd^{e_2} ... \rhd^{e_n} b_n) \\
= \;& a_0 \rhd^{d_1} a_1 \rhd^{d_2} ... \rhd^{d_m} a_m \rhd^{-e_n} b_n \rhd^{-e_{n-1}} ... \rhd^{-e_1} b_1 \rhd^{e_0} b_0 \rhd^{e_1} b_1 \rhd^{e_2} ... \rhd^{e_n} b_n.
\end{aligned}$$

Proof. The identity given in the lemma is a consequence of repeated application of the left association identity just given ■

As in the case of involutory quandles, any product of elements of a quandle can be expressed in the canonical left-associated form by repeated application of Lemma 4.4.8. The following theorem for quandles corresponds to the result obtained for involutary quandles in Theorem 4.4.4.

Theorem 4.4.9 *A free quandle Q generated by S is an algebraic structure in which the elements are all left-associated products*

$$a_0 \rhd^{e_1} a_1 \rhd^{e_2} ... \rhd^{e_n} a_n, \; n \geqslant 0,$$

where $a_i \in S$, $a_0 \neq a_1$, and whenever $a_i = a_{i+1}$, then $e_i = e_{i+1}$, $1 \leqslant i \leqslant n-1$. Distinct left-associated products represent distinct elements of Q. The multiplication of left-associated products in Q is given using Lemma 4.4.8 followed by the use of cancellation and idempotency to remove adjacent occurrences of the same generator where possible.

Note Joyce describes free quandles in terms of groups [Joyce1982b].

Proof. The argument is analogous to the argument used in the proof of Theorem 4.4.4. ■

Remark 4.4.10 (*Fundamental asymmetry.*) *We conclude this section with a discussion of the fundamental asymmetry between the right and left arguments of quandle operations $\triangleright$, $\triangleright^{-1}$. As we have shown, any product of elements of a quandle can be expressed in the canonical left-associated form. No corresponding canonical right-associated form for products of quandle elements exists.*

First, consider symmetry between right and left arguments in the axioms defining quandles. Although the first axiom, $x \triangleright x = x$, possesses a symmetry between left and right arguments of the operation $\triangleright$, the second axiom, $(x \triangleright y) \triangleright^{-1} y = x = \left(x \triangleright^{-1} y\right) \triangleright y$, has no such symmetries. Exchanging roles of right and left arguments in this second axiom, we obtain

$$y \triangleright^{-1} (y \triangleright x) \stackrel{?}{=} x \stackrel{?}{=} y \triangleright \left(y \triangleright^{-1} x\right).$$

The first quandle given in Figure 4.4.1 does not satisfy this identity. Thus axioms defining quandles show fundamental asymmetry between right and left arguments. Since the example given in Figure 4.4.1 is an involutary quandle, axioms defining involutary quandles are also asymmetric.

Finally, we examine the possible existence of the right-associated canonical form for products of elements of quandles. The second quandle in Figure 4.4.1 provides a counterexample to such existence. The elements

a, b, c generate this quandle. In particular, $f = (c \rhd a) \rhd b$. However, f cannot be written as a right-associated product of generators a, b, c. Since this quandle is also an involutory quandle, we conclude there is no right-associated canonical form for products of elements for both quandles and involutary quandles.

$\rhd, \rhd^{-1}$	a	b
a	a	a
b	b	b

$\rhd, \rhd^{-1}$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	b	b	b	b	b	b
c	d	e	c	c	c	c
d	c	f	d	d	d	d
e	f	c	e	e	e	e
f	e	d	f	f	f	f

Figure 4.4.1. *Left-right asymmetry in quandles*

4.5 Diagramming of Presentations of Involutory Quandles - Tracing Relations

In this section and the two that follow, we present an algorithm for constructing diagrams for involutory quandles directly from their presentations. The method is formalized in Definition 4.7.1 with the aid of Definitions 4.5.3 and 4.6.10. Correctness of our algorithm is discussed in Sections 4.9-4.10. In Section 4.11, we extend our method to arbitrary quandles. The method partially overcomes the difficulties alluded to in Remark 4.2.5 and Exercise 4.5.12.

A main feature of the diagramming method is it allows construction of such a diagram directly from the given presentation of a quandle and it does not require, as the intermediate step does, construction of the multiplication table for the quandle. In fact, as shown below, the diagram contains all the information present in a multiplication table.

Given the diagram of an involutory quandle Q generated by the set S and two vertices labeled by p, q in the diagram, the vertex $p \triangleright q$ is found as follows. From the diagram, read an expression for q as a left-associated product of generators $q = g_0 \triangleright g_1 \triangleright ... \triangleright g_n$. From the vertex p, follow a sequence of arcs g_n, g_{n-1}, ..., g_1, g_0, g_1, ..., g_n connecting vertex p and vertex q, that is

$$p \triangleright g_n \triangleright ... \triangleright g_1 \triangleright g_0 \triangleright g_1 \triangleright ... \triangleright g_n = p \triangleright q.$$

Example 4.5.1 *Consider the diagram given in Figure* 4.5.1. *To calculate $d \triangleright e$, read from the diagram*

$$e = a \triangleright b \triangleright a.$$

Then,

$$d \triangleright e = d \triangleright (a \triangleright b \triangleright a) = d \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a.$$

Applying Lemma 4.4.2 we obtain

$$d \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a = c$$

by following the sequence of arcs a, b, a, b, a from d to arrive at c.

Remark 4.5.2 *For any $q \in Q$, at least one expression for q as a product of the generators of Q is read from the diagram. For instance, q can be expressed as a left-associated product of a corresponding initial vertex v and the sequence of arcs in the diagram connecting v to q.*

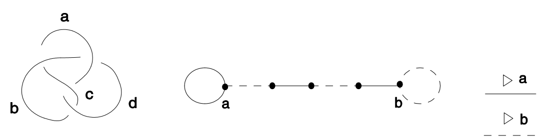


Figure 4.5.1. Figure eight knot and diagram of involutory quandle

Now we present our construction of a diagram of Q directly from the corresponding presentation of Q. The construction of the diagram from the presentation of an involutory quandle $Q = IQ\langle S : R \rangle$ generated by S starts from the rudimentary partial diagram having one vertex labeled g for each generator $g \in S$ and no arcs (see Figure 4.5.2, *top). This partial diagram is then expanded and completed by successively incorporating the relations given in the presentation of Q and their algebraic consequences. The process of incorporating a relation into the diagram of Q is called* tracing *the relation.*

Definition 4.5.3 *In the construction of the diagram of an involutory quandle Q generated by the set S,* trace *the relation,*

$$a_0 \triangleright a_1 \triangleright ... \triangleright a_m = b_0 \triangleright b_1 \triangleright ... \triangleright b_n, \quad a_i,\ b_i \in S,$$

by performing the following sequence of operations:

1. *Locate the vertex $a_0 \triangleright a_1 \triangleright ... \triangleright a_m$ in the existing partial diagram. That is, locate the vertex a_0 and trace from that vertex along arcs labeled $a_1, ..., a_m$ to the desired vertex. If some vertex $a_0 \triangleright a_1 \triangleright ... \triangleright a_m$, $0 < i < m$, is not met by any arc labeled a_{i+1}, adjoin the required sequence of arcs $a_{i+1}, ..., a_m$ and corresponding new vertices to the diagram.*

2. *In the same manner, find the vertex $b_0 \triangleright b_1 \triangleright ... \triangleright b_n$.*

3. *Merge* (*make identical*) *the two vertices located in steps* (1) *and* (2).

4. *For any two merged vertices p, q and for every $s \in S$, merge the vertices $p \triangleright s$ and $q \triangleright s$* (*if both are present in the partial diagram*).

Remark 4.5.4 *Tracing relations has the following properties:*

(i) *The two merged vertices represent algebraically equal elements of Q.*

(ii) *Any further merged vertices also follows algebraically.*

(iii) *The relations satisfied before tracing a given relation in the previous diagram will still hold in the new diagram.*

Remark 4.5.5 *Step* (4) *may be repeated a number of times during a single tracing depending on given circumstances. This suggests a major difficulty in diagramming–how to determine whether two seemingly distinct vertices must be merged at some point. For infinite diagrams, the effect of an infinite number of identifications must be considered. Under certain circumstances, unsolvable problems may arise due to the absence of a solution for the word problem concerning groups. Fortunately, there is a solution for the word problem concerning knot groups [Waldhausen 1968], so we will not encounter such difficulties in this paper.*

We now produce an actual diagram by tracing relations.

Example 4.5.6 *We shall construct a diagram for the involutory quandle given by the presentation*

$$IQ\langle a, b : (a \rhd b) \rhd a = b, \ \ (b \rhd a) \rhd b = a\rangle.$$

We start with the partial diagram with two vertices labeled with the generators a, b (see Figure 4.5.2*, top). Trace the relation $(a \rhd b) \rhd a = b$ (see Figure* 4.5.2*, center). Since the vertices $a \rhd b$ and $(a \rhd b) \rhd a$ are not present in the partial diagram, we add them to the existing partial diagram together with the corresponding arcs. The vertex $(a \rhd b) \rhd a$ is then identified with the vertex labeled b since the relation $(a \rhd b) \rhd a = b$ is present in the presentation of the quandle. As shown in the diagram, $a \rhd b = b \rhd a$. Algebraically, this is a consequence of the traced relation. In general, various consequences of the original relations become evident as the diagram is produced.*

Next, trace the other relation $(b \rhd a) \rhd b = a$. The vertex $(b \rhd a) \rhd b$ is already present in the partial diagram (since $a \rhd b = b \rhd a$) and, in fact, is the vertex a. Thus tracing the second relation produces no further effect on the existing diagram. Finally, we trace the relations $a \rhd a = a$ and $b \rhd b = b$, which are the instances of axiom 1 (Figure 4.5.2, bottom).

The resulting diagram satisfies the elementary conditions stated in Remark 4.3.6. Although not yet proven, the resulting diagram is actually the complete diagram for the given presentation. We still must prove the given relations and all algebraic consequences are already incorporated into this diagram. The necessary proof is straightforward after full discussion of our diagramming method (see Sections 4.5-4.7) and once correctness of the method is proved in general (see Sections 4.9-4.10).

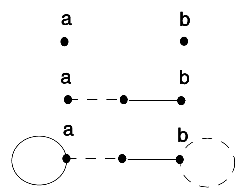


Figure 4.5.2. *Diagram of* $IQ\langle a, b : a \triangleright b \triangleright a = b, b \triangleright a \triangleright b = a\rangle$

Now we show how to streamline the process of tracing relations.

Remark 4.5.7 *In tracing a relation, the number of identifications of existing vertices can be minimized as follows. First, the intermediate step between constructing a new vertex and merging can be omitted (Figure* 4.5.3*, top). Second, both construction and merging may further be reduced applying Lemma 4.1.6 to rewrite the relation before tracing it. For example, the relation* $a \triangleright b \triangleright c \triangleright b = e$ *can be rewritten as* $a \triangleright b = e \triangleright b \triangleright c$*. If the vertex* $e \triangleright b \triangleright c$ *is already present in the partial diagram, use of the latter form simplifies the construction and merging (Figure* 4.5.3*, bottom).*

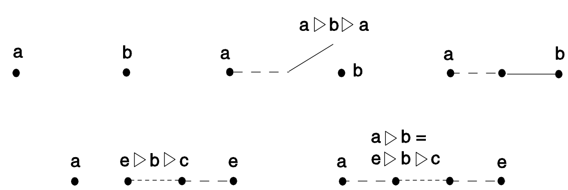


Figure 4.5.3. Streamlined tracing (Remark *4.5.7*)

Remark 4.5.8 *When using this streamlined method of tracing, beware of incorporating an extraneous relation that does not follow from the relation being traced. For example, consider tracing of the relation* $a \triangleright b \triangleright a \triangleright b = a$ *in the diagram at the top of Figure* 4.5.4*. The diagram at the center of Figure* 4.5.4 *is not the* *correct result, even though the relation $a \triangleright b \triangleright a \triangleright b = a$ is satisfied in this diagram. Unfortunately, the extraneous*

relation $a \triangleright b = a$ is incorporated. The diagram at the bottom of Figure 4.5.4 is the correct result of tracing the relation. Similar caution is given for a more subtle case in Remark 4.7.6

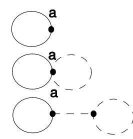


Figure 4.5.4. Incorrect and correct tracing of the relation $a \triangleright b \triangleright a \triangleright b = a$ (Remark 4.5.8)

Now we construct diagrams for the involutory quandles of certain knots.

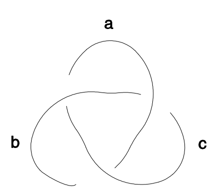
Example 4.5.9 *The trefoil knot and the presentation of its involutory quandle is shown in Figure 4.5.5. In order to obtain a more readable diagram, it is desirable to eliminate generators wherever possible. In this example, the generator c can be eliminated using the first relation, so the remaining relations become*

$$b \triangleright (a \triangleright b) = a \text{ and } (a \triangleright b) \triangleright a = b.$$

Elimination of generators is described in general in Remark 4.5.10. Left association (Lemma 4.4.1) yields

$$b \triangleright a \triangleright b = a \text{ and } a \triangleright b \triangleright a = b.$$

The above relations are examined in Example 4.5.6. The diagram (Figures 4.5.2, 4.5.5) is identical to the one obtained from the multiplication table in Example 4.3.4.



Generators: a, b, c

Relations:

$a \triangleright b = c$

$b \triangleright c = a$

$c \triangleright a = b$

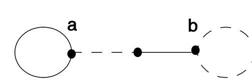
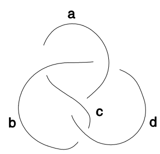


Figure 4.5.5. The trefoil knot, generators and relations, and diagram of its involutory quandle

Remark 4.5.10 (*Elimination of generators.*) *The presentation $IQ\langle S : g = t,\ R\rangle$, where S is a set of generators $g \in S$, t is a product over $S - \{g\}$, and R is a set of relations, defines the same involutary* *quandle as the quandle given by the presentation $IQ\langle S : R'\rangle$, where R' is obtained from R by replacing every occurrence of g by t in every relation of R.* (*This Remark applies to presentation quandles, $n-$quandles, groups, and involutory quandles.*)



Generators:

a, b, c, d

Relations:

$a \rhd b = c$

$b \rhd a = d$

$b \rhd d = c$

$a \rhd c = d$

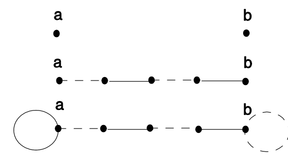


Figure 4.5.6. The figure-eight knot, presentation of its involutary quandle and its corresponding diagram

Remark 4.5.11 *The figure-eight knot and the presentation of its involutory quandle is shown in Figure* 4.5.6*. The generators c, d can be eliminated using the first two relations. The remaining two relations become:*

$$b \rhd (b \rhd a) = a \rhd b \text{ and } a \rhd (a \rhd b) = b \rhd a$$

or, when written in the left-associated form,

$$b \rhd a \rhd b \rhd a = a \rhd b \text{ and } a \rhd b \rhd a \rhd b = b \rhd a.$$

On the initial diagram consisting of vertices labeled a, b, we trace the above relations then the relations $a \rhd a = a$, $b \rhd b = b$ (Figure 4.5.6*). As in Example 4.5.6, the second of the relations traced, namely $a \rhd b \rhd a \rhd b = b \rhd a$, is already incorporated into the diagram by tracing the first relation.* (*This phenomenon is not universal in knot and link quandles; see Examples 4.6.1 and 4.6.9.*) *The diagram is the same as the one obtained from the multiplication table given in Example 4.3.5. As with Example 4.5.6, proof all derivable relations have been incorporated must await development of the necessary machinery.*

Exercise 4.5.12 *Construct the multiplication table for the preceding example as an interesting exercise in understanding the power of the diagramming method. Begin by writing the known products in a table form* (*Figure* 4.5.7). *Dots in Figure* 4.5.7 *reflect the general necessity of the table as it is developed. Next, derive other product values by use of the known values in the table, axioms, and lemmas, such as Lemma 4.1.6, and enter these into the table. The product $c \rhd a$ cannot be assigned any of the elements a, b, c, d, so* *let* $c \triangleright a = e$ *be an additional element which we put into the table. Now further calculations close off the table without the need for naming additional elements. Verify the table produced in such a way satisfies the axioms for an involutory quandle in all instances* (*Definition 4.1.2*) . *Correctness of the table follows from Lemma 4.2.7.*

	a	b	c	d	.	.	.
a	a	c	d				
b	d	b		c			
c			c				
d				d			
.					.		
.						.	
.							.

Figure 4.5.7. Computation of a quandle multiplication table (Remark *4.5.12*)

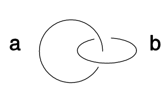
In the next section, we focus on more complex instances of applying the diagramming method.

4.6 Diagramming with Multiple Components and Secondary Relations

In this section, we construct diagrams more complex than those of the preceding section. The complexities are of three types. First, a diagram may have two or more connected components. Such diagrams arise from the involutory quandles of links (Example 4.6.1). Second, tracing of the given relations and idempotency may not suffice to complete the diagram (Example 4.6.9). Additional derived relations, called *secondary relations* (Definition 4.6.10), must be traced. Given relations, secondary relations, and idempotency suffice to complete the diagram for any presentation of an involutory quandle independent of the order in which they are traced (Theorem 4.10.14). Third, we construct an infinite diagram (Example 4.6.9). The process of diagramming reveals the repeating pattern that defines the infinite structure.

Example 4.6.1 *Diagram the involutory quandle of the simplest nontrivial link* (*Figure* 4.6.1)*. None of the generators* (*see Figure* 4.6.1) *can be eliminated. The construction of the diagram is straightforward*: *trace the relations and* $a \triangleright a = a$, $b \triangleright b = b$. *The involutory quandle has two generators as in Examples 4.5.6 and 4.5.11, but in the the present example, the second relation is not automatically incorporated by tracing the first relation. The diagram is not connected, a point now examined in detail.*

Definition 4.6.2 *The* components *of the diagram of an involutory quandle are the connected components of the underlying graph.*



Generators:

a, b

Relations:

$a \triangleright b = a$

$b \triangleright a = b$

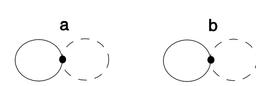


Figure 4.6.1. *The simplest nontrivial link, generators and relations, and diagram of the involutory quandle*

Remark 4.6.3 *In the diagram of any involutory quandle* Q *generated by* $S \subset Q$*, the nodes* p, $q \in Q$ *lie in the same connected component of the diagram iff* $q = p \triangleright g_1 \triangleright g_2 \triangleright ... \triangleright g_n$ *for some* $g_1, g_2, ..., g_n \in S$.

Remark 4.6.4 *In the diagram of an involutory quandle Q generated by S, each connected component contains at least one vertex labeled by a generator $g \in S$. (To see this, write any element q of the component as a left-associated product of generators and apply Remark 4.6.3.)*

To diagram link quandles in a useful manner, relate the components of the link diagram to the algebraic components (Definition 4.1.12).

Lemma 4.6.5 *The connected components of the diagram of an involutory quandle Q generated by S correspond to the algebraic components of Q.*

Proof. If p, q are in the same connected component of the diagram of Q, then evidently they are in the same algebraic component. Conversely, suppose p, q are in the same algebraic component. Then

$$q = p \rhd r_1 \rhd r_2 \rhd \ldots \rhd r_n, \quad r_i \in Q.$$

Writing each r_i as a left-associated product of generators and then left associating to the resulting product (Lemma 4.4.2), we obtain

$$q = p \rhd g_1 \rhd g_2 \rhd \ldots \rhd g_m$$

for some $g_1, g_2, \ldots, g_m \in S$. Thus p, q are in the same connected component of the diagram by Remark 4.6.3.

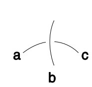
■

Remark 4.6.6 *The algebraic components of an involutory quandle Q are in one-to-one correspondence with the elements of the homomorphic image Q' formed by imposing the relation $x \rhd y = x$ for all x, $y \in Q$.*

Remark 4.6.7 *Any homomorphic image Q' of an involutory quandle Q contains no more components than does Q. In particular, if p, q are in the same connected component of Q, then their images are in the same component of Q'.*

Lemma 4.6.8 *If a tame link L has n components, then its involutory quandle $IQ(L)$ has n components.*

Proof. There are no more algebraic components of $IQ(L)$ than generators (see Remark 4.6.3). Generators associated with the same component of the link are placed in the same component of $IQ(L)$ by the relations:

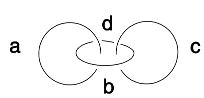


$$a \rhd b = c$$

Thus $IQ(L)$ has at most n components. The involutory quandle Q' consisting of one element for each component of the link L and satisfies relations $x \triangleright y = x$ for all x, y is a homomorphic image of $IQ(L)$ (because it satisfies all relations read from the projection of L) and has n components. Hence, $IQ(L)$ has exactly n components. ■

Now we focus on the construction of an infinite diagram with multiple components. Tracing just the relations read from the link projection will not be sufficient in constructing the diagram. Other derived relations called *secondary relations* must be traced to obtain the correct diagram.

Example 4.6.9 *We examine the link of three components, including the generators and relations, in Figure* 4.6.2. *The generator d can be eliminated using the relation $b \triangleright a = d$, so the relation $b \triangleright c = d$ becomes $b \triangleright c = b \triangleright a$. The two remaining relations are unchanged. Begin the diagram with the three vertices a, b, c and trace the given relations and idempotency* (*Figure* 4.6.3)*.*



Generators:

a, b, c, d

Relations:

$$\begin{array}{c} a \triangleright b = a \\ c \triangleright b = c \\ b \triangleright a = d \\ b \triangleright c = d \end{array} \quad \left\{ \begin{array}{c} a \triangleright b = a \\ c \triangleright b = c \\ b \triangleright a = b \triangleright c \end{array} \right\}$$

Figure 4.6.2. *A three-component link, generators, and relations for the involutory quandle*

These operations do not produce the final diagram in that the elements $a \triangleright c$, $b \triangleright$, and $c \triangleright a$ are not present in this diagram. Furthermore, certain relations can be derived algebraically but are not incorporated into the diagram. For example, the relation $c \triangleright b = c$ implies

$$a \triangleright (c \triangleright b) = a \triangleright c$$

or, through left association,

$$a \triangleright b \triangleright c \triangleright b = a \triangleright c,$$

but this relation is not incorporated in the diagram. In order to avoid this kind of omission we introduce the notion of *secondary relations*.

Definition 4.6.10 *Let $IQ\langle S : R\rangle$ be a presentation of an involutory quandle. The primary relations of the presentation are represented as R. The* secondary relations *for the presentation are the relations*

$$x \rhd s = x \rhd t \textit{ for all } x \in Q,$$

whenever the relation $s = t$ is an element of R. We denote the set of secondary relations by R_S. The idempotency relations of the presentation are the relations $g \rhd g = g$, $g \in S$. We denote this set of relations by I_S. In most contexts, it is convenient to assume the relations of R, R_S are in the canonical left-associated, right-cancelled form (Lemma 4.4.2, Remark 4.4.6). The variable x ranges over all left-associated, right-cancelled products in the generators.

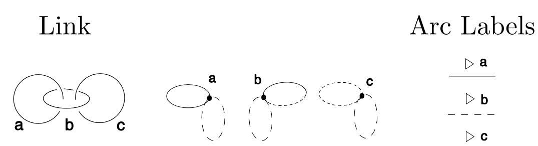


Figure 4.6.3. *Partial diagram of $IQ(L)$ for the link in Figure* 4.6.2

In our example, we have the following secondary relations corresponding to the primary relation $c \rhd b = c$.

$$x \rhd (c \rhd b) = x \rhd c \text{ for all } x \in Q,$$

or, using the left-associated form,

$$x \rhd b \rhd c \rhd b = x \rhd c \text{ for all } x \in Q.$$

The relation captures the instance $a \rhd b \rhd c \rhd b = a \rhd c$ missing in the partial diagram given above.

We shall trace the primary relations as well as the secondary relations for each vertex x in the diagram. This procedure will incorporate more of the necessary relations, but will it incorporate all the derivable relations? For example, do we also need to trace the relations $x' \rhd (x \rhd b \rhd c \rhd b) = x' \rhd (x \rhd c)$ for all x, $x' \in Q$ and so on ad infinitum? As shown in Sections 4.9-4.10, tracing the secondary relations in addition to the primary relations and idempotency is sufficient as all other derivable relations are incorporated automatically. We now obtain and trace the secondary relations for our example.

Primary relation	*Secondary relation*
$a \triangleright b = a$	$x \triangleright b \triangleright a \triangleright b = x \triangleright a$, $x \in Q$
$c \triangleright b = c$	$x \triangleright b \triangleright c \triangleright b = x \triangleright c$, $x \in Q$
$b \triangleright c = b \triangleright a$	$x \triangleright c \triangleright b \triangleright c = x \triangleright a \triangleright b \triangleright a$, $x \in Q$.

Trace the first of these secondary relations at $x = b$–that is, trace the relation $b \triangleright b \triangleright a \triangleright b = b \triangleright a$. This completes the "$b$" component (Figure 4.6.4, bottom, left). The reader may verify all three secondary relations are satisfied at both vertices of this component.

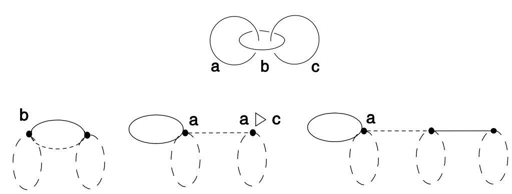


Figure 4.6.4. *Diagram of the involutory quandle of a link*

Next, we turn to the "a" component. Consider first $x = a$; the first of the secondary relations is already satisfied at this vertex (Figure 4.6.3). However, the second relation is not. Trace $a \triangleright b \triangleright c \triangleright b = a \triangleright c$ to obtain the partial diagram of Figure 4.6.4, bottom, center. Note the products $a \triangleright b \triangleright c$ and $a \triangleright c$ correspond to the same vertex; tracing simply adds a dashed loop. Next, consider $x = a \triangleright c$. Trace the first secondary relation at the vertex,

$$(a \triangleright c) \triangleright b \triangleright a \triangleright b = (a \triangleright c) \triangleright a,$$

to obtain the partial diagram in Figure 4.6.4, right. The second relation is already satisfied for $x = a \triangleright c$. Continuing vertex by vertex in this fashion, we obtain the entire infinite "a" component shown in Figure 4.6.5.

The reader may verify each of the three secondary relations is satisfied at each vertex x of this component. The "c" component is constructed similarly. Correctness of the completed diagram given in Figure 4.6.5 follows from Theorem 4.10.14.

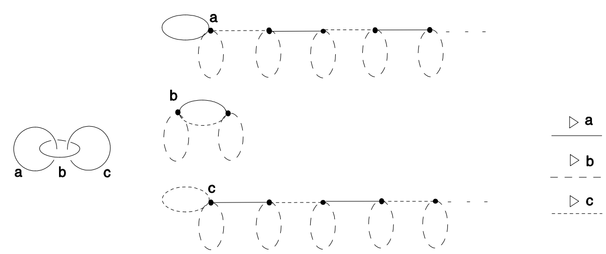


Figure 4.6.5. *Diagram of the infinite involutory quandle of a link*

Remark 4.6.11 *In the link of Example 4.6.9 and Figures* 4.6.3, 4.6.5, *the arcs* b, d *correspond to distinct elements of the involutory quandle* (*hence to distinct elements of the quandle as well*). *This follows from distinctness of* b, $b \triangleright a = d$ *in the diagram. By contrast, the meridians at* b, d *are homotopic and, therefore, equal in the knot group* (*Figure* 4.6.6).

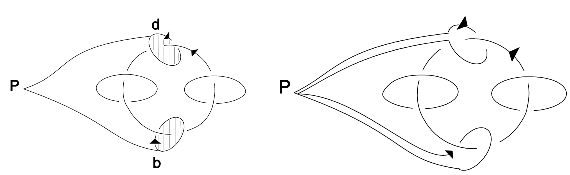


Figure 4.6.6.*Distinct quandle elements b,d, homotopic meridians*

We have now seen the essentials of the diagramming method. The method is summarized and applied to a more difficult example in the following section.

4.7 The Diagramming Method Defined and Applied to the Borromean Rings

In this section, we provide a formal definition of the diagramming method (Definition 4.7.1). Then we state lemmas making computation more efficient. Finally, we diagram the involutory quandle of the Borromean rings, the prime link of fewest crossings with an infinite involutory quandle (Example 4.7.5). The diagramming method is extended to arbitrary, not necessarily involutory, quandles in Section 4.11.

Definition 4.7.1 *A* diagram *of a quandle given by the presentation $IQ\langle S : R\rangle$ is a labeled undirected graph obtained as follows. Begin with the graph consisting of one vertex labeled g for each generator $g \in S$ and no arcs. Well-order the union of the following sets: the primary relations R, the secondary relations R_S, and the idempotency relations I_S (Definition 4.6.10). Trace these relations in the chosen order (Definition 4.5.3).*

We shall show (Theorem 4.10.14) the diagram constructed using the *presentation* $IQ\langle S : R\rangle$ is independent of the choice of well-ordering and is, in fact, the diagram of the involutory *quandle* $Q = IQ\langle S : R\rangle$ generated by S (Definition 4.3.1). Note Definition 4.7.1 streamlines the computation implicit in Definition 4.3.1 and in the definition of presentation of an algebraic structure (Section 4.2). In particular, a finite diagram can be constructed applying a finite number of tracings even when the number of relations derivable from the given set R is infinite.

Remark 4.7.2 *The notion of the limiting partial diagram corresponding to a limit ordinal is implicit in the notion of tracing an infinite well-ordered set of relations. Such a limit is formed as follows. The set of vertices V of the limit diagram consists of those vertices present in the initial diagram and those constructed prior to the limit ordinal with the omission of each vertex merged with a previously constructed vertex. The set of arcs of the limit diagram contains each arc present between vertices of V at any step prior to the limit ordinal.*

This notion is used for the Borromean rings in Example 4.7.5, in which the idempotency relations I_S are traced after all other relations in the infinite set $R \cup R_S$.

Now we turn to some conputationally useful results.

Lemma 4.7.3 *The following primary relations are equivalent for an involutory quandle.*

1. $p \triangleright q_1 \triangleright ... \triangleright q_n = r$
2. $p \triangleright q_1 \triangleright ... \triangleright q_i = r \triangleright q_n \triangleright q_{n-1} \triangleright ... \triangleright q_{i+1}$ *for* $1 \leqslant i \leqslant n-1$
3. $p = r \triangleright q_n \triangleright q_{n-1} \triangleright ... \triangleright q_1$.

Proof. The proof is shown by repeated applications of Lemma 4.1.6. ■

Any primary relation

$$p \triangleright q_1 \triangleright ... \triangleright q_n = r$$

and the corresponding secondary relation

$$x \triangleright q_n \triangleright ... \triangleright q_1 \triangleright p \triangleright q_1 \triangleright ... \triangleright q_n = x \triangleright r \text{ (for all } x\text{)}$$

are related in the diagram as shown in Figure 4.7.1. The primary relation, a chain traceable only from the specific point p, "opens out" into a loop traced anywhere in the diagram.

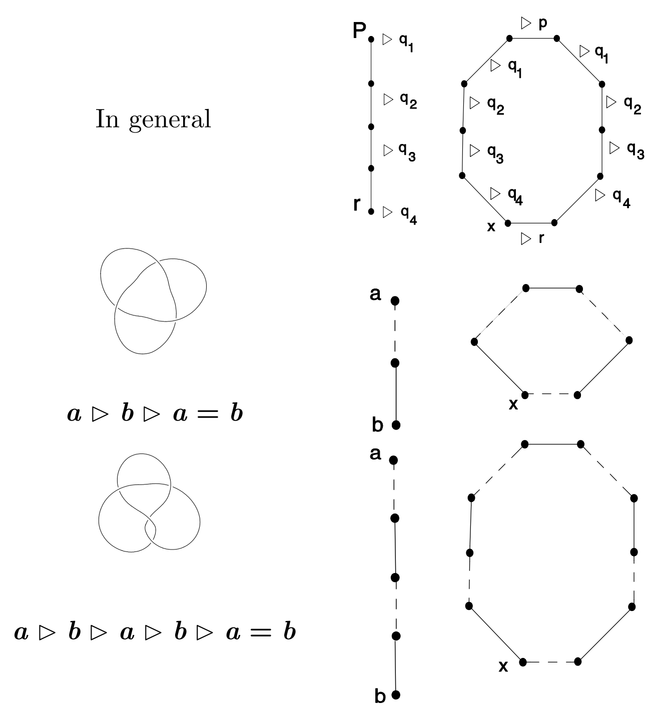


Figure 4.7.1. *Primary and secondary relations*

Lemma 4.7.4 (Equivalent forms for a secondary relation) *For an involutory quandle Q, the following relations are equivalent.*

1. $x \triangleright q_1 \triangleright ... \triangleright q_n = x$, $x \in Q$
2. $x \triangleright q_1 \triangleright ... \triangleright q_1 = x \triangleright q_n \triangleright q_{n-1} \triangleright ... \triangleright q_{i+1}$ (*any* $1 \leqslant i \leqslant n-1$), $x \in Q$
3. $x \triangleright q_{i+1} \triangleright q_{i+2} \triangleright ... \triangleright q_n \triangleright q_1 \triangleright ... \triangleright q_i = x$ (*any* $1 \leqslant i \leqslant n-1$), $x \in Q$
4. $x \triangleright q_n \triangleright q_{n-1} \triangleright ... \triangleright q_1 = x$, $x \in Q$.

Proof. Repeated application of Lemma 4.1.6 yields equivalence of the first, second, and fourth forms. The third form is obtained from the second by substituting $x \triangleright q_{i+1} \triangleright q_{i+2} \triangleright ... \triangleright q_n$ for x in the second form and then cancelling. The second form can be obtained similarly from the third. ■

Note there is no distinction among q, q_i, r in the above lemma. Our notation is chosen to emphasize the cyclic rearrangements, rather than to emphasize the connection with the primary relations. The last two equivalent forms correspond to the rotational and reflectional symmetries of an n-sided polygon. We illustrate $n = 6$ diagrammatically in Figure 4.7.2.

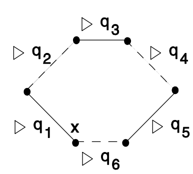


Figure 4.7.2. *Secondary relations and polygonal symmetry*

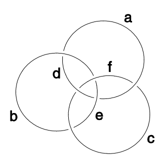
Now we address the first major question posed in our investigation: Does a prime knot or link with an infinite involutory quandle exist? We shall diagram the infinite involutory quandle of the Borromean rings. The study of this quandle led to the development of the diagramming method presented here. In the next section, we address the question of minimality in terms of the number of crossings, for an infinite involutory quandle.

Example 4.7.5 *The Borromean rings and a presentation of its involutory quandle are illustrated in Figure* 4.7.3. *The three relations in the center of the figure express the generators d, e, f in terms of*

a, b, c. *Eliminating* d, e, f *from the presentation and expressing relations in the left-associated form yields:*

	Primary relations		Corresponding secondary relations are obtained by left multiplication by x
1.	$a \triangleright b \triangleright c \triangleright b \triangleright c = a$	$1s$	$x \triangleright c \triangleright b \triangleright c \triangleright b \triangleright a \triangleright b \triangleright c \triangleright b \triangleright c = x \triangleright a$
2.	$b \triangleright c \triangleright a \triangleright c \triangleright a = b$	$2s$	$x \triangleright a \triangleright c \triangleright a \triangleright c \triangleright b \triangleright c \triangleright a \triangleright c \triangleright a = x \triangleright b$
3.	$c \triangleright a \triangleright b \triangleright a \triangleright b = c$	$3s$	$x \triangleright b \triangleright a \triangleright b \triangleright a \triangleright c \triangleright a \triangleright b \triangleright a \triangleright b = x \triangleright c$

We now construct the diagram of the involutory quandle $IQ(L)$ of the Borromean rings generated by $S = \{a, b, c\}$ with the aid of the above relations. The diagram will consist of three components corresponding to the three components of the link (see Lemma 4.6.8). Each component of the diagram will include exactly one vertex labeled with a generator–a, b, or c respectively. Focus on the component containing vertex a; the other two components are obtained from the component containing vertex a by cyclic permutation of the roles of a, b, c.



$$a \triangleright b = d \qquad d \triangleright e = a$$
$$b \triangleright c = e \qquad e \triangleright f = b$$
$$c \triangleright a = f \qquad f \triangleright d = c$$

Figure 4.7.3. *Borromean rings, generators and relations for the involutory quandle*

To begin construction of the "a" component, we trace the primary relation (1) $a \triangleright b \triangleright c \triangleright b \triangleright c = a$ (Figure 4.7.4, left). The two remaining primary relations (2), (3) affect the "a" component indirectly through the secondary relations (2_S), (3_S). Note the delay in tracing idempotency relations $a \triangleright a = a$ for easier construction (Remark 4.7.7). Trace the secondary relation (1_S), which yields the polygon (Lemma 4.7.4) illustrated in the center of Figure 4.7.4. When (1_S) is traced at the vertex $x_1 = a \triangleright b \triangleright c$ of the partial diagram (Figure 4.7.4, left) it gives rise to the second diamond (Figure 4.7.4, right). We illustrate the pinching induced by (1) $a \triangleright b \triangleright c \triangleright b \triangleright c = a$; the merging of $a \triangleright b \triangleright c \triangleright a$ and $a \triangleright c \triangleright b \triangleright a$; and the reversal of the lower diamond (for a more pleasant appearance of the diagram later). In the same manner, trace the relation (1_S) for $x_2 = a \triangleright c$ and $x_3 = a \triangleright c \triangleright b \triangleright a \triangleright c$ (Figure 4.7.5). The next step, closing the near-octagon in Figure 4.7.5, cannot be done by simply tracing (1_S) once more.

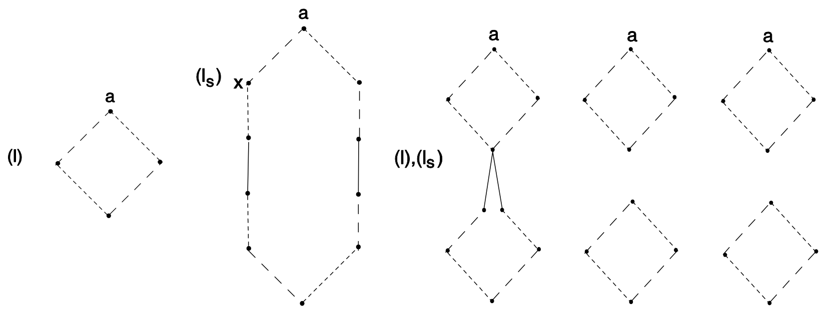


Figure 4.7.4. *Tracing relations* (1) *and* (1_S)

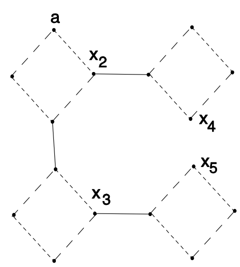


Figure 4.7.5. *Further tracing of relation* (1_S) *for* x_2, x_3

Remark 4.7.6 (*Tracing of the relation*) (1_S) $x \triangleright c \triangleright b \triangleright c \triangleright b \triangleright a \triangleright b \triangleright c \triangleright b \triangleright c = x \triangleright a$ *does not suffice to show* $x_4 \triangleright a = x_5$ *in Figure* 4.7.5. *If* (1_S) *is traced at* x_4*, an additional diamond is created* (*Figure* 4.7.6*, left*)*, but* (1_S) *does not justify merging this new diamond with the* x_5 *diamond.*

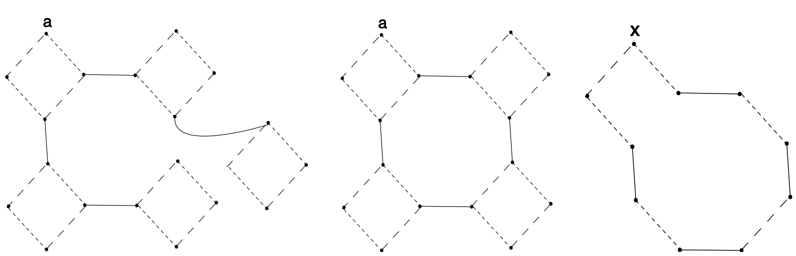


Figure 4.7.6. *Closing of octagon by relation* (2_S)

In order to connect x_4 to x_5, closing the octagon, we rewrite relation (2_S) applying Lemma 4.7.4 as

$$x \triangleright c \triangleright a \triangleright c \triangleright a = x \triangleright b \triangleright c \triangleright a \triangleright c \triangleright a \triangleright b$$

and we trace this relation for $x = a$ (Figure 4.7.6, right). We now have a pattern for the diamond-and-octagon tiling of the plane (Figure 4.7.7). The entire tiling is obtained by repeatedly tracing (1_S) to adjoin diamonds both horizontally and vertically and (2_S) close octagons.

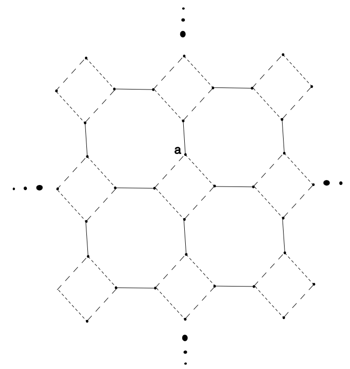


Figure 4.7.7. *Diamond-and-octagon tiling of the plane, obtained in the process of diagramming the involutory quandle of the Borromean rings.*

Remark 4.7.7 *The relations* (1_S) , (2_S) , *and* (3_S) *hold throughout the partial diagram shown in Figure* 4.7.7. *By the symmetry of the diagram it suffices to verify* $(1_S) - (3_S)$ *at just one vertex. For instance, at* $x = a$. *We have delayed tracing of idempotency in order to retain this useful symmetry.*

With the secondary relations verified throughout, we trace idempotency relations $a \triangleright a = a$. This tracing merges not only $a \triangleright a$ with a, but also $a \triangleright a \triangleright g_1 \triangleright ... \triangleright g_n$ with $a \triangleright g_1 \triangleright ... \triangleright g_n$, $g_i \in \{a, b, c\}$. Diametrically opposite pairs of vertices are merged, rolling the diagram into the cone illustrated in the flattened form in Figure 4.7.8. This is the "a" component of the diagram. The "b" and "c" components are obtained from it by cyclic permutation of the roles of a, b, c. (Change the label of the topmost vertex and change solid arcs to dashed, dashed to dotted, and dotted to solid.) The involutory quandle thus constructed clearly has an infinite cardinality.

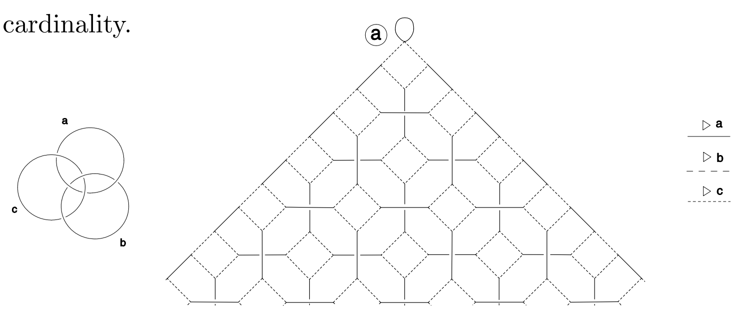


Figure 4.7.8. The Borromean rings and diagram of the involutory quandle "a" component

4.8 Examples

In this section, we examine the involutory quandles of two-bridge (rational) knots, certain three-bridge knots, and certain Whitehead links. Through this examination, we find a prime knot K with infinite involutory quandle $IQ(K)$ must have at least 8 crossings in the projection of K. For certain three-bridge knots K, called pretzel knots, the diagram of $IQ(K)$ is related to a tiling of a plane, hyperbolic plane, or sphere with polygons. Finally, we distinguish certain Whitehead links with homeomorphic complements by examining their involutory quandles.

We begin by examining n-bridge knots and links in general (Figure 4.8.1). Such a knot or link K consists of a braid of $2n$ strands capped at each end by n arcs. $Q(K), IQ(K), Q_m(K)$ is presented using only n generators as follows. Label the left-hand n cap arcs with the generators $a_1, ..., a_n$. At each undercrossing x under y, label the new arc with the corresponding product in the generators, $x \triangleright y$ or $x \triangleright^{-1} y$. Read off a relation at each right-hand capping arc. Examples of such presentations for 2- and 3-bridge knots are provided below. We conjecture the presentations employ the minimum possible number of generators.

Conjecture 4.8.1 *A tame knot or link K is n-bridge but not $(n-1)$-bridge iff its quandle $Q(K)$ (respectively, $IQ(K)$, respectively, $Q_m(K)$) is generated by some set of n generators but by no set of $n-1$ generators.*

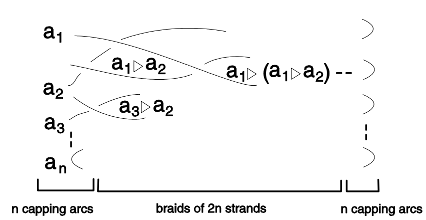


Figure 4.8.1. *General 2n-strand knot. The involutory quandle can be presented as follows. The n capping arcs at left are labeled $a_1, ..., a_n$. Arcs in the braid are labeled with products of these. A relation is read at each of n capping arcs at the right.*

K is the trivial knot if and only if it is 1-bridge. The conjecture for $n = 1$ is proved in [Joyce1982a] for $Q(K)$ and below for $IQ(K), Q_m(K)$ (Theorem 5.2.5).

The following notation will be useful in our discussion. We write $x\,(\triangleright\, y \triangleright z)^n$ where n is a repetition factor

$$x\,(\triangleright\, y \triangleright z)^0 = x,\ x(\triangleright\, y \triangleright z)^{n+1} = x\,(\triangleright\, y \triangleright z)^n \triangleright y \triangleright z.$$

For $n \leqslant 0$,

$$x\,(\triangleright\, y \triangleright z)^{n-1} = x\,(\triangleright\, y \triangleright z)^n \triangleright^{-1} z \triangleright^{-1} y,$$

and recall $\triangleright$ and $\triangleright^{-1}$ are the same in involutory quandles.

Now we discuss the involutory quandles of four-strand (viergeflecht) knots and links K omitting details of computation. Figure 4.8.2 is a typical example. In general, a 4-strand braid is capped at both ends. At one end, we label the capping arcs a, b. The involutory quandle $IQ\,(K)$ is finite, or "abelian", and has order equal to the knot determinant $\mid K \mid$, see [Joyce1982a]. However, finiteness of $IQ\,(K)$ does not imply that K is a 4-strand braid (see Example 4.8.2 below).

What is the minimum number of crossings for a prime knot with infinite involutory quandle? Every prime knot of 7 or fewer crossings is 4-strand and, therefore, has finite involutory quandle. On the other hand, $IQ\,(8_{16})$ is infinite, establishing 8 as the minimum. Construction of the involved diagram of $IQ\,(8_{16})$ is deferred to another paper. The infinite $IQ\,(9_{35})$ is diagrammed in Example 4.8.3 below.

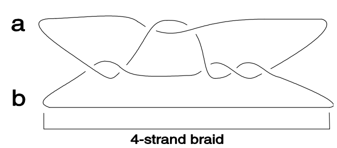


Figure 4.8.2. A 4-strand knot.

Among prime links with infinite involutory quandle, the Borromean rings (Example 4.7.5) has the minimal number of crossings. All prime links with fewer crossings are 4-strand.The smallest non-prime knot and link each have infinite involutory quandle. The diagram for the smallest non-prime link is illustrated in Example 4.6.9 above, while those for the square and granny knots are left to the reader. Any disconnected link (e.g. two unlinked circles) also has infinite involutory quandle.

Now we turn to the involutory quandles of pretzel knots and links $K_{k,m,n}$ (Figure 4.8.3). The integers k, m, n count half-twists. For pretzel knots, apply the convention that positive values of k, m, n denote twists of the form , while negative values of k, m, n denote . (Note, while k, m, n positive yields an alternating knot, all crossings thus obtained may not be positive in the same sense used in Section 3.1. Consider $K_{2,3,5}$ of Figure 4.8.3). We assume none of k, m, n is zero or ± 1 to avoid degenerate cases. Any permutation of k, m, n yields the same knot or link. For example, $K_{2,3,5}$, $K_{3,2,5}$ and $K_{3,5,2}$ are the same knot.

The pretzel knots and links fall into four classes based on evenness/oddness of k, m, n.

1. k, m, n are all odd, $K_{k,m,n}$ is a knot

2. exactly one of k, m, n is even, $K_{k,m,n}$ is a knot

3. exactly two of k, m, n are even, $K_{k,m,n}$ is a link of two components

4. k, m, n are all even, $K_{k,m,n}$ is a link of three components.

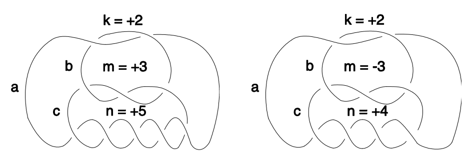


Figure 4.8.3. Pretzel knots and links

Now we state general results concerning the involutory quandles of pretzel knots and links. $IQ\,(K_{k,m,n})$ is infinite if and only if

$$\frac{1}{k} + \frac{1}{m} + \frac{1}{n} \leqslant 1.$$

(For reasons of space, we omit the proof, which involves polygonal tiling, spherical, planar, and hyperbolic surfaces and case analysis on evenness/oddness of k, m, n.) We obtain a convenient presentation for $IQ\,(K_{k,m,n})$ by labeling the three arcs to the left of the twists with a, b, c as in Figure 4.8.3. All generators, except a, b, c, can be eliminated. For case (1) k, m, n odd, we obtain the following primary relations.

- $a\,(\rhd\, a \rhd c)\,\frac{n+1}{2} = b\,(\rhd\, c \rhd b)\,\frac{m-1}{2}$

- $b\,(\rhd\, b \rhd a)\,\frac{k+1}{2} = c(\rhd\, a \rhd c)\frac{n-1}{2}$

- $c\,(\rhd\, c \rhd b)\,\frac{m+1}{2} = a\,(\rhd\, b \rhd a)\,\frac{k-1}{2}$

Calculation of similar formulas for the three remaining cases is left to the reader. In all cases, the secondary relations can be written

- $x\,(\triangleright\, a \triangleright b)^k = x\,(\triangleright\, b \triangleright c)^m = x\,(\triangleright\, c \triangleright a)^n$ for all x.

A certain homomorphic image of $IQ\,(K_{k,m,n})$ yields particularly pleasing diagrams and is obtained by setting

$$x\,(\triangleright\, a \triangleright b)^k = x\,(\triangleright\, b \triangleright c)^m = x\,(\triangleright\, c \triangleright a)^n = x \text{ for all } x.$$

Tracing this relation throughout one component produces a partial diagram (a tiling of a sphere, plane, or hyperbolic plane according to whether $\frac{1}{k} + \frac{1}{m} + \frac{1}{n}$ is greater than, equal to, or less than one) composed of polygons of $2k$, $2m$, and $2n$ sides. (If the three quantities in the secondary relation are not equated to x, a third dimension of depth appears in the diagram but is well-behaved. The polygons of $2k$, $2m$, $2n$ sides lift to parallel helices. The criterion for infiniteness remains the same, but the proof involves analysis of the four evenness/oddness cases and is omitted here.) Now we turn to examples of specific $K_{k,m,n}$.

Example 4.8.2 *The knot* $8_5 = K_{2,3,3}$ *is shown at left in Figure* 4.8.4. *Primary relations for* $Q = IQ\,(K_{2,3,3})$ *are*

- $c \triangleright a \triangleright c = a \triangleright b \triangleright a,$
- $a \triangleright c \triangleright a \triangleright c = b \triangleright c \triangleright b$, and
- $b \triangleright a = c \triangleright b \triangleright c \triangleright b.$

Secondary relations are

- $x\,(\triangleright\, a \triangleright b)^2 = x\,(\triangleright\, b \triangleright c)^3 = x\,(\triangleright\, c \triangleright a)^3$ for all x.

Obtain the diagram of the homomorphic image Q' of Q by setting

$$x\,(\triangleright\, a \triangleright b)^2 = x\,(\triangleright\, b \triangleright c)^3 = x\,(\triangleright\, c \triangleright a)^3 = x \text{ for all } x.$$

When this relation is diagrammed, the result is a spherical tiling (Figure 4.8.4, center). Tracing of the primary relations and idempotency yields the diagram of Figure 4.8.4, right. The mapping $Q \to Q'$ can be shown to be 7-to-1 pointwise. Making the diagram for Q is a more complicated version than shown for Q'.

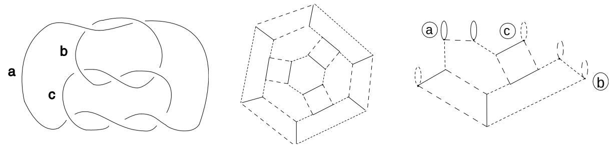


Figure 4.8.4. *The knot* $8_5 = K_{2,3,3}$ *with diagrams of homomorphic images of secondary relations and of involutory quandle* (*Example 4.8.2*).

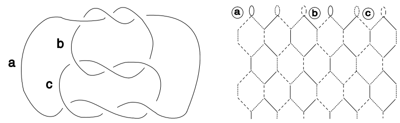


Figure 4.8.5. *The knot* $9_{35} = K_{3,3,3}$ *and diagram of homomorphic image of* $IQ\,(9_{35})$ (*Example 4.8.3*). *The left edge of the diagram shown is to be glued to the right edge to form a cylinder, which extends downward to infinity.*

Example 4.8.3 *The knot* $9_{35} = K_{3,3,3}$ *is shown at left in Figure* 4.8.5. *The primary relations for* $Q = IQ\,(K_{3,3,3})$ *have the form given above for case* (1) k, m, n *odd. The secondary relations have the general form given in Figure* 4.8.5 *for pretzel knots. We obtain the diagram of the homomorphic image* Q' *of* Q *by setting*

$$x\,(\rhd a \rhd b)^3 = x\,(\rhd b \rhd c)^3 = x\,(\rhd c \rhd a)^3 = x \text{ for all } x.$$

When this relation is diagrammed, the result is the familiar tiling of the plane with hexagons. Tracing of the primary relations yields the infinite diagram of Figure 4.8.5, *right. Note the left edge is to be "glued" to the right edge to form a cylinder. The homomorphic mapping* $Q \to Q'$ *is infinite-to-one at all points. The diagram of* Q *has an added dimension* (*perpendicular to the page or thickening the cylinder obtained after gluing*), *and the hexagons lift to helices.*

Example 4.8.4 *The knot* $K_{2,3,7}$ *is shown in Figure* 4.8.6. *The primary relations for* $Q = IQ\,(K_{2,3,7})$ *are as follows.*

- $(a \rhd c)\,(\rhd a \rhd c)^3 = b \rhd c \rhd b$
- $a \rhd b \rhd a = c\,(\rhd a \rhd c)^3$
- $b \rhd a = c \rhd b \rhd c \rhd b$

The secondary relations have the general form for pretzel knots. We obtain the diagram of the homomorphic image Q' of Q by setting

- $x\,(\triangleright\, a \triangleright b)^2 = x\,(\triangleright\, b \triangleright c)^3 = x\,(\triangleright\, c \triangleright a)^7 = x$ for all x.

Diagramming this relation yields a tiling of the hyperbolic plane with squares, hexagons, and 14-sided polygons. Tracing of the primary relations yields the infinite diagram of Figure 4.8.7. Note the two copies of the illustrations are to be taken one above the other and the edges be knitted together left to left and right to right to form an infinite cylinder.

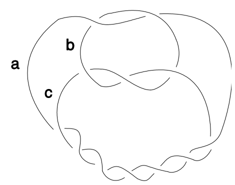


Figure 4.8.6. *The knot* $K_{2,3,7}$.

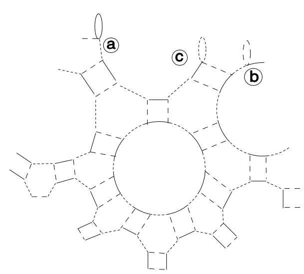


Figure 4.8.7. *Half diagram of a homomorphic image of* $IQ\,(K_{2,3,7})$ *(Example 4.8.4). To obtain the correct diagram from the half shown here: Form identical second half just over that shown. Glue the left edges of the two halves together and glue the right edges together. The result is a cylinder (flared cylinder of hyperbolic surface). Vertices* a, b *lie in the lower half;* c *in the upper half. The diagram contains an infinite number of vertices.*

Example 4.8.5 *Finally, we distinguish certain links with homeomorphic complements. Consider the class of links illustrated in Figure* 4.8.8. *The complements are homeomorphic as follows. The parameter* n, *the number of half-twists, can be adjusted by* ± 2 *by cutting along the shaded disk, twisting* 360^0, *and re-pasting. Thus the complements for* n *even are all homeomorphic. The complements for* n *odd are all homeomorphic* *to the complements for n even because n, $-n-1$ half-twists yield mirror-image links. By contrast, these links are all distinguished by the involutory quandle (up to mirror images). The primary relations are*

- $b \rhd a = b \rhd c$,
- $c\,(\rhd\, c \rhd a)^{\lceil \frac{n}{2} \rceil} \rhd b \rhd c \rhd b = a\,(\rhd\, c \rhd a)^{\lfloor \frac{n}{2} \rfloor}$, and
- $(c \rhd b)\,(\rhd\, a \rhd c)^{n} \rhd a = a \rhd b$.

where $\lceil \frac{n}{2} \rceil$ is the smallest integer $\geqslant \frac{n}{2}$ and $\lfloor \frac{n}{2} \rfloor$ is the largest integer $\leqslant \frac{n}{2}$. The secondary relations are

- $x \rhd a \rhd b \rhd a = x \rhd c \rhd b \rhd c$ (for all x),
- $x\,(\rhd\, b \rhd c)^2\,(\rhd\, c \rhd a)^{2\lceil \frac{n}{2} \rceil}\,(\rhd\, b \rhd c)^2\,(\rhd\, c \rhd a)^{2\lfloor \frac{n}{2} \rfloor} = x$, and
- $x\,(\rhd\, a \rhd c)^{n+1}\,(\rhd\, c \rhd b)^2\,(\rhd\, a \rhd c)^{n}\,(\rhd\, a \rhd b)^2 = x$.

The diagram is given in Figure 4.8.9. There are two components–the "b" component, always of 4 elements, and the "a, c" component of $\mid 8n+4 \mid$ elements (whether n is even or odd). The order of the involutory quandle suffices to distinguish the links except for mirror images, which the involutory quandle cannot distinguish.

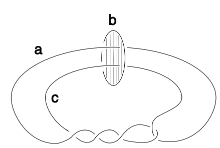


Figure 4.8.8. *Whitehead link.*

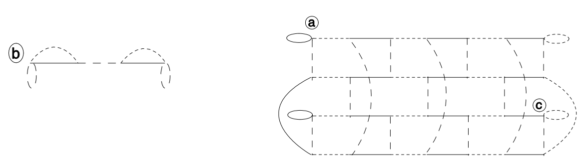


Figure 4.8.9. *Involutory quandle of Whitehead link* (*Example 4.8.5*)

The diagram consists of two components: one of four vertices and the other of $8n+4$ *vertices where* n *is the number of half-twists* (*Figure* 4.8.8).

In the latter component, a basic pattern of 8 *vertices is repeated horizontally* n *times* ($n \geqslant 0$) *or* $\mid n-1 \mid$ *times* ($n < 0$).

We offer the following hints for the reader who wishes to construct the diagram by the method of Definition 4.7.1. For the "b" component, trace the first primary relation and idempotency. Apply the first of the secondary relations repeatedly to show $x \rhd a = x \rhd c$ throughout this component. Then, apply the second of the secondary relations. For the "a, c" component, first trace $a \rhd a = a$. Next, develop the brick-wall pattern by repeated use of the first of the secondary relations. Trace the second primary relation, the relation $c \rhd c = c$, and the third primary relation. Finally, sew the edges together by repeated use of the first secondary relation.

We conclude Section 4.8 and proceed to prove correctness of the diagramming method.

4.9 Correctness of the Diagramming Method: Diagrams and d-derivations

In this section and the next, we prove the correctness (Theorem 4.10.14) of the diagramming method presented in Sections 4.3 - 4.7 (Definition 4.7.1). The diagramming method and proof of correctness are extended to arbitrary quandles in Section 4.11.

Throughout this section, we are concerned with an arbitrary presentation of an involutory quandle $IQ\langle S : R\rangle$ with generating set S and set of relations R on $U(S, \triangleright)$ (Definition 4.2.3). We may assume the relations in R are all canonically left-associated without loss of generality (Lemma 4.4.2). Note the proofs apply when S, R are infinite, even though both S and R are finite for knots and links.

The task is to prove a diagram constructed from the presentation $IQ\langle S : R\rangle$, in accordance with Definition 4.7.1, is the same as the diagram of the involutory quandle $Q = IQ\langle S : R\rangle$ (Definition 4.3.1 and Section 4.2). The central theme of the proof is the question, which aspects of involutory quandle structure are encoded intrinsically in every diagram, and which remain to be incorporated into a particular diagram by means of tracing? The diagrams build in the right multiplication property of equality; $p = q$ implies

$$p \triangleright x = q \triangleright x.$$

However, the diagram does not build in the left multiplication property; $p = q$ implies

$$x \triangleright p = x \triangleright q.$$

A detailed discussion of this matter precedes Lemma 4.10.9.

Our proof is algebraic in nature. The first task, to which Section 4.9 is devoted, is to algebraically formalize the diagramming method while focusing on the operation of tracing (Definition 4.5.3) that merges formerly distinct vertices of a partial diagram. The tracing operation is formalized in terms of algebraic derivation of equalities, and we discuss such derivations in detail accordingly. Begin by defining the universe within which the derivations are made. This universe is restricted to left-associated expressions due to the pervasive role of left association in diagramming.

Definition 4.9.1 *The d-*universe *(diagrammatic universe) $U_d(S, \triangleright)$ of words over the generating set S with operation $\triangleright$ is the subset of the full universe $U(S, \triangleright)$ (Definition 4.2.3), which consists of all left-associated* *words and contains no sub-expression of the form*

$$(x \rhd g) \rhd g,\ x \in U(S,\ \rhd),\ g \in S.$$

Remark 4.9.2 *In general, the words of $U_d(S,\ \rhd)$ are not in canonical form with respect to idempotency (axiom 1). For example, $g \rhd g$ and g are distinct words of $U_d(S,\ \rhd)$. Our reasons for defining $U_d(S,\ \rhd)$ in this way are as follows. First, idempotency is not intrinsic in diagrams and still must be traced. Therefore, the statement of Lemma 4.9.7, which relates $U_d(S,\ \rhd)$ to diagrams, is simplified by making idempotency not intrinsic in $U_d(S,\ \rhd)$. Second, it is useful to delay tracing of idempotency in the construction of actual diagrams . In the case of the Borromean Rings (Example 4.7.5), tracing of idempotency breaks the symmetry of the partial diagram. It is considerably easier to check satisfaction of the secondary relations before the symmetry is broken. For this reason, our theory includes partial diagrams lacking idempotency and, correspondingly, a universe lacking idempotency.*

Remark 4.9.3 *We have the following (elaborated) sequence of canonical surjective maps.*

$$\begin{array}{ccccccc} U(S,\ \rhd) & \underset{u_d}{\rightarrow} & U_d(S,\ \rhd) & \underset{u_f}{\rightarrow} & IQ\langle S:\phi\rangle & \underset{h}{\rightarrow} & IQ\langle S:R\rangle \\ & & \downarrow \delta & & & & \\ & & D & & & & \end{array}$$

The map u_d sends each word in $U(S,\ \rhd)$ to its canonical left-associated, right-cancelled form in $U_d(S,\ \rhd)$ (Lemma 4.4.2, Remark 4.4.6). The map u_d does not canonicalize with respect to idempotency. That aspect of displaying quandles in canonical form is done by the map u_f, a two-to-one map sending both $g_0 \rhd g_0 \rhd g_1 \rhd \ldots \rhd g_n$ and $g_0 \rhd g_1 \rhd \ldots \rhd g_n$ to the same element of the free involutory quandle $IQ\langle S:R\rangle$ generated by S. Finally, we discuss the map δ from $U_d(S,\ \rhd)$ to a diagram D constructed from the presentation $IQ\langle S:R\rangle$ according to Definition 4.7.1. This map is induced by the operation of location (tracing, Definition 4.5.3, point 1); $\delta(w)$ is the vertex of D obtained by locating the word w in D.

Now the goal of our correctness proof is stated in terms of maps. We show $IQ\langle S : R\rangle$ and D induce the same equivalence relation on $U(S, \ \rhd)$;

$$u_d \circ u_f \circ h(v) = u_d \circ u_f \circ h(w)$$

iff

$$u_d \circ \delta(v) = u_d \circ \delta(w), \quad v, \ w \in U(S, \rhd).$$

Equivalently, we show

$$u_f \circ h(v) = u_f \circ h(w)$$

iff

$$\delta(v) = \delta(w), \quad v, \ w \in U_d(S, \rhd).$$

The first step in capturing the diagramming method algebraically is using any partial diagram D to induce an equivalence relation on the corresponding d-universe.

Definition 4.9.4 *Let D be any (involutory) diagram, partial or complete, on the generating set S. The equivalence relation $\tilde{D}$ induced by D on the d-universe $U_d(S, \ \rhd)$ is the relation in which $v\tilde{D}w$ iff v, w label the same vertex when located in the diagram D. We shall abbreviate $\tilde{D}$ as $\tilde{}$ when the choice of D is clear from context.*

Note a partial diagram may not contain a vertex for every word $w \in U_d(S, \ \rhd)$. For this reason, we include the phrase "when located" (Definition 4.5.3, tracing, point 1) in Definition 4.9.4.

The notion of d-derivation is defined in order to study this equivalence relation.

Definition 4.9.5 *Let S be any generating set and $\hat{R}$ be any set of (canonicalized) relations in $U_d(S, \ \rhd)$. A d-*derivation *(diagrammatic derivation) from $\hat{R}$ in $U_d(S, \ \rhd)$ is a sequence of steps, each of which derives a relation by one of the following rules.*

1. Reflexivity. For any word $w \in U_d(S, \ \rhd)$, $w = w$.

2. Symmetry. From the relation $p = q$, either given in $\hat{R}$ or derived in a previous step, derive the relation $q = p$.

3. Transitivity. Derive the relation $p = r$ from the given or derived relations $p = q$, $q = r$.

4. Right multiplication. Derive the relation $p \rhd g = q \rhd g$ from the given or derived relation $p = q$ and for any generator $g \in S$.

Right cancellation, $x \rhd g \rhd g = x$ for $g \in S$, is assumed to be applied wherever possible. A relation $p = q$ will be called *d-derivable* from $\hat{R}$ over S if there is a d-derivation of $p = q$ from $\hat{R}$ over S.
The rule of reflexivity is supplied only so $w = w$ can be d-derived in a single step. Reflexivity is not used in longer, nontrivial derivations. Arbitrary derivations are defined in the next section (Definition 4.10.1) in connection with the discussion of presentation involutory quandles.

Example 4.9.6 (*A d-derivation.*) *The following d-derivation employs a secondary relation corresponding to the primary relation $a \rhd b = c$ together with one instance of idempotency to derive the relation*

$$b \rhd a \rhd b = b \rhd c.$$

1.	$b \rhd b \rhd a \rhd b = b \rhd c$	$b \rhd (a \rhd b) = b \rhd c$, *secondary relation for* $a \rhd b = c$
2.	$b \rhd b = b$	*idempotency*
3.	$b = b \rhd b$	2, *right multiplication by* b (*or* 2, *symmetry of equality*)
4.	$b \rhd a = b \rhd b \rhd a$	3, *right multiplication*
5.	$b \rhd a \rhd b = b \rhd b \rhd a \rhd b$	4, *right multiplication*
6.	$b \rhd a \rhd b = b \rhd c$	5, 1, *transitivity of equality*

Each relation derived (steps $1 - 6$) holds in the involutory quandle $IQ\langle a,\ b,\ c\ :\ a \rhd b = c\rangle$ and in any involutory quandle for which the relation $a \rhd b = c$ holds (including those of the trefoil [Examples 4.1.3, 4.5.9] and the figure-eight knot [Examples 4.1.4, 4.5.11]). The relation derived in step 6 is obtained in a different manner in Examples 4.10.4 and 4.10.6 of the next section.

Now we capture the notion of partial diagram and the effect of tracing of relations in algebraic form with the aid of d-derivation (Definition 4.5.3).

Lemma 4.9.7 *Let S be a set of generators and $\hat{R}$ be a set of relations in $U_d(S,\ \rhd)$. Let D be the partial diagram obtained by tracing each relation of $\hat{R}$ (in any order). Let v, w be any words of $U_d(S,\ \rhd)$. Then, in the equivalence relation $\tilde{D}$ (Definition 4.9.4), $v \sim w$ iff the relation $v = w$ is d-derivable from $\hat{R}$ over S.*

Proof. $\Longleftarrow$: Assume $v \nsim w$ and obtain a contradiction. Given $v = w$ is d-derivable, examine one such d-derivation $\triangle$. Since every relation $p = q$ in R has been traced, $p \sim q$ for each such relation. Find the earliest step of the d-derivation $\triangle$ such that for the relation $x = y$ derived in that step $x \nsim y$. Now we do a case analysis based on the d-derivation rule used in that step.

1. *Reflexivity.* The derived relation is $x = x$ for some $x \in U_d(S, \ \triangleright)$. However, $x \sim x$ for all x in $U_d(S, \ \triangleright)$, which contradicts the assumption $x \nsim x$.

2. *Symmetry.* The derived relation is $x = y$ and is derived from $y = x$. By assumption, we have $y \sim x$ but $x \nsim y$, a contradiction.

3. *Transitivity.* The derived relation is $x = y$, which is derived from $x = z$ and $z = y$ for some $z \in U_d(S, \ \triangleright)$. By assumption $x \sim z$, $z \sim y$, but $x \nsim y$, a contradiction.

4. *Right multiplication.* The derived relation has the form

$$p \triangleright g = q \triangleright g, \quad g \in S \text{ (assuming no cancellation occurs).}$$

By assumption, $p \sim q$, but $p \triangleright g \nsim q \triangleright g$. This contradicts the definition of tracing (Definition 4.5.3), whereby if p, q label the same vertex, then $p \triangleright g$, $q \triangleright g$ must label the same vertex. A similar argument holds when cancellation occurs.

$\Longrightarrow$: In the initial diagram D_0 in which no relations have been traced, $v \sim w$ iff v, w are the same word of $U_d(S, \ \triangleright)$. Whenever vertices labeled v, w are merged by tracing, the relation $v = w$ is d-derivable from the relations traced. Step 3 of tracing (Definition 4.5.3) merges v, w for a given relation $v = w$ of $\hat{R}$. Step 4 of tracing merges $p \triangleright g$, $q \triangleright g$ for already merged vertices p, q and various $g \in S$. In this case, d-derivability of $p = q$ implies

$$p \triangleright g = q \triangleright g.$$

In this case, d-derivability of $p = q$ implies

$$p \triangleright g = q \triangleright g.$$

Rules (2) and (3) of d-derivation, symmetry and transitivity, may be required to merge multiple vertices.

A similar argument applies when v, w are not represented by actual vertices in the partial diagram but are represented by the same vertex when located. ■

Corollary 4.9.8 *Let the diagram D be constructed from the involutory quandle presentation $IQ\langle S : R\rangle$ according to Definition 4.7.1. Then two words $v, w \in U_d\,(S, \ \triangleright)$ label the same vertex of D iff the relation $v = w$ is d-derivable from the set of relations $R \cup R_S \cup I_S$. The diagram is independent of the order in which these relations are traced.*

Proof. Set $\hat{R} = R \cup R_S \cup I_S$ in Lemma 4.9.7. The d-derivation is independent of the order of tracing, and the criterion completely specifies the diagram. Therefore, the diagram is independent of the order. ■

The following lemma gives a still tighter form for d-derivation and closes this section. The proof of this lemma introduces the style of proof to be used in Lemma 4.10.12 in a simpler context.

Lemma 4.9.9 (*Reordering in d-derivation.*) *Given any d-derivation of a relation $v = w$ from a set $\hat{R}$ of relations over a universe $U\,(S, \ \triangleright)$, there exists a related d-derivation of $v = w$ in which all applications of right multiplication precede all applications of symmetry of equality. The applications of symmetry of equality in turn precede all applications of transitivity.* (*The resulting derivation simply applies symmetry and transitivity to various right-multiplied variations of the initial relations.*)

Proof. First, we show how to move applications of right multiplication to the beginning. Two adjacent steps independent of one another can be exchanged at will. The difficulty arises when right multiplication is applied to a relation derived by symmetry or transitivity. In the first of these cases, i.e.

(a)	$p = q$	(given or derived earlier)
(b)	$q = p$	a, symmetry of equality
(c)	$q \triangleright g = p \triangleright g$	b, right multiplication

the application of right multiplication may be placed before the application of symmetry while deriving the same formulas (and one additional one) as follows.

(a)	$p = q$	
$(a\prime)$	$p \triangleright g = q \triangleright g$	a, right multiplication
(b)	$q = p$	a, symmetry of equality
(c)	$q \triangleright g = p \triangleright g$	a', symmetry of equality

A similar device is used when an application of right multiplication is preceded by the application of transitivity. When the relation $p = r$ is derived from $p = q$ and $q = r$ and $p \triangleright g = r \triangleright g$ is derived from $p = r$, instead derive

$$\begin{aligned} p \triangleright g &= q \triangleright g, \\ q \triangleright g &= r \triangleright g \end{aligned}$$

by right multiplication followed by

$$p \triangleright g = r \triangleright g$$

and (if needed), $p = r$ by transitivity. In this case, two additional steps are inserted in the d-derivation. Repeated application of these maneuvers places all applications of right multiplication before those of symmetry and transitivity. Finally, we show how to place an application of symmetry before one of transitivity. When $p = r$ is derived from $p = q$ and $q = r$ and $r = p$ is derived from $p = r$, instead derive

$$\begin{aligned} q &= p, \\ r &= q \end{aligned}$$

by symmetry of equality followed by

$$r = p$$

and (if needed), $p = r$ by transitivity. Repetition of this maneuver places all applications of symmetry before all those of transitivity. ■

4.10 Correctness of the Diagramming Method: Presentation Involutory Quandles, Derivation, and d-derivation

In this section, we complete the proof of correctness began in the preceding section. First, we examine our task in terms of mappings (Remark 4.9.3).

$$\begin{array}{ccccccc} U(S, \ \triangleright) & \underset{u_d}{\rightarrow} & U_d(S, \ \triangleright) & \underset{u_f}{\rightarrow} & IQ\langle S : \phi\rangle & \underset{h}{\rightarrow} & IQ\langle S : R\rangle \\ & & \downarrow \delta & & & & \\ & & D & & & & \end{array}$$

The map δ is characterized in terms of d-derivation in Corollary 4.9.8. The maps $u_d \circ u_f \circ h$ and $u_f \circ h$ remain to be characterized, which we do by means of derivation (Definition 4.10.1) and derivation with canonicalization (Definition 4.10.5) respectively. These notions must be related to d-derivation (Lemmas 4.10.10 through 4.10.13).

Alternatively, we may view our task as performing a series of recastings. The series begins with the notion of presentation of an algebraic structure defined in Section 4.2 for arbitrary algebras, and here we deal with involutory quandles. This definition is totally general for universal algebra and makes use of none of the special properties of involutory quandles. Correspondingly, the definition of a presentation is not suited for computation and indeed is not even stated in computational terms. The highly specific and highly computational diagramming method and its characterization in terms of d-derivation (preceding section) is at the end of the series.

The transition between these two very different entities is made in the following stages. First, the definition of presentation involutory quandle is given a more computational quality through derivation (Definition 4.10.1). Derivation is then recast as derivation with canonicalization (Definition 4.10.5) which, because it derives canonically left-associated formulas, is easier to deal with notationally and is closer to d-derivation . The final stage–transition from derivation with canonicalization to d-derivation (Lemmas 4.10.10-4.10.13)–is also the most difficult. This recasting eliminates the operation of left multiplication and adjoins the secondary relations to the initial set of relations through compensation. These modifications reflect right multiplication is built into the diagrams but left multiplication is not. Therefore, left multiplication must be incorporated by tracing the secondary relations. The entire sequence of recasting is summarized in the statement of Theorem 4.10.14.

Now we turn to the first stage of the recasting.

Definition 4.10.1 *Let S be a set of generators and $\hat{R}$ be a set of relations in $U(S, \rhd)$. A derivation in $U(S, \rhd)$ from $\hat{R}$ is a sequence of steps through which a relation is derived by one of the following rules.*

1. *Reflexivity.* For any word $w \in U(S, \rhd)$, $w = w$.

2. *Symmetry.* From the relation $p = q$ (either given in $\hat{R}$ or derived in a previous step), derive the relation $q = p$.

3. *Transitivity.* From the given or derived relations $p = q$, $q = r$ derive the relation $p = r$.

4. *Right multiplication.* From the given or derived relation $p = q$ and for any word $w \in U(S, \rhd)$, derive the relation $p \rhd w = q \rhd w$.

5. *Left multiplication.* From the given or derived relation $p = q$ and for any word $w \in U(S, \rhd)$, derive the relation $w \rhd p = w \rhd q$.

Reflexivity is used only for a one-step derivation of $w = w$ and is not applied in longer, nontrivial derivations.

Note the definition refers to $\hat{R}$ and not R because R is not the initial set of relations employed. This raises the following question. What set of initial relations is employed given a presentation $IQ\langle S : R\rangle$? The following definition will be useful in supplying a precise answer.

Definition 4.10.2 *Let A_S denote the set of all instances of the involutory quandle axioms over the generating set S. Therefore, A_S consists of all relations of the forms*

$$\begin{aligned} x \rhd x &= x, \\ (x \rhd y) \rhd y &= x, \\ (x \rhd y) \rhd z &= (x \rhd z) \rhd (y \rhd z) \quad \textit{for } x,\ y,\ z \in U(S, \rhd). \end{aligned}$$

Now we answer the question posed above.

Remark 4.10.3 *Two words v, w of the universe $U(S, \ \triangleright)$ represent the same word of the involutory quandle $IQ\langle S : R\rangle$ iff the relation $v = w$ is derivable from the set of relations $R \cup A_S$ (according to Definition 4.10.1).*

This Remark may be verified by inspection of the appropriate definitions. Since the characterization of the map $u_d \circ u_f \circ h$ (Remark 4.9.3) and the first stage of the recasting discussed above are accomplished, we proceed with an example of a derivation. The example is used to illustrate subsequent stages of the recasting.

Example 4.10.4 (*A derivation.*) *The following derivation employs the relation $a \triangleright b = c$ and various instances of the involutory quandle axioms to derive the relations*

$$(b \triangleright a) \triangleright b = b \triangleright c$$

and

$$(c \triangleright a) \triangleright ((b \triangleright a) \triangleright b) = (c \triangleright a) \triangleright (b \triangleright c).$$

1.	$a \triangleright b = c$	*given relation*
2.	$b \triangleright (a \triangleright b) = b \triangleright c$	1, *left multiplication*
3.	$(b \triangleright b) = b$	*instance of axiom* 1
4.	$(b \triangleright b) \triangleright (a \triangleright b) = b \triangleright (a \triangleright b)$	3, *right multiplication*
5.	$(b \triangleright b) \triangleright (a \triangleright b) = b \triangleright c$	4, 2, *transitivity of equality*
6.	$(b \triangleright a) \triangleright b = (b \triangleright b) \triangleright (a \triangleright b)$	*instance of axiom* 3
7.	$(b \triangleright a) \triangleright b = b \triangleright c$	6, 5, *transitivity of equality*
8.	$(c \triangleright a) \triangleright ((b \triangleright a) \triangleright b) = (c \triangleright a) \triangleright (b \triangleright c)$	7, *left multiplication*

Each relation derived (steps $2-8$) thus holds in the involutory quandle $\langle a, b, c : a \triangleright b = c\rangle$ and in any involutory quandle for which the relation $a \triangleright b = c$ holds (including those of the trefoil [Examples 4.1.3, 4.5.9] and the figure eight knot [Examples 4.1.4, 4.5.11]).

As the preceding example suggests, the parenthesis structure of the terms in a derivation can become quite complex. For convenience of notation, we prefer to work with the notion of *derivation with canonicalization.*

Definition 4.10.5 *A* derivation with canonicalization *is a derivation* (*Definition 4.10.1*) *in which each initial relation of* $\hat{R}$ *is considered to be in left-associated, right-cancelled form* (*that is, of the form* $p = q$ *for* $p, q \in U_d(S, \triangleright)$) *and in which each derived relation is immediately placed in the canonical form.*

Note d-derivation is a special case of *derivation with canonicalization.*

Example 4.10.6 (*A derivation with canonicalization.*) *The following derivation with canonicalization represents a step-by-step recasting of Example 4.10.4 The relation* $a \triangleright b = c$ *and various instances of the involutory quandle axioms are employed to derive the relations*

$$b \triangleright a \triangleright b = c$$

and

$$c \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a \triangleright b = c \triangleright a \triangleright c \triangleright b \triangleright c$$

The latter relation is the canonicalized form of Example 4.10.4, step 8.

1.	$a \triangleright b = c$	given relation
2.	$b \triangleright (a \triangleright b) = b \triangleright c$	1, left multiplication
	$b \triangleright b \triangleright a \triangleright b = b \triangleright c$	immediately canonicalized
3.	$b \triangleright b = b$	instance of axiom 1
4.	$(b \triangleright b) \triangleright (a \triangleright b) = b \triangleright (a \triangleright b)$	3, right multiplication
	$b \triangleright a \triangleright b = b \triangleright b \triangleright a \triangleright b$	immediately canonicalized
5.	$b \triangleright a \triangleright b = b \triangleright c$	4, 2, transitivity of equality
6.	$(b \triangleright a) \triangleright b = (b \triangleright b) \triangleright (a \triangleright b)$	instance of axiom 3
	$b \triangleright a \triangleright b = b \triangleright a \triangleright b$	immediately canonicalized, becomes trivial
7.	$b \triangleright a \triangleright b = b \triangleright c$	6, 5, transitivity of equality
8.	$(c \triangleright a) \triangleright (b \triangleright a \triangleright b) = (c \triangleright a) \triangleright (b \triangleright c)$	either 5 or 7, left multiplication
	$c \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a \triangleright b = c \triangleright a \triangleright c \triangleright b \triangleright c$	immediately canonicalized

Several features of the recasting of Example 4.10.4 include the following. First, one of the two desired relations is obtained in step 5 of the recasting, as well as in in step 7 (as in the original). Thus, a derivation with canonicalization may be shorter than the corresponding ordinary derivation. Second, the instance of axiom 3 in step 6 becomes trivial upon canonicalization. This is a general phenomenon (Lemma 4.10.8 below). Third, the derivation of Example 4.10.4 maps step for step to the corresponding derivation with canonicalization. These observations are incorporated in the following two lemmas.

Lemma 4.10.7 (*Recasting of derivation.*) *Let $\triangle$ be a derivation from a set of canonicalized relations $\hat{R}$ in $U(S, \;\triangleright)$. Then $\triangle$ can be recast as a derivation with canonicalization from $\hat{R}$ in which the same relations derived in $\triangle$ are derived in left-associated, right-cancelled form.*

Proof. The recasting is straightforward. For each step of $\triangle$, simply canonicalize the resulting relation while retaining the same derivation rule with reference to the same preceding step(s). Definition 4.10.1 shows the canonicalization of a derived relation does not interfere with subsequent application of the derivation rules. ∎

Lemma 4.10.8 *Two words v, w of the universe $U(S, \triangleright)$ represent the same element of the involutory quandle $IQ\langle S : R\rangle$ iff the relation $v^1 = w^1$ is derivable with canonicalization from the set of relations $R \cup I_S$, where v^1, w^1 are the left-associated, right-cancelled forms of v, w and the relations R are assumed to be canonicalized as well.*

Proof. $\Longrightarrow$: If v, w represent the same element of $IQ\langle S : R\rangle$, then a derivation of $v = w$ from $R \cup A_S$ exists (Remark 4.10.3). Recast this derivation as a derivation with canonicalization (preceding lemma). Relations of A_S, which are instances of axioms 2 and 3, become trivial when canonicalized (Remark 4.4.6, proof of Theorem 4.4.4). The remaining considerations are instances of axiom 1, $x \triangleright x = x$. By canonicalization, we may assume

$$x = g_0 \triangleright g_1 \triangleright ... \triangleright g_n \text{ for some } g_0, ..., g_n \in S.$$

When canonicalized, the relation

$$x \triangleright x = x$$

becomes

$$g_0 \triangleright g_0 \triangleright g_1 \triangleright ... \triangleright g_n = g_0 \triangleright g_1 \triangleright ... \triangleright g_n.$$

This relation can be obtained from the relation $g_0 \triangleright g_0 = g_0$in I_S by a derivation with canonicalization involving repeated right multiplication.

$\Longleftarrow$: Given a derivation with canonicalization from $R \cup I_S$, a corresponding ordinary derivation is readily obtained by applying canonicalization explicitly by derivation. (Canonicalization by explicit derivation requires use of axioms 2 and 3, as in the proof of Lemma 4.4.1. Therefore, the explicit derivation depends on A_S. Right and left multiplication are applied to embed instances of canonicalization within larger expressions; symmetry and transitivity of equality combine multiple applications of canonicalization.). ■

Now we review the recasting accomplished thus far and outline what remains to be accomplished. To review, the original task is to establish the relationship between the diagramming method (Definition 4.7.1) and the notion of presentation involutory quandle (Section 4.2). Both entities–the diagramming method in terms of d-derivation (Corollary 4.9.8) and the notion of presentation involutory quandle in terms of derivation with canonicalization (preceding lemma)–have been recast. The desired relationship between these two types of derivation must be established.

Examination of Definition 4.9.5, 4.10.1, and 4.10.5 shows d-derivation is a restricted case of derivation with canonicalization. Specifically, right multiplication is restricted to generators, and left multiplication is excluded entirely. These restrictions on d-derivation reflect diagrams build in right multiplication but not left multiplication. As will be seen in Lemma 4.10.10, the restriction on right multiplication does not actually reduce derivation power but may lead to a longer derivation. On the other hand, the absence of left multiplication makes d-derivation strictly weaker than derivation with canonicalization. How is the weakness of d-derivation counterbalanced?

The lesser power of d-derivation is counterbalanced using a larger set of initial relations: $R \cup R_S \cup I_S$ as compared with $R \cup I_S$. The larger set of initial relations does not overcompensate (Theorem 4.10.14, $(iv) \implies (iii)$) because d-derivation is not made stronger than derivation with canonicalization. The major difficulty lies in showing the addition of R_S is sufficient to compensate for the absence of left multiplication. The necessary recasting techniques and proof are given in Lemmas 4.10.11-4.10.13 and Theorem 4.10.14. We begin by illustrating the recasting process.

Example 4.10.9 (*Derivation with canonicalization recast as d-derivation.*) *We recast Example 4.10.6 in* *terms of d-derivation. Left multiplication is removed. By way of compensation, various instances of the secondary relation*

$$x \triangleright (a \triangleright b) = x \triangleright c$$

are introduced and correspond to the primary relation $a \triangleright b = c$. We develop the recasting step by step.

1. $a \triangleright b = c$ given relation

The first step needs no modification.

(2. $b \triangleright b \triangleright a \triangleright b = b \triangleright c$ 1, left multiplication)

We must remove this left multiplication; instead we cite the corresponding secondary relation.

2. $b \triangleright b \triangleright a \triangleright b = b \triangleright c$ secondary relation $b \triangleright (a \triangleright b) = b \triangleright c$

3. $b \triangleright b = b$ idempotency

(4. $b \triangleright a \triangleright b = b \triangleright b \triangleright a \triangleright b$ 3, right multiplication by $a \triangleright b$)

This step must be recast in terms of right multiplication by generators. Lemma 4.10.10 below covers the general case. Here three steps are required.

4a. $b = b \triangleright b$ 3, right multiplication by b

4b. $b \triangleright a = b \triangleright b \triangleright a$ 4a, right multiplication by a

4c. $b \triangleright a \triangleright b = b \triangleright b \triangleright a \triangleright b$ 4b, right multiplication by b

Step 5 can be retained unmodified.

5. $b \triangleright a \triangleright b = b \triangleright c$ 4c, 2, transitivity of equality

(6, 7 omitted)

(8. $c \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a \triangleright b = c \triangleright a \triangleright c \triangleright b \triangleright c$ 5, left multiplication by $c \triangleright a$)

The final step illustrates the difficulty of removing arbitrary left multiplication. Left multiplication applied to a given primary relation (as in Step 2) is easily handled by employing the corresponding secondary relation. At this point, left multiplication is applied to a derived relation. Generally, for such cases, left multiplication must be pushed toward the beginning of the proof to reach a more tractable situation. However, in this case, we can derive the necessary relation from two secondary relations as follows.

8a.	$(c \triangleright a) \triangleright b \triangleright a \triangleright b = (c \triangleright a) \triangleright c$	relation given in R_S
8b.	$c \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a = c \triangleright a \triangleright c \triangleright a$	$8a$, right multiplication
8c.	$c \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a \triangleright b = c \triangleright a \triangleright c \triangleright a \triangleright b$	$8b$, right multiplication
8d.	$c \triangleright a \triangleright c \triangleright a \triangleright b = c \triangleright a \triangleright c \triangleright b \triangleright c$	relation given in $R_8 : (c \triangleright a \triangleright c \triangleright b) \triangleright (a \triangleright b) = (c \triangleright a \triangleright c \triangleright b) \triangleright c$
8e.	$c \triangleright a \triangleright b \triangleright a \triangleright b \triangleright a \triangleright b = c \triangleright a \triangleright c \triangleright b \triangleright c$	$8c$, $8d$, transitivity

The exemplified recasting is accomplished by the techniques given in the proofs of the following four lemmas. Note the application of the techniques to step 8 of this example gives rise to superfluous yet valid steps.

Lemma 4.10.10 *Let $\triangle$ be any derivation with canonicalization containing no application of left multiplication. Let the initial set of relations for $\triangle$ be $\hat{R}$ and the universe $U(S, \ \triangleright)$. Then $\triangle$ can be recast as a d-derivation from $\hat{R}$ in $U_d(S, \ \triangleright)$ deriving all relations derived by $\triangle$.*

Proof. The power of derivation with canonicalization exceeds that of d-derivation only in the rules of left and right multiplication. Application of left multiplication is absent by hypothesis. Application in $\triangle$ of right multiplication by an arbitrary word $w \in U(S, \ \triangleright)$ can be recast as right multiplication by generators, as required by d-derivation. Suppose $\triangle$ derives $p \triangleright w = q \triangleright w$ from the relation $p = q$. Placing this relation is canonical form causes w to be canonicalized as well. Therefore, in $w.l.o.g.$, w has the form $g_0 \triangleright g_1 \triangleright ... \triangleright g_n$, $g_i \in S$. The same canonicalized relation is obtained by successive right multiplication by $g_n, ..., g_1, g_0, g_1, ..., g_n$. ■

Lemma 4.10.11 *Let $p = q$ be any relation in $U(S, \ \triangleright)$ and x, y be any words of $U(S, \ \triangleright)$. Then the relation*

$$y \triangleright (x \triangleright p) = y \triangleright (x \triangleright q)$$

is d-derived (in canonicalized form) from two instances of the secondary relation

$$w \triangleright p = w \triangleright q$$

for $w \in U_d(S, \ \triangleright)$. That is, left-multiplied forms of secondary relations are d-derived from secondary relations.

Proof. We give a derivation in which left multiplication is not used.

1.	$y \rhd p = y \rhd q$	secondary relation
2.	$y \rhd p \rhd x = y \rhd q \rhd x$	1, right multiplication
3.	$y \rhd p \rhd x \rhd p = y \rhd q \rhd x \rhd p$	2, right multiplication
4.	$(y \rhd q \rhd x) \rhd p = (y \rhd q \rhd x) \rhd q$	secondary relation
5.	$y \rhd p \rhd x \rhd p = y \rhd q \rhd x \rhd q$	3, 4, transitivity

Step 5 is the desired relation

$$y \rhd (x \rhd p) = y \rhd (x \rhd q)$$

with one application of canonical left association to each side. Recast this derivation as a derivation with canonicalization (Lemma 4.10.7) and then as a d-derivation (Lemma 4.10.10) ■

Lemma 4.10.12 *Let $\triangle$ be any derivation with canonicalization from a set of relations $\hat{R} \cup \hat{R}_S$ in $U_d(S, \rhd)$ (a set of relations and the corresponding secondary relations), which is a d-derivation except for the final step of left multiplication. Then $\triangle$ is recast as a d-derivation from $\hat{R} \cup \hat{R}_s$ to derive the same relations (and possibly others).*

Proof. The reader may wish to review the similar but simpler proof of Lemma 4.9.9. The present proof is by induction on the length (number of steps) of the derivation $\triangle$. The lemma is vacuously true for a derivation of length one because, in the final step, left multiplication must be applied to the result of a previous step. Now assume the lemma is true for derivations of less than n steps. We wish to prove the lemma for derivations of n steps.

In the final step, left multiplication must be applied to the result of the m^{th} step for some $1 \leqslant m < n$. If $m < n-1$, we may simply reorder the steps so the left multiplication is the $m+1^{st}$ step and apply the induction hypothesis to the sub-derivation consisting of steps 1 through $m+1$. Consider left multiplication applied to the result of step $m = n-1$ and perform a case analysis on the nature of step $n-1$. If step $n-1$ simply states a relation $p = q$ given in $\hat{R}$, then the result

$$x \rhd p = x \rhd q$$

of step n can be stated as given in $\hat{R}_S$.

If step $n-1$ states a relation

$$x \triangleright p = x \triangleright q$$

given in R_S, then the result of step n can be d-derived from $\hat{R}_S$ by the preceding lemma.

We may assume step $n-1$ does not apply reflexivity, which is only of use in the trivial derivation of $w = w$.

If step $n-1$ applies symmetry of equality to the relation $p = q$ to derive $q = p$, then step n derives

$$x \triangleright q = x \triangleright p.$$

Recast so step $n-1$ derives

$$x \triangleright p = x \triangleright q$$

by left multiplication and steps n, $n+1$ derive

$$\begin{aligned} x \triangleright q &= x \triangleright p, \\ q &= p \end{aligned}$$

by symmetry. Apply the induction hypothesis to recast the sub-derivation consisting of steps 1 through $n-1$ as a d-derivation .

If step $n-1$ applies transitivity to the relations

$$\begin{aligned} p &= q, \\ q &= r \end{aligned}$$

to derive

$$p = r,$$

then step n derives

$$x \triangleright p = x \triangleright r.$$

Consider the following two related derivations.

1. Retain steps 1 through $n-2$ of $\triangle$ but derive $x \triangleright p = x \triangleright q$ by left multiplication as the final $(n-1)^{st}$ step.

2. Derive $x \triangleright q = x \triangleright r$ similarly in $n-1$ steps.

Apply the induction hypothesis to each of these two derivations to show the relations $x \triangleright p = x \triangleright q$, $x \triangleright q = x \triangleright r$ are d-derivable from $\hat{R} \cup \hat{R}_S$. The relations $x \triangleright p = x \triangleright r$, $p = r$ are then d-derivable by transitivity.

If step $n-1$ applies right multiplication to the relation $p = q$ to derive $p \rhd g = q \rhd g$, then step n derives $x \rhd (p \rhd g) = x \rhd (q \rhd g)$ or, closer to the canonical form, $x \rhd g \rhd p \rhd g = x \rhd g \rhd q \rhd g$. Recast so step $n-1$ derives

$$(x \rhd g) \rhd p = (x \rhd g) \rhd q$$

by left multiplication and steps n, $n+1$ derive

$$\begin{aligned} x \rhd g \rhd p \rhd g &= x \rhd g \rhd q \rhd g, \\ p \rhd g &= q \rhd g \end{aligned}$$

by right multiplication. Apply the induction in order to recast the sub-derivation consisting of steps 1 through $n-1$ as a d-derivation. ■

We note certain technicalities involved in the induction argument presented above thereby pointing out the need for the three preceding lemmas and the following lemma. First, the total length of a derivation may increase when recast. Therefore, care is taken in phrasing the induction hypothesis based on derivation length. Second, the recasting yields two new left multiplications when left multiplication follows transitivity. Each new left multiplication is dealt with individually and ultimately removed. This technicality stands in the way of combining the induction proof above with the proof of Lemma 4.10.13 because the latter induction depends on reduction of the number of left multiplications. Fortunately, neither of these points need concern the practical "diagrammer," who requires only the result stated in Theorem 4.10.14 below and not the details of its proof.

Lemma 4.10.13 *Let $\triangle$ be any derivation with canonicalization from a set of relations $\hat{R}$ in $U_d(S, \rhd)$. The $\triangle$ can be recast as a d-derivation from $\hat{R} \cup \hat{R}_S$ deriving the same relations as $\triangle$ (and possibly others as well).*

Proof. The proof is by induction on the number of applications of left multiplication in $\triangle$. If there are no applications of left multiplication, apply Lemma 4.10.10 to complete the proof. Otherwise, find the first application and let n be the number of that step. Apply Lemma 4.10.10 to recast steps 1 through $n-1$ in terms of d-derivation. Apply Lemma 4.10.12 to recast the d-derivation combined with the first left multiplication as a d-derivation from $\hat{R} \cup \hat{R}_S$. One application of left multiplication has been removed. Repeat until all applications of left multiplication are removed. Apply Lemma 4.10.10 to recast the remainder of the derivation as d-derivation. ■

Although removal of the first left multiplication acts on $\hat{R}$ to bring in an element of $\hat{R}_S$, subsequent left multiplications may act on the element brought about in this manner. Therefore, Lemma 4.10.11 and the full power of Lemma 4.10.12 are required.

Now we state and prove the main theorem of Sections 4.9 - 4.10.

Theorem 4.10.14 (*Correctness of the diagramming method for involutory quandles.*) *Let $Q = IQ\langle S : R\rangle$ be a presentation involutory quandle. Then any diagram D produced according to Definition 4.7.1 is the diagram of Q (over the generating set S).*

In addition, suppose the relations R are in canonical form (Lemma 4.4.2). Let p, q be any words of the universe $U(S, \triangleright)$ and p^1, q^1 their canonicalized correspondents in $U_d(S, \triangleright)$ (Definition 4.9.1). Then the following are equivalent.

(*i*) p, q represent the same element of the involutory quandle $IQ\langle S : R\rangle$.

(*ii*) The relation $p = q$ is derivable from the set of relations $R \cup A_S$ (Definitions 4.10.1, 4.10.2).

(*iii*) The relation $p^1 = q^1$ is derivable from the set of relations $R \cup I_S$ (Definitions 4.6.10, 4.10.5).

(*iv*) The relation $p^1 = q^1$ is d-derivable from the set of relations $R \cup R_S \cup I_S$ (Definitions 4.6.10, 4.9.6).

(*v*) p^1, q^1 label the same vertex of the diagram D.

Proof. Proof that (i)$\Longleftrightarrow$(v) is sufficient to prove D is the diagram of Q.

(i)$\Longleftrightarrow$(ii): Remark 4.10.3.

(i)$\Longleftrightarrow$(iii): Lemma 4.10.8.

(iii)$\Longrightarrow$(iv): In Lemma 4.10.13, set $\hat{R} = R \cup I_S$. $\hat{R}_S$ then contains R_S. In fact, $\hat{R}_S = R_S$ because the secondary relation $w \triangleright (g \triangleright g) = w \triangleright g$ canonicalizes to the trivial relation $w \triangleright g = w \triangleright g$ for any relation $g \triangleright g = g$ in I_S and any word $w \in U_d(S, \triangleright)$. Thus

$$\hat{R} \cup \hat{R}_S = R \cup R_S \cup I_S$$

as required for this application of Lemma 4.10.13

(iv)$\Longrightarrow$(iii): Every d-derivation is a derivation with canonicalization. The difficulty lies in the initial unavailability of the secondary relations R_S for the desired derivation with canonicalization. However, every secondary relation required from R_S can be obtained in the context of derivation with canonicalization by applying the appropriate left multiplication to the corresponding primary relation from R.

(iv) $\Longleftrightarrow$ (v): Corollary 4.9.8. $\blacksquare$

4.11 Diagramming of Arbitrary Quandles

The diagram of the arbitrary quandle is obtained similar to involutory quandles. The following modifications are necessary.

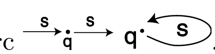
1. The diagram is a directed graph (Definition 4.3.9, Example 4.3.10) rather than an undirected graph (Definition 4.3.1). An arc labeled b and directed from a to c indicates $a \rhd b = c$ and $c \rhd^{-1} b = a$.

2. Modify the definition of *tracing* for the directed graph.

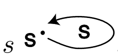
3. $\rhd^{-1}$ is distinguished from $\rhd$ in left association (Lemmas 4.4.7, 4.4.8).

Theorem 4.11.6 is the analogue of Theorem 4.10.14 for arbitrary quandles. The necessary definitions and some examples are provided. An application is found in Section 4.12, in which the 5-quandles of the square and granny knots are shown to be non-isomorphic. Review of Definition 4.3.9 and Example 4.3.10 before proceeding is useful.

Remark 4.11.1 *The diagram of a (non-involutory) quandle Q over a generating set $S \subset Q$ has the following properties.*

1. For every $q \in Q$ and every $s \in S$, there is exactly one arc labeled s and directed to q and exactly one arc labeled s and directed from q. These may be the same arc 

.

2. For every $s \in S$, there is a vertex labeled s met by both ends of an arc labeled s 

.

As in Remark 4.3.8, the aforementioned properties do not guarantee a directed graph is the diagram of some quandle.

Review of left association in arbitrary quandles from Lemmas 4.4.7 and 4.4.8 is useful. As with involutory quandle diagrams (Section 4.5), left association is useful in calculating multiplication table entries. In Example 4.3.10 we have

$$c \rhd d = (a \rhd b) \rhd \left(a \rhd^{-1} b\right)$$

(by the diagram)

$$= a \rhd b \rhd b \rhd a \rhd^{-1}$$

(Lemma 4.4.7)

$$= a$$

(by the diagram).

Given a presentation $Q\langle S : R\rangle$ for a non-involutory quandle Q, the diagram of Q over S may be obtained by a method analogous to Sections 4.5 - 4.7. Tracing is key to the method.

Definition 4.11.2 *In the construction of the diagram of a presentation quandle Q with generating set S, to* trace *the relation*

$$a_0 \rhd^{e_1} a_1 \rhd^{e_2} ... \rhd^{e_m} a_m = b_0 \rhd^{f_1} b_1 \rhd^{f_2} ... \rhd^{f_n} b_n, \quad a_i b_i \in S, \quad e_i, f_i = \pm 1,$$

is to perform the following sequence of operations.

1. *Locate* the vertex $a_0 \rhd^{e_1} a_1 \rhd^{e_2} ... \rhd^{e_m} a_m$ in the existing partial diagram. In other words, locate the vertex a_0 and, beginning at the vertex, a_0, trace along arcs labeled $a_1, ..., a_m$ (and directed by corresponding exponents $e_1 ... e_m$) to the desired vertex. If some vertex $a_0 \rhd^{e_1} a_1 \rhd^{e_2} ... \rhd^{e_i} a_i$, $0 \leqslant i < m$, is not met by any arc labeled a_{i+1} and directed outward ($e_{i+1} = +1$) or inward ($e_{i+1} = -1$) as necessary, then adjoin the required sequence of arcs $a_{i+1}, ..., a_m$ (appropriately directed) and corresponding new vertices to the diagram.

2. Locate $b_0 \rhd^{e_1} b_1 \rhd^{e_2} ... \rhd^{e_n} b_n$ in the same manner.

3. Merge (make identical) the two located vertices.

4. For any two merged vertices p, q and every $s \in S$, merge the vertices $p \rhd s$, $q \rhd s$ (if both are present in the partial diagram). Similarly merge $p \rhd^{-1} s$, $q \rhd^{-1} s$ (if both are present).

Definition 4.11.3 *Let $Q\langle S : R\rangle$ be any quandle presentation. The primary relations of the presentation are the relations R. The secondary relations R_S are, for each relation $s = t$ of R, the relations*

$$x \rhd s = x \rhd t$$

for all $x \in U\left(S,\ \rhd,\ \rhd^{-1}\right)$.

Definition 4.11.4 *A* diagram *of a quandle presentation $Q\langle S : R\rangle$ is any labeled directed graph D obtained as follows. Begin with the graph consisting of one vertex labeled g for each $g \in S$ and no arcs. Well-order the set of relations $R \cup R_S \cup I_S$. Finally, trace these relations in the chosen order.*

The reader may wish to state and prove the analogues of Lemmas 4.7.3 and 4.7.4 (rewriting of primary and secondary relations) for the case of non-involutory quandles.

Remark 4.11.5 *We alter the notion of d-derivation (Definition 4.9.5) for arbitrary presentation quandles as follows. The d-universe in question is $U_d\left(S,\ \rhd,\ \rhd^{-1}\right)$ rather than $U_d\left(S,\ \rhd\right)$. The d-universe consists of left-associated words in which right cancellation (axiom 2) is applied wherever possible. Operation (4), right multiplication, must be expanded to admit both $\rhd$ and $\rhd^{-1}$multiplication; from $p = q$, either*

$$p \rhd g = q \rhd g \text{ or } p \rhd^{-1} g = q \rhd^{-1}$$

may be derived for $g \in S$.

Now we state the analogue of Theorem 4.10.14 for arbitrary quandles.

Theorem 4.11.6 (*Correctness of the diagramming method for arbitrary quandles.*) *Let $Q = Q\langle S : R\rangle$ be any presentation quandle (in general, non-involutory). Then any diagram D produced according to Definition 4.11.4 is the diagram of Q.*

In addition, suppose the relations R are in canonical form (Lemma 4.4.8). Let p, q be any words of the universe $U\left(S,\ \rhd,\ \rhd^{-1}\right)$ and p^1, q^1 be their canonicalized correspondents in $U_d\left(S,\ \rhd,\ \rhd^{-1}\right)$

(Remark 4.11.5). Then the following are equivalent.

(i) p, q represent the same element of the quandle $Q = Q\langle S : R\rangle$.

(ii) The relation $p^i = q^i$ is d-derivable from the set of relations $R \cup R_S \cup I_S$ (Definition 4.11.3, Remark 4.11.5).

(iii) p^i, q^i label the same vertex of the diagram D.

Proof. Analogous to the proof of Theorem 4.10.14. ■

Corollary 4.11.7 *Let $Q = Q\langle S : R\rangle$ be any presentation quandle. If the relations $R \cup R_S \cup I_S$ are d-derivable from a subset R' of $R \cup R_S \cup I_S$, then tracing the relations R' is sufficient to diagram Q.*

The following result is convenient for diagramming n-quandles (Section 4.12) and completes this section.

Corollary 4.11.8 *In diagramming an n-quandle $Q_n\langle S : R\rangle$, the relations $x \rhd^n y = x$ need only be traced for $y \in S$, and the corresponding secondary relations need not be traced.*

Proof. We show the relations

$$x \rhd^n y = x, \text{ where } y \in U\left(S, \quad \rhd, \quad \rhd^{-1}\right),$$

are d-derivable from

$$x \rhd^n y = x, \text{ where } y \in S,$$

and need not be traced (preceding corollary). Write y as a canonicalized product in the generators,

$$y = g_0 \rhd^{e_1} g_1 \rhd^{e_2} ... \rhd^{e_m} g_m, \; g_i \in S, \; e_i = \pm 1.$$

Let

$$x_1 = x \rhd^{-e_m} g_m \rhd^{-e_{m-1}} ... \rhd^{-e_1} g_1, \; x_2 = x_1 \rhd^n g_0 \rhd^{e_1} g_1 \rhd^{e_2} ... \rhd^{e_m} g_m.$$

Then

$$x \rhd^n y = x_2$$

(by left association and right cancellation). The relation $x \rhd^n y = x$, or $x_2 = x$, is d-derived from $x_1 \rhd^n g_0 = x_1$ in m right multiplications, namely $\rhd^{e_1} g_1 \rhd^{e_2} ... \rhd^{e_m} g_m$. The corresponding secondary relations,

$$z \rhd (x \rhd^n y) = z \rhd x,$$

can be d-derived from primary relations by recasting the following as a d-derivation:

$$z \rhd (x \rhd^n y) = z \rhd^{-n} y \rhd x \rhd^n y = z \rhd^{-n} y \rhd x = z \rhd x.$$

The recasting procedure is analogous to Lemmas 4.10.7 and 4.10.10. ■

The reader may now verify the diagram in Example4.3.10 by tracing the relations

$$R = \{a \rhd b \rhd a = b,\ b \rhd a \rhd b = a\}$$

$$R_S = \{x \rhd^{-1} a \rhd^{-1} b \rhd a \rhd b \rhd a = x \rhd b \text{ for all } x,$$

$$x \rhd^{-1} b \rhd^{-1} a \rhd b \rhd a \rhd b = x \rhd a \text{ for all } x\}$$

$$I_S = \{a \rhd a = a,\ b \rhd b = b\}$$

and, in addition,

$$\begin{aligned} x \rhd^3 a &= x, \\ x \rhd^3 b &= x \end{aligned}$$

for all vertices x in a diagram.

4.12 The Square Knot Distinguished from the Granny Knot by Examination of Their 4-quandles

In this section, the diagramming method is used to show the 4-quandle of the granny knot is non-isomorphic to that of the square knot. Our criterion is algebraic.

Proposition 4.12.1 *Let Q^* denote the quandle $Q_4\langle a,\ b : a \rhd b \rhd a = b\rangle$, T the trefoil knot, and $\bar{T}$ its mirror image; $T \# T$, $T \# \bar{T}$ are the granny and square knots respectively. There exists a homomorphism from $Q_4 (T \# T)$ onto Q^*, but no homomorphism exists from $Q_4 (T \# T)$ onto Q^*.*

This proposition, once verified, yields a quandle-theoretic proof of the (well-known) distinctions of the two knots. Therefore, the proposition yields the (also well-known) distinctness of the trefoil knot T from its mirror image $\bar{T}$.

In order to verify the proposition we will diagram

$$Q^* = Q_4\langle a, b : a \rhd b \rhd a = b\rangle.$$

The diagram (Figure 4.12.4) is conveniently finite. We will use it to verify inequality in Q^* of various pairs of expressions in a, b. Note such inequality information cannot be obtained by simply deriving algebraic identities from the relations for Q^*.

We begin verification by obtaining presentations of the n-quandles $Q_n (T \# T)$ and $Q_n \left(T \# \bar{T}\right)$. The case $n = 4$ is the present focus. The two knots are illustrated in Figure 4.12.1. In order to present the quandle of a knot an orientation, a direction of travel along the knot is chosen (Section 3.4). The orientation is chosen arbitrarily for the knot $T \# T$ because the two oppositely oriented forms are ambient isotopic via a 180° rotation along a horizontal axis (Figure 4.12.2). Similarly, the orientation for $T \# \bar{T}$ is chosen arbitrarily. Orient the knots as shown in Figure 4.12.1, label the arcs as shown, and obtain the following presentations (after eliminating all but three generators).

$$Q_n(T \# T) = Q_n\langle a_1, b_1, c_1 : a_1 \rhd b_1 \rhd a_1 = b_1,$$

$$a_1 \rhd c_1 \rhd^{-1} a_1 = c_1,$$

$$b_1 \rhd a_1 b_1 = c_1 \rhd^{-1} a_1 \rhd^{-1} c_1\rangle$$

$$Q_n\left(T \# \bar{T}\right) = Q_n\langle a_2, b_2, c_2 : a_2 \rhd b_2 \rhd a_2 = b_2,$$

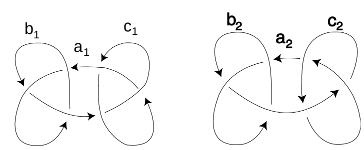
$$a_2 \rhd c_2 \rhd a_2 = c_2,$$

$$b_2 \rhd a_2 \rhd b_2 = c_2 \rhd a_2 \rhd c_2\rangle$$

Obtain the homomorphism $h : Q_4\left(T \# \bar{T}\right) \to Q^*$ by letting

$$h(a_2) = a, \; h(b_2) = h(c_2) = b.$$

We verify these assignments extend to a well-defined homomorphism. Figure 4.12.3 is a geometric interpretation of the homomorphism h based on the mirror symmetry of the square knot. Corresponding quandle elements on opposite sides of the plane of symmetry are identified. The image quandle Q^* has the same presentation as $Q_4(T)$, the 4-quandle of the trefoil knot, except one relation is omitted.



Granny T # T Square T # T

Figure 4.12.1. *Granny knot and square knot*

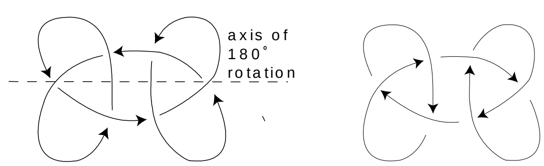


Figure 4.12.2. *Ambient isotopy of the two choices of orientation for the granny knot*

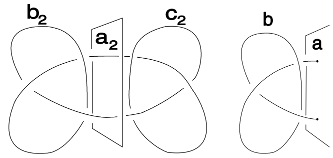


Figure 4.12.3. *Geometric interpretation of homomorphism from* $Q_4\left(T \# \bar{T}\right)$ *onto* Q^*

Now we verify h is well-defined by applying Definition 4.2.10 and Remark 4.2.11. Examine the image of each defining relation for $Q_4\left(T \# \bar{T}\right)$. The images are

$$a \rhd b \rhd a = b \text{ (twice)}$$

and

$$b \rhd a \rhd b = b \rhd a \rhd b$$

and obviously are satisfied in Q^*. Note h is surjective (onto) because both generators of Q^* are in the image of h.

A longer computation is needed to verify there is no homomorphism from $Q_4(T \# T)$ onto

$$Q^* = Q_4\langle a, b : a \rhd b \rhd a = b\rangle.$$

We begin by diagramming Q^*, Figure 4.12.4. This diagram is readily produced by the methods of Section 4.11. We use this diagram to show various relations do not hold in Q^*; these non-relations block various possible homomorphisms.

Now we consider the possibilities for homomorphisms from $Q_4(T \# T)$ onto Q^*. Any homomorphism h is determined by the mapping of a set of generators. Thus, we consider the possible values for $h(a_1)$, $h(b_1)$, $h(c_1)$ in Q^* and check if a well-defined homomorphism results in each case.

The number of cases to be examined is reduced since $h(a_1)$ may be set equal to a without loss of generality. The inner automorphism group Inn Q^* (Definition 4.1.11) is transitive on the elements of Q^*. That is, for every element $q \in Q^*$, there is an inner automorphism $\phi \in Q^*$ mapping q to a. (Note the diagram of Q^* is connected as a graph, and the arcs of the graph represent the action of the inner automorphisms $\phi_a : x \to x \rhd a$ and $\phi_b : x \to x \rhd b$.) Therefore, if surjective homomorphism $h : Q_4(T \# T) \to Q^*$ existed, a surjective homomorphism $h^1 : Q_4(T \# T) \to Q^*$with $h^1(a_1) = a$,where $h^1 = h \circ \phi$ for some $\phi \in$ Inn Q^* would exist also.

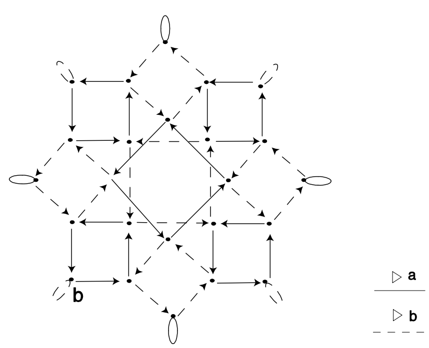


Figure 4.12.4. *Diagram of* $Q^* = Q_4\langle a, b : a \rhd b \rhd a = b\rangle$

We next consider the possible values of $h(b_1)$ given $h(a_1) = a$. Because

$$a_1 \rhd b_1 \rhd a_1 = b_1,$$

we must have

$$a \rhd h(b_1) \rhd a = h(b_1) \text{ (Definition4.1.7).}$$

The possibilities for $h(b_1)$ can be read from the diagram of Q^* and tested by diagrammatic multiplication (Sections 4.5, 4.11). Only $h(b_1) = a, hb_1) = b \rhd^k a, k = 0, 1, 2, 3$ prove satisfactory (where k has been reduced mod 4). Similarly, $a_1 \rhd^{-1} c_1 \rhd^{-1} a_1 = c_1$ implies $a \rhd^{-1} h(c_1) \rhd^{-1} a = h(c_1)$, whence $h(c_1) = a$ or $h(c_1) = a \rhd^{-1} b \rhd^m a$, $m = 0, 1, 2, 3$. Finally, because

$$b_1 \rhd a_1 \rhd b_1 = c_1 \rhd^{-1} a_1 \rhd^{-1} c_1,$$

the following must be true:

$$h(b_1) \rhd a \rhd h(b_1) = h(c_1) \rhd^{-1} a \rhd^{-1} h(c_1).$$

With $h(b_1), h(c_1)$ selected from the admissible values just given, equality results only when

$$h(b_1) = h(c_1) = a.$$

Although these values yield a homomorphism, it is not surjective. This completes the verification.

We note the diagram of Q^* is used repeatedly in this verification to show inequality of elements. Derivation of algebraic identities from the relations given for Q^* is not sufficient to show such inequality. Rather, demonstration of inequality concerning correctness of the diagramming method rests on Theorem 4.11.6.

To close this section, consider possible generalizations of Proposition 4.12.1. Let K be any tame knot. The image quandle Q^* of the Proposition generalizes as follows. Let

$$Q(K) = Q\langle S : R\rangle$$

be the knot quandle presented as in Section 3.3. Let R' be a set of relations obtained from R by deleting any one of the given relations. Let

$$\begin{aligned} Q^*(K) &= Q\langle S : R\rangle, \\ Q_n^*(K) &= Q_n\langle S : R\rangle. \end{aligned}$$

Then Q^* of the Proposition is $Q_4^*(T)$. $Q^*(K)$, $Q_n^*(K)$ are knot invariants (proof is omitted for reasons of space). We now have the following generalization of one half of Proposition 4.12.1.

Proposition 4.12.2 *A surjective homomorphism* $h : Q_n\left(K \# \bar{K}\right) \to Q_n^*(K)$ *exists for every tame knot* K *and every* $n \geqslant 2$.

The homomorphism merges symmetrically opposite quandle elements as in Figure 4.12.3.

Generalization for $Q_n(K \# K) \to Q_n^*(K)$ is more complex. Existence/nonexistence of a surjective homomorphism varies with both K and n. When K is invertible $\left(K = \bar{K}\right)$, the homomorphism for $Q_n\left(K \# \bar{K}\right)$ serves for $Q_n(K \# K)$. On the other hand, for the trefoil $K = T$, existence varies with n. For $n = 2, 3$

$$Q_n(T \# T) = Q_n\left(T \# \bar{T}\right)$$

(proof omitted) yielding existence of a surjective homomorphism. Nonexistence holds for $n = 4$ (Proposition 4.12.1) and $n = 5$ (a lengthier but similar verification).

Conjecture 4.12.3 *No surjective homomorphism* $Q_n(T \# T) \to Q_n^*(T)$ *exists for* $n \geqslant 4$.

Does there exist, for every non-invertible tame knot K, an n such that no surjective homomorphism $Q_n(T \# T) \to Q_n^*(T)$ exists? That is, can Proposition 4.12.1 be generalized to demonstrate $K \neq \bar{K}$ for every non-invertible K?

5 Quandles, Groups, and Branched Covering Spaces

At this point, the reader may suspect close interrelationships exist between quandles and groups. For example, there is similarity between a knot quandle element and a meridian in the knot group (Section 3.3) and a similarity of quandle diagrams (Sections 4.3 - 4.11) to Cayley diagrams of groups. In the present section, we discuss those interrelationships between groups and quandles centered around the conjugate group and the n-conjugate groups (Definitions 5.1.1, 5.1.2); the n-fold cyclic branched covering spaces of knots and links; and the corresponding n-quandles (Theorem 5.2.2). By examining these interrelationships in detail, we show (Theorem 5.2.5) that a tame knot with trivial involutory quandle or trivial n-quandle (for some $n \geqslant 2$) must be the trivial knot.

5.1 The Conjugate and n-conjugate Groups of a Quandle

In this section, we define the conjugate group and the n-conjugate groups of an arbitrary quandle. These groups are obtained in an invariant manner and, therefore, are knot invariants for knot quandles. The fundamental groups of cyclic branched covering spaces are obtained from these groups (Sections 5.2).

The first group considered is the *conjugate group Conj* Q of a quandle Q and is obtained by interpreting quandle multiplication as group theoretic conjugation.

Definition 5.1.1 *The conjugate group Conj* Q *of a quandle* Q *is the presentation group*

$$\langle \bar{q} \text{ for } q \in Q : \bar{q}^{-1}\bar{p}\bar{q} = \bar{r} \text{ whenever } p \rhd q = r \text{ in } Q \rangle.$$

The presentation given in Definition 5.1.1 is unwieldy when Q *has more than a few elements. Theorem 5.1.7 offers a more convenient presentation of Conj* Q *for any presentation quandle* Q. *Note Joyce refers to Conj* Q *as Adconj* Q *[Joyce1982b].*

For any tame knot (or link) K, $Conj\ Q(K)$ is the knot (or link) group (Corollary 5.1.8). For any $q \in Q(K)$, $\bar{q}$ is the meridian path (Definition 3.3.2) of the quandle element q. Different quandles may have the same conjugate group. For example, the square and granny knots have non-isomorphic quandles and 4-quandles (Section 4.12) but have isomorphic knot groups $Conj\ Q(K)$.

Distinct elements q, $r \in Q$ may yield the same element $\bar{q} = \bar{r}$ in $Conj\ Q$. Consider $Q\left(T \# \bar{T}\right)$, the quandle of the square knot presented in Section 4.12. The elements a_2 and $q = b_2 \rhd a_2 \rhd b_2$ are distinct (they are distinct even in the homomorphic image Q^*, Section 4.12) but have the same meridian $\bar{a}_2 = \bar{q}$ (Figure 5.1.1).

The n-conjugate groups $n \geqslant 2$ can be defined for any quandle but are particularly useful in connection with n-quandles.

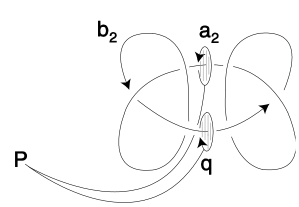


Figure 5.1.1. *Distinct quandle elements, same meridian*

Definition 5.1.2 *The n-conjugate group, $Conj_n\ Q$ of a quandle Q where $n \geqslant 2$, is the presentation group*

$$\langle \bar{q} \text{ for } q \in Q : \bar{q}^{-1}\bar{p}\bar{q} = \bar{r} \text{ whenever } p \rhd q = r \text{ in } Q, \bar{q}^n = 1 \text{ for } q \in Q\rangle.$$

$Conj_2\ Q$ is also referred to as the involutory conjugate group *$IConj\ Q$.*

The n-conjugate group is the image of the canonical homomorphism of the conjugate group. The generators are involutions in the involutory conjugate group. Thus the relations

$$\bar{q}^{-1}\bar{p}\bar{q} = \bar{r}$$

can be rewritten as

$$\bar{q}\bar{p}\bar{q} = \bar{r}$$

for this group. Generally, the involutory conjugate group has non-involutory elements as well (Example 5.1.12). Similarly, the n-conjugate group has elements x with $x^n \neq 1$. Joyce refers to $Conj_n\ Q$ as $AdQ_n\ Q$ [Joyce1982b].

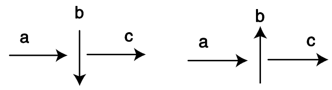
Exponent-zero subgroups of the above groups are defined below and are useful in Section 5.2.

Definition 5.1.3 *Let Q be any quandle. The exponent of an element of $Conj\ Q$ represented as a product of powers of generators $\bar{q}_1^{e_1}\bar{q}_2^{e_2}...\bar{q}_k^{e_k}$ is the sum of the exponents $e_1+e_2+...+e_k$. The exponent of an element of $Conj_n\ Q$ is the corresponding sum* mod *n. The* exponent-zero *subgroups $E^0Conj\ Q$, $E^0Conj_n\ Q$ consist of all elements of exponent-zero.*

The exponent-zero subgroups are quandle invariants and, therefore, are knot invariants for knot quandles. Moreover, K, $E^0Conj_n\ Q_n(K)$ is the fundamental group of the n-fold cyclic branched covering space of K for a tame knot or link (Theorem 5.2.2). In regard to this connection, we have the following.

Remark 5.1.4 *Let K be any tame oriented knot or link. The the exponent of any element $\alpha \in Conj\ \ Q(K) \cong \pi_1\left(S^3 - K\right)$ (Corollary 5.1.11) is the linking number of the corresponding path with K. The exponent of any $\beta \in Conj_n\ Q(K)$ is a linking number* mod *n–the linking number of any path α mapping to β under the canonical homomorphism $Conj\ Q(K) \to Conj_n\ Q(K)$. Thus the exponent-zero subgroups consist of paths of linking number zero (or zero* mod *n) with K.*

Remark 5.1.5 *Let K be any tame knot. Then $E^0 Conj\ Q\,(K)$ is the commutator subgroup of $Conj\ Q\,(K)$, and $E^0 Conj_n\ Q_n\,(K) \cong E^0 Conj_n\ Q\,(K)$ (Corollary 5.1.10) is the commutator subgroup of $Conj_n\ Q_n\,(K)$ for every $n \geq 2$.*



Q	$a \rhd b = c$	$a \rhd^{-1} b = c$
$ConjQ$	$\bar{b}^{-1}\bar{a}\bar{b} = c$	$\bar{b}\bar{a}\bar{b}^{-1} = \bar{c}$
π_1	$b^{-1}ab = c$	$bab^{-1} = c$

Figure 5.1.2. Presentations for $Q\,(K)$, $Conj\ Q\,(K)$, $\pi_1\left(S^3 - K\right)$

Now we develop the corresponding presentations for Conj Q and Conj$_n$ Q for any presentation quandle $Q = Q\langle S : R\rangle$. The corresponding presentations are considerably more convenient for computation than the presentation given in Definition 5.1.1. Furthermore, the convenient presentations lead to additional interesting results. A conjugation-based recasting of words over quandle generators to words over group generators must be defined.

Definition 5.1.6 *Let w be any word on a generating set S with operations $\rhd$, $\rhd^{-1}$ (Definition 4.2.4). Let $\bar{S} = \{\bar{s} : s \in S\}$. Then, $\bar{w}$ is the group theoretic word obtained from w by replacing $u \rhd v$ by $\bar{v}^{-1}\bar{u}\bar{v}$ and $u \rhd^{-1} v$ by $\bar{v}\bar{u}\bar{v}^{-1}$ throughout.*

For example, $\overline{(a \rhd b) \rhd c}$ is $\bar{c}^{-1}\bar{b}^{-1}\bar{a}\bar{b}\bar{c}$.

Theorem 5.1.7 *Let $Q = Q\langle S : R\rangle$ be any presentation quandle. Then, $Conj\ Q \cong \langle \bar{S} : \bar{R}\rangle$ where $\bar{S} = \{\bar{s} :\in S\}$ and $\bar{R} = \{\bar{r} = \bar{s} : r = s$ is a relation in $R\}$.*

Example 5.1.8

$$Q = Q\langle a, b : a \rhd b \rhd a = b, b \rhd a \rhd b = a\rangle;$$

$$ConjQ = \langle a, b : \bar{a}^{-1}\bar{b}^{-1}\bar{a}\bar{b}\bar{a} = \bar{b},\ \bar{b}^{-1}\bar{a}^{-1}\bar{b}\bar{a}\bar{b} = \bar{a}\rangle.$$

Proof. We show

1. $Conj\ Q$ is a homomorphic image of $\langle S : R\rangle$; and

2. the homomorphism is an isomorphism.

To prove (1), map every $\bar{q} \in \bar{S}$ to $\bar{q} \in Conj\ Q$. This mapping extends to a well-defined homomorphism h by Remark 4.2.11 for groups. Definitions 5.1.6 and 5.1.1 show $w = q$ in Q implies $\bar{w} = \bar{q}$ in $Conj\ Q$.

Proof of part (2) is an exercise in derivations (Section 4.2). First, prove if the relation $r = s$ is derivable from R, then $\bar{r} = \bar{s}$ is derivable from $\bar{R}$. Examine each derivation rule; only left and right multiplication present difficulty. If $r = s$ is the relation $u \triangleright v = u \triangleright w$ derived from $v = w$ by left multiplication, then $\bar{r} = \bar{s}$ is the relation

$$\bar{v}^{-1}\bar{u}\bar{v} = \bar{w}^{-1}\bar{u}\bar{w}$$

derivable as

$$\bar{v}^{-1}\bar{u}\bar{v} = \bar{w}^{-1}\bar{u}\bar{v} = \bar{w}^{-1}\bar{u}\bar{w}$$

by right multiplication, left multiplication, and transitivity. Right $\triangleright$ multiplication and $\triangleright^{-1}$are dealt with similarly.

When p, q, $r \in Conj\ Q$ are expressed in terms of the generators in $\bar{S}$, derivability of $p \triangleright q = r$ from R implies derivability of $\overline{p \triangleright q} = \bar{r}$. That is,

$$\bar{q}^{-1}\bar{p}\bar{q} = \bar{r},$$

from $\bar{R}$. Therefore, h is an isomorphism. ■

The following corollary describes $Conj\ Q\,(K)$ for knot and link quandles.

Corollary 5.1.9 *[Joyce1982a] Let K be any tame knot or link. Then $Conj\ Q\,(K)$ is the knot or link group $\pi_1\left(S^3 - K\right)$. For any disk with path q, $\bar{q}$ is the corresponding meridian path.*

Proof. Compare the presentations of $Q\,(K)$, $Conj\ Q\,(K)$ with the Wirtinger presentation of the knot group (Figure 5.1.2, Theorem 5.1.7, and Sections 3.3, 3.4). ■

Corollary 5.1.10 *If $Q = Q\langle S : R\rangle$ then*

$$Conj_n Q \cong \langle \bar{S} : \bar{R}, \bar{q}^n = 1 \text{ for } \bar{q} \in \bar{S}\rangle.$$

$$Conj_n Q\langle S : R\rangle \cong Conj_n Q_n\langle S : R\rangle.$$

Proof. For the first $\cong$,

$$\bar{q}^n = 1 \text{ for } \bar{q} \in \bar{S}$$

implies

$$\bar{q}^n = 1 \text{ for } q \in Q$$

because the remaining $\bar{q}$ are conjugates of those in $\bar{S}$. For the second $\cong$, the relations

$$x \rhd^n y = x \text{ for } Q_n$$

yield

$$\bar{y}^{-n} \bar{x} \bar{y}^n = \bar{x},$$

which reduces to

$$\bar{x} = \bar{x}$$

in the presence of $\bar{y}^n = 1$. ■

Therefore, $Conj_n$ only reflects quandle structure mod $x \rhd^n y = x$. This statement is useful in the next section.

Corollary 5.1.11 *Let K be any tame knot or link. Then, $Conj_n\ Q(K)$ is $\pi_1\left(S^3 - K\right)$* mod $\mu^n = 1$ *for meridians μ.*

Example 5.1.12 *Recall that $IConj$ is $Conj_2$ and IQ is Q_2. For the trefoil knot:*

$$IQ(K) = IQ\langle a, b : a \rhd b \rhd a = b, b \rhd a \rhd b = a\rangle;$$

$$IConjIQ(K) = \langle \bar{a}, \bar{b} : \bar{a}\bar{b}\bar{a}\bar{b}\bar{a} = \bar{b}, \bar{b}\bar{a}\bar{b}\bar{a}\bar{b} = \bar{a}, \bar{a}^2 = \bar{b}^2 = 1\rangle$$

is just the symmetric group S_3, and, in this group, $\bar{a}\bar{b}$ is a non-involutory element.

The following section relates $Conj_n$ groups to branched covering spaces.

5.2 Knots, n-fold Branched Covers and n-quandles

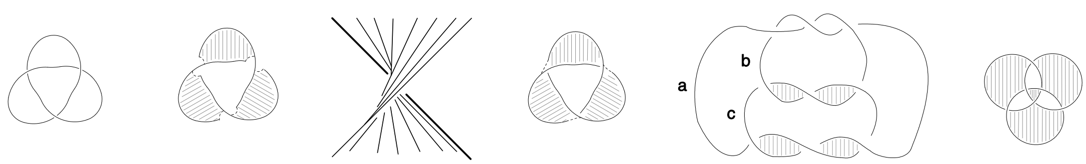
This final section demonstrates the n-quandles—$Q_n(K)$ of a tame knot or link $K \subset S^3$, $n \geq 2$—yield considerable information about the n-fold cyclic branched covering spaces $\tilde{M}_n(K)$ of K. Specifically,

$$E^0 Conj_n Q_n(K) = \pi_1\left(\tilde{M}_n(K)\right);$$

the exponent-zero subgroup of the n-conjugate group of the n-quandle is isomorphic to the fundamental group of the n-fold branched cover (Theorem 5.2.2). A tame knot with a trivial n-quandle $n \geq 2$ is trivial (Theorem 5.2.5).

A description of $\pi_1\left(\tilde{M}_n(K)\right)$ is required and begins with a brief review of covering spaces. Then, a discussion of $n = 2$ is followed by an outline of the generalization to $n > 2$.

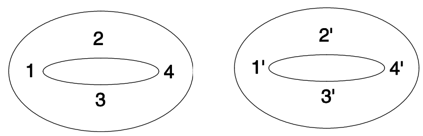
The two-fold covering space is constructed with the aid of a spanning surface F for K and is obtained as follows (Figure 5.2.1). Given a tame projection of K (item (i) of the Figure), color it in checkerboard fashion (ii). The dark areas become portions of the spanning surface F. Span each crossing (iii) with a twisted sheet to obtain F (iv). Now cut S^3 along the surface F. Glue two copies of the resulting manifold together along the cuts, as indicated schematically in Figure 5.2.2. The two copies of K are merged. The resulting manifold is the two-fold branched covering space $\tilde{M}_2(K)$. The two-fold unbranched covering space $M_2(K)$ is obtained by removing the copy of K from $\tilde{M}_2(K)$. Both $\tilde{M}_2(K)$ and $M_2(K)$ are invariants of K.



i ii iii iv v vi

Figure 5.2.1. *Spanning surface for trefoil knot* $(i - iv)$. *Checkerboard coloring is illustrated also for the knot* $9_{35}(v)$ *and the Borromean rings* (vi).

Now we outline the generalization to $n > 2$. Construct an *orientable* (contains no embedded Möbius band) spanning surface F. If K is a link, it must be oriented for this construction. The construction, involving *Seifert circles*, is more complicated. Again S^3 is cut along F, and n copies of the resulting manifold are glued together back to front along the cuts (Figure 5.2.3). The n copies of K are merged to yield the n-fold branched cover $\tilde{M}_2(K)$ or deleted to yield the n-fold unbranched cover $M_2(K)$. Both $\tilde{M}_2(K)$ and $M_2(K)$ are knot invariants.



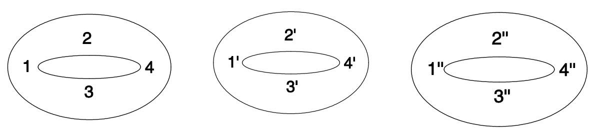
Glue 1 to 1‘

2 to 4‘

3 to 3‘

4 to 2‘

Figure 5.2.2. *Gluing of two copies of a manifold to form* $\tilde{M}_2(K)$. *Schematically* 1, 3 *represent the knot* K*;* 2, 4 *represent cut along spanning surface* F*.*



Glue 1 to 1‘ to 1"

3 to 3‘ to 3"

2 to 4‘

2‘ to 4"

2" to 4

Figure 5.2.3. *Gluing of* n *copies of a manifold to form* $\tilde{M}_2(K)$. *Schematically 1, 3 represent the knot* K*, and 2, 4 represent cut along spanning surface* F*.*

Now we examine the fundamental groups of these coverings. Employ the projection $\pi : \tilde{M}_2(K) \to S^3$, which re-merges the n sheets of the cover. The projection π maps $M_n \subset \tilde{M}_n$ onto $S^3 - K$. Any path α in M_n projects onto a path $\pi(\alpha)$ in $S^3 - K$. Figure 5.2.4 gives such a projected path for the case $n = 2$. In the figure, the solid portion of $\pi(\alpha)$ lifts to one sheet of M_2, and the dotted portion lifts to the other sheet. The path α passes from one sheet to the other whenever $\pi(\alpha)$ passes through the spanning surface F. The path must both start and end in the sheet of the basepoint $P \in M_2$. Therefore, for any path α in M_2, $\pi(\alpha)$ passes through F an even number of times. Equivalently, $\pi(\alpha)$ is evenly linked with K and is homotopic to the product of an even number of meridians. In general, $n\ \pi(\alpha)$ has a linking number (with K) congruent to zero mod n.

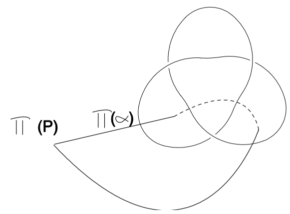


Figure 5.2.4. *Projection of a path α in $\tilde{M}_2$ onto a path $\pi(\alpha)$ in $S^3 - K$.*

Conversely, any product α_0 of an even number of meridians (for $n = 2$) or (for arbitrary n) with exponent zero mod n lifts to a path α in M_n. Two paths α, β in M_n are homotopic, $\alpha \sim \beta$, iff $\pi(\alpha) \sim \pi(\beta)$ in $S^3 - K$ because homotopies project and lift. Hence

$$\pi_1(M_n) \cong E^0 \pi_1\left(S^3 - K\right) \cong E^0 ConjQ(K),$$

the subgroup of the knot group consisting of all elements of exponent zero mod n.

Now consider $\pi_1\left(\tilde{M}_n\right)$. Any path in $M_n \subset \tilde{M}_n$ is a path in $\tilde{M}_n$, and any path in $\tilde{M}_n$ is homotopic to a path in M_n. But isomorphism of the fundamental groups does not follow because there are additional homotopies available in $\tilde{M}_n$. In particular, if $\pi(\alpha) = \mu^n$ for some meridian μ of $K \subset S^3$, then $\alpha \sim 1$ in $\tilde{M}_n$ (Figure 5.2.5). Since $\mu^n = 1$ for all meridians μ of $K \subset S^3$, $\alpha \sim 1$ in $\tilde{M}_n$ (Figure 5.2.5). That the relations $\mu^n = 1$ for meridians μ of K are the only relations added by the presence of the lift of K is proved in [ ].

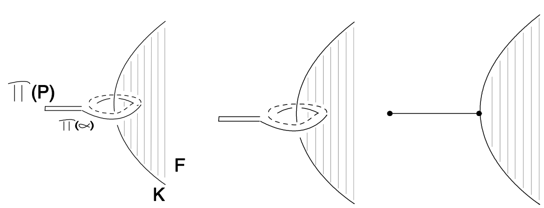


Figure 5.2.5. *Homotopy of path α (in $\tilde{M}_2$) to trivial path, when $\pi(\alpha) \sim \mu^2$, a squared meridian*

We have obtained $\pi_1\left(\tilde{M}_n(K)\right)$ as follows. In $Conj\ Q(K) \cong \pi_1\left(S^3 - K\right)$, form the subgroup of elements of exponent zero mod n. In this subgroup, mod *out* by $\mu^n = 1$ for meridians μ to obtain $\pi_1\left(\tilde{M}_n(K)\right)$. Interchange the order of operations; mod *out* by $\mu^n = 1$ and form the subgroup of exponent zero mod n. The following lemma guarantees the same group results.

Lemma 5.2.1 *Let G be any group and $H < G$ any subgroup. Let C be any subset of H closed under conjugation in G. Let $\langle C \rangle$ be the subgroup of H generated by C. Then $\langle C \rangle$ is normal in H and in G, and $H/\langle C \rangle$ is isomorphic to the subgroup of $G/\langle C \rangle$ generated by all elements (cosets) $\langle C \rangle h$, $h \in H$.*

Proof. $\langle C \rangle$ is normal in G, H because C is closed under conjugation in G, H. Normality and $\langle C \rangle \subset H$ yield the result. ■

We apply the lemma by letting

$$\begin{aligned} G &= ConjQ\left(K\right) \cong \pi_1\left(S^3 - K\right), \\ H &= \pi_1\left(M_n\right), C \end{aligned}$$

the set of all n^{th} powers of meridians. Therefore,

$$\pi_1\left(\tilde{M}_n\right) \cong \pi_1\left(M_n\right)/\langle C \rangle \cong E^0 Conj_n Q\left(K\right) \cong E^0 Conj_n Q_n\left(K\right)$$

(Corollaries 5.1.10, 5.1.11). Therefore, the following is proved.

Theorem 5.2.2 *Let K be any tame oriented knot or link in S^3. Let $\tilde{M}_n$ be the corresponding n-fold cyclic branched covering space, $n \geqslant 2$. Then the fundamental group $\pi_1\left(\tilde{M}_n\right)$ is isomorphic to the exponent-zero subgroup $E^0 Conj_n Q\left(K\right) \cong E^0 Conj_n Q\left(K\right)$.*

Note

1. $\pi_1\left(\tilde{M}_n\right)$ can be obtained given just the n-quandle, and

2. it is not necessary K be oriented for $n = 2$.

Example 5.2.3 (*Illustration of Theorem 5.2.2 for K the trefoil knot, $n = 2$.*) *We have*

$$\begin{aligned} Q_2\left(K\right) &= IQ\left(K\right) = IQ\langle a,\ b : a \triangleright b \triangleright a = b,\ b \triangleright a \triangleright b = a \rangle. \\ Conj_2 Q_2\left(K\right) &= IConj\ IQ\left(K\right) = \langle \bar{a},\ \bar{b} : \bar{a}\bar{b}\bar{a}\bar{b}\bar{a} = \bar{b},\ \bar{a}^2 = \bar{b}^2 = 1 \rangle, \end{aligned}$$

which is just the symmetric group S_3. The exponent-zero subgroup is generated by $\bar{a}\bar{b}$ and is isomorphic to $\mathbb{Z}_3$. Therefore, the fundamental group of the two-fold cyclic branched covering space of the trefoil knot is $\mathbb{Z}_3$.

The Smith Conjecture [Smith 1965] is required to prove our final result and is stated as follows.

Theorem 5.2.4 (*Smith Conjecture*). *Let K be any tame knot in S^3. If the n-fold cyclic branched covering space of $K \subset S^3$ has trivial fundamental group for some $n \geqslant 2$, then K is the trivial knot.*

The proof for $n = 2$ appears in [Waldhausen 1969]. The general case is proved, but no summary paper has appeared.

Our final result is proven with the aid of the following theorem.

Theorem 5.2.5 *Let K be any tame knot in S^3. If K has trivial n-quandle $Q_n(K)$ for some $n \geqslant 2$, then K is the trivial knot.*

Proof. $Q_n(K)$ trivial implies $Conj_n Q_n(K) \cong \mathbb{Z}_n$. The exponent-zero subgroup is trivial; $\pi_1\left(\tilde{M}_n\right)$ is trivial by Theorem 5.2.2; K is trivial by the Smith Conjecture. ■

Theorem 5.2.5 represents a first step toward deriving knot structure information from involutory quandle and n-quandle structure.

References

[Alexander and Briggs1976] Alexander, J. W. and G. B. Briggs, On Types of Knotted Curves, *Ann. of Math.* 28 (1926), pp. 562-566.

[Crowell and Fox1977] Crowell R. and R. H. Fox, *An Introduction to Knot Theory*, 2nd edition, Springer-Verlag, 1977.

[Graetzer and Taylor]

[Joyce1979] Joyce, D., *An Algebraic Approach to Symmetry With Applications to Knot Theory*, Ph. D. Thesis, University of Pennsylvania, 1979.

[Joyce1982a] Joyce, D., A Classifying Invariant of Knots, the Knot Quandle, *Journal of Pure and Applied Algebra* 23 (1982) 37-65.

[Joyce1982b] Joyce, D., Simple Quandles, *Journal of Algebra* 79 (1982) pp. 307-318.

[Rolfsen 1976] Rolfsen, D., *Knots and Links*, Publish or Perish, Berkeley, CA, 1976.

[Smith 1965] Smith, P., Periodic Transformations of 3-manifolds, *Ill. J. Math.* 9 (1965) 343-348.

[Waldhausen 1968] Waldhausen, F., On Irreducible 3-manifolds Which Are Sufficiently Large, *Ann. of Math.* 87 (1968) pp. 56-88.

[Waldhausen 1969] Waldhausen, F., Uber Involutionen der 3-sphäre, *Topology* 8 (1969) pp. 81-91.