BOUNDED AND DIVERGENT TRAJECTORIES AND EXPANDING CURVES ON HOMOGENEOUS SPACES

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Abstract. Suppose g_t is a 1-parameter Ad-diagonalizable subgroup of a Lie group G and $\Gamma < G$ is a lattice. We study the dimension of bounded and divergent orbits of g_t emanating from a class of curves lying on leaves of the unstable foliation of g_t on the homogeneous space G/Γ. We obtain sharp upper bounds on the Hausdorff dimension of divergent on average orbits and show that the set of bounded orbits is winning in the sense of Schmidt (and, hence, has full dimension). The class of curves we study is roughly characterized by being tangent to copies of $\mathrm{SL}(2, \mathbb{R})$ inside G, which are not contained in a proper parabolic subgroup of G.

We describe applications of our results to problems in Diophantine approximation by number fields and intrinsic Diophantine approximation on spheres. Our methods also yield the following result for lines in the space of square systems of linear forms: suppose $\varphi(s) = sY + Z$ where $Y \in \mathrm{GL}(n, \mathbb{R})$ and $Z \in M_{n,n}(\mathbb{R})$. Then, the dimension of the set of points s such that $\varphi(s)$ is singular is at most $1/2$ while badly approximable points have Hausdorff dimension equal to 1.

Contents

1. Introduction

1.1. **Summary of the results.** The purpose of this article is to study the Hausdorff dimension of bounded and divergent orbits of diagonalizable flows emanating from curves on homogeneous spaces. The motivation for studying these problems comes from the theory of Diophantine approximation. The class of curves we study is roughly characterized by being tangent to maximal representations of $\mathrm{SL}(2, \mathbb{R})$ into the ambient Lie group G. These are representations whose images are not contained in a proper parabolic subgroup G. See Definition 10.1 for a precise description. In

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this setting, we provide a sharp upper bound on the dimension of divergent on average trajectories (Definition 2.1) and show that bounded orbits are winning for a Schmidt game on intervals of the real line (see Section 5 for detailed definitions). Moreover, we establish, in a quantitative form, the non-divergence of push-forwards of shrinking curve segments (cf. Proposition 6.1).

For concreteness, we state our results in the introduction in the examples which are most relevant to applications in Diophantine approximation, deferring the more general statements to Theorems 4.3, 10.7, and 11.6. These concrete examples include homogeneous spaces of products of real rank 1 Lie groups (Theorem A), a more general class of curves on homogeneous spaces of products of $\mathrm{SO}(n,1)$ (Theorem B and 11.6), and actions of $\mathrm{SL}(2,\mathbb{R})$ on any homogeneous space of finite volume (Theorem C). Curves on more general arithmetic homogeneous spaces are studied in Section 10.

In Section 3, we present applications of our results to problems in intrinsic Diophantine approximation on spheres (Corollary 3.1), Diophantine approximation by number fields (Corollary 3.2), and Diophantine properties of lines in the space of square systems of linear forms $M_{n,n}$ (Corollary 3.3).

1.2. **Historical context.** To the best of our knowledge, the problem of the dimension of divergent orbits starting from curves has not been previously addressed in the literature. Among the motivations for studying this problem is a well-known deep conjecture, due to Wirsing, concerning the approximability of transcendental numbers by algebraic numbers of bounded degree. By the work of Bugeaud and Laurent, the Hausdorff dimension of the set of counterexamples to Wirsing's conjecture in degree n is bounded above by the dimension of singular vectors in $M_{1,n} \cong \mathbb{R}^n$ lying on the Veronese curve $\left\{(\xi,\xi^2,\dots,\xi^n) : \xi \in \mathbb{R}\right\}$ [BL05].

To place our results in context, we briefly survey the history of the subject. In [Dan86, Dan89], Dani studied the problem of bounded orbits in two settings: orbits of diagonalizable flows on homogeneous spaces of rank 1 Lie groups and orbits in $\mathrm{SL}(m+n,\mathbb{R})/\mathrm{SL}(m+n,\mathbb{Z})$ of the form $g_t u_Y \Gamma$, where, for $t \in \mathbb{R}$ and $Y \in M_{m,n}$ an $m \times n$ real matrix,

$$g_t = \mathrm{diag}(e^{t/m},\dots,e^{t/m},e^{-t/n},\dots,e^{-t/n}), \quad u_Y = \begin{pmatrix} \mathrm{I}_m & Y \\ \mathbf{0} & \mathrm{I}_n \end{pmatrix}. \tag{1.1}$$

We refer to g_t as a diagonal element with weight $(1/m,\dots,1/m,1/n,\dots,1/n)$. It is shown that bounded orbits of diagonalizable flows on rank 1 homogeneous spaces have full Hausdorff dimension. It is also shown that orbits of the form $(g_t u_Y \Gamma)_{t\geqslant 0}$ are bounded if and only if Y is badly approximable, i.e., there exists $\delta > 0$ such that for all $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$, with $\mathbf{q} \neq 0$,

$$\|Y\mathbf{q} - \mathbf{p}\|^n \, \|\mathbf{q}\|^m > \delta.$$

Using the results of Schmidt on badly approximable systems of linear forms [Sch69], this implies that bounded orbits for g_t as in (1.1) have full dimension. These results were generalized in [KW10, KW13] where bounded orbits of *non-quasiunipotent* flows were shown to have full dimension.

All of these results were obtained by showing that bounded orbits are winning for variants of a game invented by Schmidt in [Sch66]. The winning property is much stronger than having full Hausdorff dimension since it is stable under countable intersections and implies thickness, i.e., the intersection of a winning set with any non-empty open set has full dimension. We refer the reader to [KW10] for more details on Schmidt's original game as well as a new variant introduced by the authors. More recently, far reaching generalizations of these results were obtained in [BPV11], in particular settling an old conjecture of Schmidt on the intersection of sets of weighted badly approximable vectors with different weights.

Dani also studied the existence and classification of divergent orbits of diagonalizable flows on homogeneous spaces in [Dan85]. Among the results obtained by Dani is the fact that divergent orbits on non-compact homogeneous spaces of a rank 1 Lie group G are *degenerate*, i.e., can be detected using the behavior of finitely many vectors in some fixed representation of G. In particular, the set of divergent orbits consists of a countable collection of immersed submanifolds in G/Γ. This result also holds for quotients of Lie groups by arithmetic lattices of rational rank 1. By contrast, quotients by higher rank arithmetic lattices always admit non-degenerate divergent orbits [Dan85, Wei04].

In a landmark paper, the precise Hausdorff dimension of divergent orbits under the flow induced by g_t in (1.1) was calculated when $(m,n) = (2,1)$ in [Che11]. This result was extended in [CC16] to the case when $\min(m,n) = 1$. These results build on earlier ideas of Cheung in [Che07] where the Hausdorff dimension of divergent orbits in $\mathrm{SL}(2,\mathbb{R})^n/\mathrm{SL}(2,\mathbb{Z})^n$ for $n \geq 2$ under the flow induced by a diagonal matrix in each coordinate was determined to be $3n - 1/2$. In [KKLM17], a sharp upper bound on the dimension of divergent orbits for general m and n was obtained by different methods. The proof in [KKLM17] relies on the powerful technique of systems of integral inequalities introduced in [EMM98] in the context of quantifying Margulis' work on the Oppenheim conjecture.

Parallel to these developments and motivated by problems in Diophantine approximation, the study of the evolution of curves on homogeneous spaces under diagonal flows attracted a lot of interest. In [KM98], Kleinbock and Margulis showed that the push-forward of certain "non-degenerate" smooth curves in the group $\{u_Y : Y \in M_{1,n}\}$ by diagonal elements similar to g_t in (1.1) do not diverge in $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SL}(n+1,\mathbb{Z})$. This allowed them to settle a conjecture due to Baker and Sprindžuk showing that the Lebesgue measure of very well approximable vectors belonging to such curves is 0. This result has been generalized in numerous directions, cf. [KLW04, BKM15, ABRdS18] for notable examples.

In [Sha09b, Sha10], using Ratner's theorems and the linearization technique, Shah extended the results of Kleinbock and Margulis by showing that the push-forwards of the parameter measure on these curves, in fact, become equidistributed towards the Haar measure on G/Γ. These results build on earlier work of Shah in [Sha09c, Sha09a] where the push-forwards of certain smooth curves on the unit tangent bundle of hyperbolic manifolds by the geodesic flow were shown to be equidistributed towards the Haar measure.

On the other hand, the problem of determining the Hausdorff dimension of bounded and divergent orbits restricted to curves as above is far less understood. In a breakthrough article, Beresnevich showed in [Ber15] that the Hausdorff dimension of finite intersections of weighted badly approximable vectors on non-degenerate analytic curves in $M_{1,n}$ is full. By means of Dani's correspondence, this implies that bounded orbits of diagonal elements similar to g_t in (1.1) with more general weights than $(1, 1/n, \ldots, 1/n)$ starting from points on curves on the group $\{u_Y : Y \in M_{1,n}\}$ is equal to 1. We refer the reader to [Ber15] for more on the history of this problem and to [ABV18] where these bounded orbits were shown to be in fact winning in the sense of Schmidt for planar curves. The dimension of bounded orbits starting from curves on other homogeneous spaces was studied in [Ara94] in rank 1 homogeneous spaces and in [EGL16] in quotients of $\mathrm{SL}(2,\mathbb{R})^r \times \mathrm{SL}(2,\mathbb{C})^s$ by irreducible lattices.

2. Main Results

2.1. **Preliminary Notions.** Before stating our main results, we need to introduce necessary definitions and notation. Given a real Lie group G, we denote by $\mathfrak{g}$ its Lie algebra. For a 1-parameter subgroup g_t of G, we say g_t is Ad-diagonalizable over $\mathbb{R}$ if $\mathfrak{g}$ decomposes over $\mathbb{R}$ under the Adjoint action of g_t into eigenspaces.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{R}} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \left\{ Z \in \mathfrak{g} : \mathrm{Ad}(g_t)(Z) = e^{\alpha t} Z \right\}.$$

We remark that the decomposition above is only an eigenspace decomposition with respect to $\mathrm{Ad}(g_t)$, not a decomposition into root spaces. Suppose that G acts on a metric space X. Our goal is to study the Hausdorff dimension of certain orbits of g_t on X with prescribed recurrence properties. For that purpose, let us make precise the recurrence notions we shall be interested in.

Definition. For a flow $g_t : X \to X$ on a metric space X and $y \in X$, we say the (forward) orbit $g_t y$ is **divergent on average**, if for any compact set $Q \subset X$, one has

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_Q(g_t y)\, dt = 0 \tag{2.1}$$

where χ_Q denotes the indicator function of Q. We say the orbit $g_t y$ is **bounded** if $\overline{\{g_t y : t > 0\}}$ is compact. The orbit $g_t y$ is said to have **linear growth** if for some base point y_0, we have

$$\limsup_{t\to\infty} \frac{d(g_t y, y_0)}{t} > 0, \tag{2.2}$$

where $d(\cdot,\cdot)$ is the metric on X.

Finally, recall that a subset A of a metric space is **thick** if the intersection of A with every non-empty open set has full Hausdorff dimension.

2.2. **Homogeneous Spaces of Products of Rank One Lie Groups.** Our first result is in the setting of homogeneous spaces of Lie groups of the form $G = G_1 \times \cdots \times G_k$, where each G_i is a real rank one Lie group. To state the result, we need some preparation.

Suppose Γ is any lattice in G. Then, we can write $\Gamma = \Gamma_1 \times \cdots \times \Gamma_l$ (up to finite index), where each Γ_j is an irreducible lattice in a sub-product of G, which we denote by H_j. By Margulis' arithmeticity theorem, if for some $1 \le j \le l$, H_j is a product of more than 1 factor (i.e. $\mathrm{rank}_{\mathbb{R}}(H_j) > 1$), then there exists a rational structure on H_j in which Γ_j is arithmetic, i.e. Γ_j is commensurable with $H_j(\mathbb{Z})$.

We say that a 1-parameter subgroup g_t of G is **split** if the projection of g_t onto each higher rank factor H_j is Ad-diagonalizable over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-structure in which Γ_j is arithmetic. The following maps into $\mathfrak{g}$ are the main object of study in this setting.

Definition. For a compact interval $B \subset \mathbb{R}$ and an Ad-diagonalizable subgroup g_t, we say a differentiable map $\varphi : B \to \mathfrak{g}$ is g_t**-admissible** if the image of φ is contained in a single eigenspace $\mathfrak{g}_\alpha$ for some $\alpha > 0$ and $[\varphi, \dot\varphi] \equiv 0$ on B. For every s, we denote by $u(\varphi(s))$ the image of $\varphi(s)$ in G under the exponential map.

Denote by $\mathfrak{g}_i$ the Lie algebra of G_i. The following is the first main theorem of this article.

Theorem A. *Suppose $G = G_1 \times \cdots \times G_k$, where each G_i is a simple Lie group of real rank 1 and finite center and Γ is any lattice in G. For each $1 \le i \le k$, let $g_t^{(i)}$ be a non-trivial 1-parameter subgroup of G_i which is* Ad*-diagonalizable over $\mathbb{R}$, and suppose $\varphi_i : B \to \mathfrak{g}_i$ is a $g_t^{(i)}$-admissible C^2-map. Let $g_t = (g_t^{(i)})_{1\le i\le k}$ and $\varphi = \oplus_{i=1}^k \varphi_i$. Assume that g_t is split and that φ is g_t-admissible. Define the following set.*

$$Z = \{s \in B : \dot\varphi_i(s) = 0 \text{ for some } 1 \le i \le k\}.$$

Then, for every $x_0 \in X = G/\Gamma$, the following hold.

(i) The Hausdorff dimension of the set of points $s \in B\backslash Z$ for which the orbit $(g_t u(\varphi(s))x_0)$ is divergent on average as $t \to \infty$ is at most $1/2$.
(ii) For any compact interval $V \subseteq B\backslash Z$, the set of points $s \in V$ for which the orbit $(g_t u(\varphi(s))x_0)_{t\geqslant 0}$ is bounded in X is winning for a Schmidt game on V induced by g_t. In particular, this set is thick in $B\backslash Z$.
(iii) For almost every $s \in B\backslash Z$, any weak-$*$ limit of the measures $\frac{1}{T}\int_0^T \delta_{g_t u(\varphi(s))x_0} ds$ is a probability measure on X.
(iv) The set of points $s \in B\backslash Z$ for which the forward orbit $(g_t u(\varphi(s))x_0)_{t\geqslant 0}$ has linear growth has Lebesgue measure 0.

Remark 2.1. In studying divergent orbits, it is necessary for our methods that the diagonalizable flow we consider in Theorem A expands the curve by the same amount in every coordinate. In Theorems B and 11.6 below, we relax this assumption, where we allow some of the coordinate flows g_t^i to be trivial. It is an interesting question as to whether similar results hold for more general diagonal flows.

We refer the reader to Section 5 for details on Schmidt games and a more precise form of part (ii) of Theorem A. Number theoretic corollaries of Theorem A concerning intrinsic Diophantine approximation on spheres are discussed in Section 3.1.

We note that the assumption in Theorem A that $\varphi = \oplus_{i=1}^k \varphi_i$ is g_t-admissible amounts to ensuring that the eigenspace of $\mathrm{Ad}(g_1^{(i)})$ containing the image of φ_i corresponds to the same eigenvalue for each i. Moreover, the restriction to the points in $B \backslash Z$ is natural since it is possible for the map φ to map a sub-interval of B onto a point whose orbit is divergent.

Remark 2.2. The proof of Theorem A is reduced to the case when Γ is an irreducible lattice in G. When $\mathrm{rank}_{\mathbb{R}} G > 1$, Γ is an arithmetic lattice by Margulis' arithmeticity theorem. In that case, Theorem A is a special case of a more general result we obtain for quotients of semisimple algebraic Lie groups by arithmetic lattices, Theorem 10.7.

In [Ara94], in the setting of rank one locally symmetric spaces, it is shown that bounded orbits under the geodesic flow restricted to non-constant C^1-maps on the unit tangent sphere around a point is winning in the sense of Schmidt. The methods in [Ara94] rely on the geometry of rank 1 locally symmetric spaces. Our proof is completely different and remains valid in more generality. Theorems B and C below are other instances where our methods also apply. We refer the reader to Theorems 4.3 and 5.2 where we show an analogous statement to Theorem A in the abstract setting of Lie group actions on metric spaces satisfying certain recurrence hypotheses.

Remark 2.3. If we assume the image of a coordinate function φ_i is contained in an abelian subspace of $\mathfrak{g}_i$, we can weaken the regularity condition on φ_i to be $C^{1+\varepsilon}$ for some $\varepsilon > 0$. In particular, Theorem A holds for $C^{1+\varepsilon}$-maps when $G_i \cong \mathrm{SO}(d_i, 1)$ for each $1 \leq i \leq k$.

Using a result in [KP17], we deduce a lower bound on the dimension of the divergent on average orbits considered in Theorem A in a special case which agrees with the upper bound we obtain. We further discuss the sharpness of this bound, as well as the bounds obtained in the results below, in Section 13.

Corollary 2.4. *In the notation of Theorem A, suppose* $G/\Gamma = (\mathrm{SL}(2,\mathbb{R})/\Gamma_1) \times (G'/\Gamma')$, *where* Γ_1 *is a non-cocompact lattice in* $\mathrm{SL}(2,\mathbb{R})$. *Assume further that* φ_1 *is non-constant. Then, for every* $x_0 \in G/\Gamma$, *the Hausdorff dimension of the set of points* $s \in B \backslash Z$ *such that the orbit* $(g_t u(\varphi(s)) x_0)_{t \geqslant 0}$ *is divergent on average is exactly* $1/2$.

2.3. **Non-maximal Curves and Restrictions of Scalars of SL(2).** In Theorem A, every coordinate of the map φ is assumed to be non-constant. However, our methods apply in more general situations. This is the content of our next result in the setting where $G = \mathrm{SL}(2,\mathbb{R})^r \times \mathrm{SL}(2,\mathbb{C})^s$ for some $r, s \in \mathbb{N}$. The motivation for studying these problems in this particular setting comes from questions in Diophantine approximation with number fields.

For $g \in G$, we denote by $U^+(g)$ the expanding horospherical subgroup of G associated with g and by $\mathrm{Lie}(U^+(g))$ its Lie algebra. We also use $u(z)$ to denote $\exp(z)$ for $z \in \mathrm{Lie}(U^+(g))$. For $t \in \mathbb{R}$ and $\mathbf{x} = (\mathbf{x}_i) \in \mathbb{R}^r \times \mathbb{C}^s$, let

$$a_t = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right)_{1 \leqslant i \leqslant r+s}, \qquad u(\mathbf{x}) = \left(\begin{pmatrix} 1 & \mathbf{x}_i \\ 0 & 1 \end{pmatrix} \right)_{1 \leqslant i \leqslant r+s}.$$

Note that $U^+(a_1) = \{u(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^r \times \mathbb{C}^s\}$ and for all $g \in G$, $U^+(g a_1 g^{-1}) = g U^+(a_1) g^{-1}$.

For each k, let us write $G_k = \mathrm{SL}(2,\mathbb{R})^{r_k} \times \mathrm{SL}(2,\mathbb{C})^{s_k}$. Thus, we can make the following identifications.

$$\mathrm{Lie}(U)^+(a_1) \cong \mathbb{R}^r \times \mathbb{C}^s \cong \bigoplus_{k=1}^{l} \mathbb{R}^{r_k} \oplus \mathbb{C}^{s_k}.$$

Given a map $\psi = (\psi_i) : B \to \mathbb{R}^a \times \mathbb{C}^b$ such that $\psi \not\equiv 0$, where $B \subset \mathbb{R}$, the **characteristic** of ψ, denoted by $\mathrm{char}(\psi)$ is defined to be

$$\mathrm{char}(\psi) = \frac{\#\{1 \leqslant i \leqslant a : \psi_i \equiv 0\} + 2 \cdot \#\{a < i \leqslant a+b : \psi_i \equiv 0\}}{\#\{1 \leqslant i \leqslant a : \psi_i \not\equiv 0\} + 2 \cdot \#\{a < i \leqslant a+b : \psi_i \not\equiv 0\}}. \tag{2.3}$$

We can now state our main result in this setting.

Theorem B. *Suppose $G = G_1 \times \cdots \times G_l$ is as above, $\Gamma = \Gamma_1 \times \cdots \times \Gamma_l$ such that Γ_k is an irreducible lattice in G_k, and g_t is a split 1-parameter subgroup which is conjugate to a_t. For $1 \leqslant k \leqslant l$, let $\varphi_k : B \to \mathbb{R}^{r_k} \oplus \mathbb{C}^{s_k}$ be a $C^{1+\varepsilon}$-map for some $\varepsilon > 0$ and let $\varphi = \oplus_k \varphi_k : B \to \mathrm{Lie}(U^+(g_1)) \cong \bigoplus_{k=1}^{l} \mathbb{R}^{r_k} \oplus \mathbb{C}^{s_k}$. Denote by $(\varphi_k)_i$ the i^{th} coordinate of φ_k and let*

$$Z = \{s \in B : (\dot{\varphi}_k)_i(s) = 0, (\dot{\varphi}_k)_i \not\equiv 0 \text{ for some } k, i\}.$$

Assume that φ is not a constant map. Then, for every $x_0 \in X = G/\Gamma$, the Hausdorff dimension of the set of points $s \in B \backslash Z$ for which the forward trajectory $(g_t u(\varphi(s))x_0)_{t \geqslant 0}$ is divergent on average is at most

$$\frac{1}{2} + \frac{1}{2} \max_{1 \leq k \leq l} \mathrm{char}(\dot{\varphi}_k).$$

Moreover, if the above quantity is strictly less than 1, then parts $(ii) - (iv)$ of Theorem A also hold in this setting.

We remark that the upper bound in Theorem B is strictly less than 1 if and only if

$$\#\{1 \leqslant i \leqslant r_k : (\dot{\varphi}_k)_i \not\equiv 0\} + 2 \cdot \#\{r_k < i \leqslant r_k + s_k : (\dot{\varphi}_k)_i \not\equiv 0\} > \frac{r_k + 2s_k}{2}, \tag{2.4}$$

for all $1 \leq k \leq l$.

Remark 2.5. An analogue of Theorem B holds for other products of real rank 1 Lie groups. The upper bound formula for the dimension of divergent orbits will depend on the factors in the product, but the rest of the proof goes through verbatim. We refer the reader to Theorem 11.6 for a result for products of copies of $\mathrm{SO}(n,1)$.

The bounded orbits in Theorem B were shown to be winning in the sense of Schmidt in [EGL16] for C^1-curves φ satisfying (2.4) and Γ an irreducible lattice. Our methods are rather different in flavor and apply to a wider class of examples. Moreover, equidistribution of translates by g_t of submanifolds of $U^+(g_1)$ of small codimension and satisfying certain curvature conditions was established in [Ubi17].

Applications of Theorem B to Diophantine approximation by number fields are discussed in Section 3.2.

2.4. **SL(2,R) Actions on Homogeneous Spaces.** The motivation for our next result comes from problems in Diophantine approximation of square systems of linear forms. In particular, Theorem C below is used to study the Hausdorff dimension of singular and badly approximable square systems of linear forms belonging to a straight line with an invertible slope (Corollary 3.3).

Theorem C. *Let $B \subset \mathbb{R}$ be an interval and suppose L is a semisimple algebraic Lie group defined over $\mathbb{Q}$, Γ an arithmetic lattice in L, and $\rho : \mathrm{SL}(2,\mathbb{R}) \to L$ a non-trivial representation. Let*

$$g_t = \rho\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right), \qquad u(\varphi(s)) = \rho\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right), s \in B.$$

Then, for every $x_0 \in X = L/\Gamma$, $(i) - (iv)$ of Theorem A hold in this setting.

Remark 2.6. An analogue of Theorem C is known for the action of $\mathrm{SL}(2,\mathbb{R})$ on strata of abelian differentials. The 1/2 upper bound on the dimension of divergent orbits was established by Masur in [Mas92]. This was recently extended in [AAE$^+$17] to show that this upper bound in fact holds for divergent on average orbits. Moreover, Kleinbock and Weiss showed that bounded orbits in that setting have full Hausdorff dimension in [KW04]. The winning property of bounded orbits was later obtained in [CCM13]. The proof of Theorem C uses the method of height functions and integral inequalities and is valid for $\mathrm{SL}(2,\mathbb{R})$ actions on general metric spaces satisfying the hypotheses of Theorem 5.2. In particular, the work of Eskin and Masur in [EM01] establishes these hypotheses in the setting of $\mathrm{SL}(2,\mathbb{R})$ actions on strata of abelian differentials.

2.5. **Paper Organization and Overview of Proofs.** In Section 3, we discuss applications of our main results to problems in Diophantine approximation. In Section 4, we prove a general result for Lie group actions on metric spaces which implies the upper bound on the dimension of divergent on average orbits as well as the almost sure non-divergence result of Theorems A (parts (i) and (iii)), B and C as soon as the assumptions are verified.

The winning property of bounded trajectories is also obtained for general Lie group actions in Section 5, where we discuss Schmidt's game in detail. Finally, part (iv) of the above theorems concerning growth of orbits is established under these abstract hypotheses in Section 6 where the quantitative non-divergence of expanding translates of shrinking curve segments is established.

These general results assume the existence of a certain "height function" encoding recurrence of orbits in the form of an integral inequality (Eq. (4.3)) roughly asserting that the average height of the push-forward of a curve tends to decrease. This idea was introduced in [EMM98] and has been used in numerous other contexts since. Our restriction on the class of curves is to insure that such an inequality holds uniformly and - more importantly - in a form that we can iterate.

The construction of these functions along with establishing their main properties is carried out in Sections § 8, § 9-11 and § 12. The proofs of Theorems A, B, and C are given in Sections 10.4, 11.1, and 12.3. Corollary 2.4 is established in Section 13.

3. Applications to Diophantine Approximation

In this section, we state number theoretic consequences of our main results, particularly to Diophantine approximation problems.

3.1. **Diophantine Approximation on Spheres.** Intrinsic Diophantine approximation on $\mathbb{S}^n$ refers to approximating vectors in $\mathbb{S}^n$ using elements of the set $\mathcal{Q} = \mathbb{Q}^{n+1} \cap \mathbb{S}^n$, as opposed to approximation by elements of all of $\mathbb{Q}^{n+1}$. Given a function $\phi : \mathbb{N} \to (0,\infty)$, we say that $\mathbf{x} \in \mathbb{S}^n$ is **intrinsically** ϕ-approximable if there exist infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^{n+1} \times \mathbb{N}$ such that $\mathbf{p}/q \in \mathbb{S}^n$ and

$$\left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| < \phi(q). \tag{3.1}$$

Following [KM15], we denote by $A(\phi, \mathbb{S}^n)$ the set of ϕ-approximable points and for $\tau > 0$, we let $\phi_\tau(x) = x^{-\tau}$. An analogue of Dirichlet's classical theorem was obtained in [KM15, Theorem 1.1] showing that $A(C_n\phi_1, \mathbb{S}^n) = \mathbb{S}^n$ for some constant $C_n > 0$. Moreover, it is shown that badly approximable points on $\mathbb{S}^n$ exist in this setting [KM15, Theorem 1.2]. We say $\mathbf{x} \in \mathbb{S}^n$ is **badly approximable** if there exists a constant $\epsilon(\mathbf{x}) > 0$ such that $\mathbf{x} \notin A(\epsilon(\mathbf{x})\phi_1, \mathbb{S}^n)$. The analogue of Khinchin's theorem was established in [KM15, Theorem 1.3].

We say that $\mathbf{x} \in \mathbb{S}^n$ is **intrinsically singular on average** if for all $\epsilon > 0$, the following holds.

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant \ell \leqslant N : \left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| < \epsilon 2^{-\ell}, 0 < |q| \leqslant 2^\ell \right\} = 1. \tag{3.2}$$

In [KM15], these Diophantine properties were connected to the dynamics of a diagonalizable flow g_t on $\mathrm{SO}(n+1,1)/\Gamma$, where Γ is an arithmetic lattice. This is done by associating to each $\mathbf{x} \in \mathbb{S}^n$, an element $u(Z_{\mathbf{x}})$ in the expanding horospherical subgroup of g_t. Then, they show that $\mathbf{x} \in \mathbb{S}^n$ is badly approximable if and only if the orbit $g_t u(Z_{\mathbf{x}})\Gamma$ is bounded in G/Γ. In [KM15, Theorem 1.5], the property of being ϕ-approximable was connected to excursions of the orbit $g_t u(Z_{\mathbf{x}})\Gamma$ into cusp neighborhoods parametrized by ϕ. Using this correspondence with dynamics, one can show that $\mathbf{x}$ is intrinsically singular on average if and only if the orbit $g_t u(Z_{\mathbf{x}})\Gamma$ is divergent on average in G/Γ. This correspondence when combined with Theorem A imply the following corollary.

Corollary 3.1. *Suppose $B \subset \mathbb{R}$ is a compact interval and $\varphi : B \to \mathbb{S}^n$ is a $C^{1+\varepsilon}$-map for some $\varepsilon > 0$ such that $\dot{\varphi}$ does not vanish on B. Then, the following hold.*

(1) *The Hausdorff dimension of the set of points $s \in B$ such that $\varphi(s)$ is intrinsically singular on average is at most $1/2$.*
(2) *The set of points $s \in B$ for which $\varphi(s)$ is intrinsically badly approximable is winning for a Schmidt game on B. In particular, this set is thick in B.*
(3) *For every $\gamma > 0$, the set of points $s \in B$ for which $\varphi(s) \in A(\phi_{1+\gamma}, \mathbb{S}^n)$ has Lebesgue measure 0.*

3.2. **Diophantine Approximation by Number Fields.** Our next application concerns a generalization of the classical notion of Diophantine approximation of a real number by rationals to approximation by elements in a number field. Suppose K is a finite extension of $\mathbb{Q}$ of degree d and let $\mathcal{O}_K$ denote its ring of integers. Denote by Σ the set of Galois embeddings of K into $\mathbb{R}$ and $\mathbb{C}$, where we choose one of the two complex conjugate embeddings. Let r (resp. s) denote the number of real (resp. complex) embeddings in Σ so that $d = r + 2s$. Denote by $K_\Sigma = \mathbb{R}^r \times \mathbb{C}^s$ and let $\Delta : K \to K_\Sigma$ be the embedding defined by

$$\Delta(x) = (\sigma(x))_{\sigma \in \Sigma}.$$

Let $G = \mathrm{SL}(2,\mathbb{R})^r \times \mathrm{SL}(2,\mathbb{C})^s$. The map Δ extends to an embedding of $\mathrm{SL}(2,\mathcal{O}_K)$ into G and we let $\Gamma = \Delta(\mathrm{SL}(2,\mathcal{O}_K))$. Then, Γ is a non-uniform irreducible lattice in G and there exists a rational structure on G so that Γ is an arithmetic lattice of $\mathbb{Q}$-rank 1. Define the following elements of G.

$$g_t = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right)_{\sigma \in \Sigma}, \qquad u(\mathbf{x}) = \left(\begin{pmatrix} 1 & \mathbf{x}_\sigma \\ 0 & 1 \end{pmatrix} \right)_{\sigma \in \Sigma}. \tag{3.3}$$

We say $\mathbf{x} = (x_\sigma)_{\sigma \in \Sigma} \in K_\Sigma$ is K-**badly approximable** if there exists $\epsilon(\mathbf{x}) > 0$ so that for all $p, q \in \mathcal{O}_K$ with $q \neq 0$,

$$\max_{\sigma \in \Sigma} \{|\sigma(p) + x_\sigma \sigma(q)|\} \max_{\sigma \in \Sigma} \{|\sigma(q)|\} \geqslant \epsilon(\mathbf{x}).$$

We say $\mathbf{x}$ is K-**very well approximable** if for some $\gamma > 0$, there exist infinitely many non-zero pairs $(p,q) \in \mathcal{O}_K^2$ such that

$$\max_{\sigma \in \Sigma} \{|\sigma(p) + x_\sigma \sigma(q)|\} \max_{\sigma \in \Sigma} \left\{|\sigma(q)|^{1+\gamma}\right\} < 1.$$

Finally, say $\mathbf{x}$ is K-**singular on average** if for all $\epsilon > 0$, the following holds.

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant \ell \leqslant N : \max_{\sigma \in \Sigma} \{|\sigma(p) + x_\sigma \sigma(q)|\} < \epsilon 2^{-\ell}, 0 < \max_{\sigma \in \Sigma} \{|\sigma(q)|\} \leqslant 2^\ell \right\} = 1. \tag{3.4}$$

Analogues of Dirichlet's theorem as well as the existence of badly approximable vectors have been established in this setting. Moreover, it is shown in [EGL16] that $\mathbf{x}$ is K-badly approximable if and only if the orbit $g_t u(\mathbf{x})\Gamma$ is bounded in G/Γ. The same correspondence implies that $\mathbf{x}$ is K-**singular on average** if and only if the orbit $g_t u(\mathbf{x})\Gamma$ is divergent on average in G/Γ. Finally, we note that the group g_t above is split in this case and, in particular, Theorem B applies and gives the following corollary.

Corollary 3.2. *Suppose $B \subset \mathbb{R}$ is a compact interval and $\varphi = (\varphi_\sigma)_{\sigma\in\Sigma} : B \to \mathbb{R}^r \times \mathbb{C}^s$ is a $C^{1+\varepsilon}$-map for some $\varepsilon > 0$ such that for each σ, either $\dot{\varphi}_\sigma \equiv 0$ or $\dot{\varphi}_\sigma$ has finitely many zeros. Assume further that*

$$\#\{\sigma \in \Sigma : \dot{\varphi}_\sigma \not\equiv 0, \sigma \text{ is real}\} + 2 \cdot \#\{\sigma \in \Sigma : \dot{\varphi}_\sigma \not\equiv 0, \sigma \text{ is complex}\} > \frac{r+2s}{2}. \tag{3.5}$$

Then, the following hold.

(1) *The Hausdorff dimension of the set of points $s \in B$ for which $\varphi(s)$ is K-singular on average is at most*

$$\frac{1}{2} + \frac{1}{2}\frac{\#\{1 \leqslant i \leqslant r : \dot{\varphi}_i \equiv 0\} + 2 \cdot \#\{r < i \leqslant r+s : \dot{\varphi}_i \equiv 0\}}{\#\{1 \leqslant i \leqslant r : \dot{\varphi}_i \not\equiv 0\} + 2 \cdot \#\{r < i \leqslant r+s : \dot{\varphi}_i \not\equiv 0\}}.$$

(2) *The set of points $s \in B$ for which $\varphi(s)$ is K-badly approximable is winning for a Schmidt game on B. In particular, this set is thick in B.*

(3) *The set of points $s \in B$ for which $\varphi(s)$ is K-very well approximable has Lebesgue measure 0.*

As stated in the introduction, the winning property of badly approximable vectors in Corollary 3.2 was obtained before in [EGL16] by different methods.

3.3. **Square Systems of Linear Forms.** Our next corollary is an application of Theorem C to the study of the Diophantine properties of square matrices regarded as systems of linear forms. In particular, we are interested in the dimension of badly approximable and singular matrices and the measure of very well approximable matrices belonging to a straight line in $M_{n,n}(\mathbb{R})$. We first recall the precise definitions of these notions. We say a matrix $Y \in M_{n,n}(\mathbb{R})$ is **badly approximable** if there exists $\epsilon(Y) > 0$ for all $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ with $\mathbf{q} \neq 0$:

$$\|\mathbf{p} + Y \cdot \mathbf{q}\| \, \|\mathbf{q}\| \geqslant \epsilon(Y),$$

where for $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, $\|v\| = \max |v_i|$. We say Y is **singular** if for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ so that for all $N \geqslant N_0$, the following inequalities hold for some $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^n \times \mathbb{Z}^n$.

$$\begin{cases} \|\mathbf{p} + Y\mathbf{q}\| \leqslant \varepsilon/N, \\ 0 < \|\mathbf{q}\| \leqslant N. \end{cases}$$

Finally, Y is **very well approximable** (VWA) if there exists $\varepsilon > 0$ and infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|Y\mathbf{q} - \mathbf{p}\| < \|\mathbf{q}\|^{-1-\varepsilon} \text{ for some } \mathbf{p} \in \mathbb{Z}^n.$$

These Diophantine properties can be studied through dynamics on the space of unimodular lattices in $\mathbb{R}^{2n}$ as follows. Let $G = \mathrm{SL}(2n, \mathbb{R})$, $\Gamma = \mathrm{SL}(2n, \mathbb{Z})$, and $X = G/\Gamma$. For $t \in \mathbb{R}$ and $Y \in M_{n,n}(\mathbb{R})$, define the following elements of G.

$$g_t = \begin{pmatrix} e^t \mathrm{I}_n & \mathbf{0} \\ \mathbf{0} & e^{-t}\mathrm{I}_n \end{pmatrix}, \qquad u_Y = \begin{pmatrix} \mathrm{I}_n & Y \\ \mathbf{0} & \mathrm{I}_n \end{pmatrix}, \tag{3.6}$$

where I_n denotes the identity matrix. As discussed in the introduction, Dani showed that Y is badly approximable if and only if the forward orbit $g_t u_Y \Gamma$ is bounded in X. Similarly, Y is singular if and only if the forward orbit $g_t u_Y \Gamma$ is divergent. Finally, by [KMW10, Proposition 3.1(a)], Y is VWA if and only if

$$\limsup_{t\to\infty} \frac{d_X(g_t u_Y \Gamma, x_0)}{t} > 0,$$

where $d_X(\cdot, \cdot)$ is the Riemannian metric on X induced by the right invariant metric on G and x_0 is any base point in X.

Using this correspondence with dynamics, Theorem C has the following corollary.

Corollary 3.3. *Suppose $\varphi : B \to M_{n,n}(\mathbb{R})$ is defined by $\varphi(s) = sY + Z$ for some $Y \in \mathrm{GL}(n, \mathbb{R})$ and $Z \in M_{n,n}(\mathbb{R})$. Then, the following hold.*

1. *The Hausdorff dimension of the set of points $s \in B$ for which $\varphi(s)$ is singular is at most $1/2$.*
2. *The set of points $s \in B$ for which $\varphi(s)$ is badly approximable is winning for a Schmidt game on the real line. In particular, this set is thick.*
3. *The Lebesgue measure of the set of points $s \in B$ for which $\varphi(s)$ is very well approximable is 0.*

In this setting, the homomorphism $\rho : \mathrm{SL}(2, \mathbb{R}) \to G$ used to obtain Corollary 3.3 from Theorem C is defined as follows.

$$\rho\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right) = g_t, \quad \rho\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right) = u_{sY}, \quad \rho\left(\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\right) = \begin{pmatrix} \mathrm{I}_n & \mathbf{0} \\ sY^{-1} & \mathrm{I}_n \end{pmatrix}.$$

Finally, one applies Theorem C to the base point $x_0 = u_Z\Gamma$.

4. The Contraction Hypothesis and Divergent Trajectories

In this section, we prove an abstract recurrence result for diagonalizable trajectories starting from admissible curves in actions of Lie groups on metric spaces. Theorem 4.3 is the main result of this section establishing, in particular, a bound on the dimension of divergent orbits. In later sections, we verify the hypotheses of this theorem in the settings of the results stated in the introduction.

4.1. The Contraction Hypothesis for Lie Group Actions.

Suppose G is a connected real Lie group with Lie algebra $\mathfrak{g}$. Consider a non-trivial 1-parameter subgroup $A = \{g_t : t \in \mathbb{R}\}$ which is Ad-diagonalizable over $\mathbb{R}$. Then, $\mathfrak{g}$ decomposes under the adjoint action of g_t into eigenspaces

$$\mathfrak{g} = \bigoplus_{\alpha \in A^*} \mathfrak{g}_\alpha.$$

where A^* denotes the group of additive homomorphisms $\alpha : A \to \mathbb{R}$. In particular, for every $t \in \mathbb{R}$, $\alpha \in A^*$ and $Y \in \mathfrak{g}_\alpha$, we have

$$\mathrm{Ad}(g_t)(Y) = e^{\alpha(t)} Y. \tag{4.1}$$

We are interested in studying g_t-admissible curves φ as defined in the introduction. Note that the vanishing set Z in the statements of the main theorems is a closed set. Since all the results stated in the introduction concerning measure and Hausdorff dimension are local, we assume without loss of generality that the curves we study are defined on a compact interval where $Z = \emptyset$. We make a further simplification requiring that φ commutes with itself. The case $[\varphi, \dot{\varphi}] \equiv 0$ of Theorem A requires very minor modifications to our proofs. The following definition makes these reductions more precise for purposes of reference in the later parts of the article.

Definition 4.1. A map $\varphi : [-1, 1] \to \mathfrak{g}$ is **$\mathbf{g_t}$-admissible** if the following holds:

1. φ is $C^{1+\gamma}$ for some $\gamma > 0$, i.e. it is continuously differentiable and the Hölder exponent of its derivative $\dot{\varphi}$ is γ.
2. The image of φ is contained in a a subspace V of a single eigenspace $\mathfrak{g}_\alpha$ for some α such that $\alpha(t) > 0$ for $t > 0$ and $[V, V] = 0$.
3. The derivative of φ does not vanish on $[-1, 1]$.

Note that we only require the span of the image of φ to be an abelian subalgebra. In particular, the ambient eigenspace $\mathfrak{g}_\alpha$ need not be an abelian subspace.

The following is the key recurrence property for the action which underlies the results stated in the introduction.

Definition 4.2 (The Contraction Hypothesis)**.** Suppose X is a metric space equipped with a G-action. A g_t-admissible curve $\varphi : [-1,1] \to \mathfrak{g}_\alpha \subset \mathfrak{g}$ is said to satisfy the β-**contraction hypothesis** on X if there exists a proper function $f : X \to (0,\infty]$ satisfying the following properties:

(1) The set $Z = \{f = \infty\}$ is G-invariant and f is bounded on compact subsets of $X \backslash Z$.
(2) f is uniformly log Lipschitz with respect to the G action. That is for every bounded neighborhood $\mathcal{O}$ of identity in G, there exists a constant $C_{\mathcal{O}} \geq 1$ such that for $g \in \mathcal{O}$ and all $x \in X$,
$$C_{\mathcal{O}}^{-1} f(x) \leqslant f(gx) \leqslant C_{\mathcal{O}} f(x). \tag{4.2}$$
(3) There exists $\tilde{c} \geq 1$ such that the following holds: for all $t > 0$, there exists $\tilde{b} = \tilde{b}(t) > 0$ such that for all $x \in X$ and all $s \in [-1,1]$,
$$\frac{1}{2}\int_{-1}^{1} f(g_t u(r\dot{\varphi}(s))x)\, dr \leqslant \tilde{c} e^{-\beta\alpha(t)} f(x) + \tilde{b}, \tag{4.3}$$
where $u(Y) = \exp(Y)$ for $Y \in \mathfrak{g}_\alpha$.
(4) For all $M \geq 1$, the sets $\overline{\{x \in X : f(x) \leq M\}}$, denoted by $X_{\leq M}$, are compact.

The function f will be referred to as a **height function**.

The notion of height functions was introduced to homogeneous dynamics in [EMM98]. It was used in [KKLM17] to obtain sharp upper bounds on the dimension of singular systems of linear forms. We note that allowing height functions to assume the value ∞ has proven useful in several important applications [BQ11, EMM15].

The following is the main result of this section.

Theorem 4.3. *Let G be a real Lie group and X be a metric space equipped with a G-action. Suppose g_t is an* Ad*-diagonalizable one-parameter subgroup of G and φ is a g_t-admissible curve satisfying the β-contraction hypothesis on X. Then, for all $x \in X \backslash \{f = \infty\}$, the following hold.*

(1) The Hausdorff dimension of the set of $s \in [-1,1]$ for which the trajectory $g_t u(\varphi(s))x$ is divergent on average is at most $1 - \beta$.
(2) For Lebesgue almost every s, any weak-$*$ limit of the measures $\frac{1}{T}\int_0^T \delta_{g_t u(\varphi(s))x}\, dt$ is a probability measure on X.

Throughout this section, we fix a metric space X equipped with a proper continuous G-action and we fix a g_t-admissible curve φ satisfying the β-contraction hypothesis on X.

We remark that if an orbit $\{g_t x : t \geq 0\}$ is divergent on average for some x with $f(x) < \infty$, then for all $M > 0$,
$$\frac{1}{T}\int_0^T \chi_M(g_t x)\, dt \to 0,$$
where χ_M is the indicator function of $X_{\leq M} = \{y \in X : f(y) \leq M\}$.

The main applications of our results are to G actions on homogeneous spaces of G of the form $X = G/\Gamma$ where Γ is a lattice in G. The following lemma shows that the β-contraction hypothesis is a property of the commensurability class of Γ and will allow us to reduce the task of establishing the β-contraction hypothesis to irreducible lattices in the case G is semisimple.

Recall that two topological spaces X_1 and X_2 are *commensurable* if they have homeomorphic finite-sheeted covering spaces.

Lemma 4.4. *Suppose φ is a g_t-admissible curve satisfying the β-contraction hypothesis for the G-action on a metric space X and for some $\beta > 0$. Denote by $\mathfrak{g}_\alpha$ the* $\mathrm{Ad}(g_t)$*-eigenspace of $\mathfrak{g}$ containing the image of φ. Then, the following hold.*

(1) Suppose X' is a metric space which is commensurable to X with a common finite cover Y. Assume that Y and X' are equipped with an action of G which is equivariant with respect

to the covering maps $Y \to X$ and $Y \to X'$. Then, φ satisfies the β-contraction hypothesis for the G-action on X' and for the same β.

(2) *Suppose G' is a Lie group with Lie algebra $\mathfrak{g}'$ and g'_t is a 1-parameter $\mathbb{R}$-diagonalizable subgroup of G'. Suppose φ' is a g'_t-admissible curve satisfying the β'-contraction hypothesis for the G'-action on a metric space X'. Let $\mathfrak{g}'_{\alpha'}$ be the $\mathrm{Ad}(g'_t)$-eigenspace of $\mathfrak{g}'$ containing the image of φ'. Assume further that $\alpha(t) = \alpha'(t)$ for all $t \in \mathbb{R}$. Then, $\varphi \oplus \varphi' : [-1,1] \to \mathfrak{g} \oplus \mathfrak{g}'$ is (g_t, g'_t)-admissible and satisfies the $(\min(\beta,\beta'))$-contraction hypothesis for the $G \times G'$-action on $X \times X'$.*

Proof. (1) Denote by $p : Y \to X$ and $p' : Y \to X'$ the covering maps. Let f the height function on X. Define a height function f' on X' by

$$f'(x') = \sum_{y : p'(y) = x'} f(p(y)).$$

Since the above sum runs over finitely many points, whose cardinality is equal to the sheetedness of the cover $Y \to X'$, then one verifies that f' satisfies all the properties in Definition 4.2.

(2) Denote by f and f' the height functions on X and X' respectively. Then, one defines a function $f + f'$ on $X \times X'$ by $(f + f')(x, x') = f(x) + f'(x')$. Then, $f + f'$ provides the desired height function on $X \times X'$. □

4.2. **Approximation by horocycles and the Markov property.** The following elementary lemma allows us to obtain an integral estimate over curves via integral estimates over tangents while simultaneously providing us with a mechanism for iterating such integral estimates. This iteration mechanism will play the same role as the Markov property in the context of random walks.

Recall that $\gamma > 0$ denotes the Hölder exponent of the derivative of φ.

Lemma 4.5. *There exists a constant $C_1 > 1$, such that for all $x \in X$, natural numbers n with $n \geq 1/\gamma$, $t > 0$ and all subintervals $J \subset [-1,1]$ of radius at least $e^{-\alpha(nt)}$, one has*

$$\int_J f(g_{(n+1)t} u(\varphi(s)) x)\, ds \leqslant C_1 \int_J \int_{-1}^{1} f(g_t u(r\dot{\varphi}(s)) g_{nt} u(\varphi(s)) x)\, dr\, ds. \tag{4.4}$$

Proof. First, we note that for all $r \in [-1,1]$, we have

$$J \subseteq J \pm r e^{-\alpha(nt)} := (J + r e^{-\alpha(nt)}) \cup (J - r e^{-\alpha(nt)}). \tag{4.5}$$

Using positivity of f, (4.5) and a change of variable, we get

$$\begin{aligned}
&\int_J f(g_{(n+1)t} u(\varphi(s)) x)\, ds \\
&= \int_0^1 \int_J f(g_{(n+1)t} u(\varphi(s)) x)\, ds\, dr \leqslant \int_0^1 \int_{J \pm r e^{-\alpha(nt)}} f(g_{(n+1)t} u(\varphi(s)) x)\, ds\, dr \\
&= \int_{-1}^1 \int_{J + r e^{-\alpha(nt)}} f(g_{(n+1)t} u(\varphi(s)) x)\, ds\, dr = \int_{-1}^1 \int_J f(g_{(n+1)t} u(\varphi(s + r e^{-\alpha(nt)})) x)\, ds\, dr.
\end{aligned}$$

Then, Fubini's theorem and the fact that φ is $C^{1+\gamma}$ imply the following.

$$\int_J f(g_{(n+1)t} u(\varphi(s)) x)\, ds \leqslant \int_J \int_{-1}^1 f(g_{(n+1)t} u(\varphi(s) + r e^{-\alpha(nt)} \dot{\varphi}(s) + O(e^{-(1+\gamma)\alpha(nt)})) x)\, dr\, ds.$$

Moreover, by definition of g_t and $u(Y)$, we have

$$g_t u(Y) g_{-t} = u(e^{\alpha(t)} Y).$$

Thus, by our assumption that $n \geqslant 1/\gamma$, we get

$$\int_J f(g_{(n+1)t}u(\varphi(s))x)\;ds \leqslant \int_J\int_{-1}^1 f(u(O(1))g_{(n+1)t}u(\varphi(s)+re^{-\alpha(nt)}\dot\varphi(s))x)\;dr\;ds$$
$$= \int_J\int_{-1}^1 f(u(O(1))g_t u(r\dot\varphi(s))g_{nt}u(\varphi(s))x)\;dr\;ds.$$

Note that $u(O(1))$ belongs to a bounded neighborhood of identity independently of t and n. Hence, by the log Lipschitz property of f, there exists a constant $C_1 > 1$ such that for all $y \in X$,

$$f(u(O(1))y) \leqslant C_1 f(y).$$

This concludes the proof. □

4.3. **Integral estimates and long excursions.** The goal of this section is to prove an upper bound on the measure of the set of trajectories with long excursions outside of fixed compact sets. We show that such a measure decays exponentially in the length of the excursion. We remark that our proof of this fact is different from the proof of a similar step in [KKLM17, Proposition 5.1]. Our method allows us to handle curves which are in general not subgroups that are normalized by g_t. The proof of [KKLM17], however, uses this point crucially.

For $x \in X$, $M, t > 0$ and natural numbers $m, n \in \mathbb{N}$, we define the following sets

$$B_x(M,t,m;n) = \left\{s \in [-1,1] : f(g_{mt}u(\varphi(s)x) < M, f(g_{(m+l)t}u(\varphi(s))x) \geqslant M,\text{ for } 1 \le l \le n\right\}.$$

For every $N \in \mathbb{N}$, let $\mathcal{P}_N$ denote the partition of the interval $[-1,1]$ into N intervals of equal length.

Proposition 4.6. *There exists a constant $c_0 \geq 1$ such that for every $t > 0$ with $e^{\alpha(t)} \in \mathbb{N}$, there exists $M_0 = M_0(t) > 0$, so that for all $M > M_0$ the following holds. For all natural numbers $m \geq 1/\gamma$ and $n \geq 1$ and all $x \in X\setminus\{f = \infty\}$, one has that*

$$|B_x(M,t,m;n) \cap J_0| \leqslant c_0^n e^{-\beta\alpha(nt)}|J_0|,$$

for every interval $J_0 \in \mathcal{P}_{e^{\alpha(mt)}}$, where $|\cdot|$ denotes the Lebesgue measure on $[-1,1]$.

Proof. Let $t > 0$ be fixed. Let $\tilde c$ and $\tilde b = \tilde b(t) > 0$ be as in (3) of Definition 4.2. Let $T = \tilde b e^{\beta\alpha(t)}/\tilde c$. Then, for all $x \in X$ with $f(x) > T$, using (4.3), we get

$$\frac{1}{2}\int_{-1}^1 f(g_t u(r\dot\varphi(s))x)\;dr \leqslant 2\tilde c e^{-\beta\alpha(t)} f(x).$$

Using (2) of Definition 4.2, we can find $\tilde C_1 \geq 1$ such that for all $x \in X$ and all $s \in [-1,1]$, we have

$$\tilde C_1^{-1} f(x) \leqslant f(u(\dot\varphi(s))x) \leqslant \tilde C_1 f(x). \tag{4.6}$$

We define c_0 and M_0 as follows

$$c_0 = 4C_1\tilde C_1\tilde c, \qquad M_0 = \tilde C_1 T,$$

where C_1 denotes the constant in Lemma 4.5. Suppose $M > M_0$. To simplify notation, for each $k \in \mathbb{N}$, we let

$$B(M,k) := B_x(M,t,m;k).$$

For purposes of induction, we also define $B(M,0)$ as follows

$$B(M,0) := \{s \in [-1,1] : f(g_{mt}u(\varphi(s))x) > T\}.$$

Let us also write $\mathcal{P}_k$ to denote $\mathcal{P}_{e^{\alpha(kt)}}$ for simplicity.

Suppose $J \in \mathcal{P}_{m+n-1}$ is such that $J \cap B(M,n-1) \neq \emptyset$ and let $s_0 \in J \cap B(M,n-1)$. Then, we have $f(g_{(m+n-1)t}u(\varphi(s_0))x) > M$. Now, consider any $s \in J$. Writing $\varphi(s) = \varphi(s_0) + O_{\dot\varphi}(|J|)$, we see that

$$f(g_{(m+n-1)t}u(\varphi(s))x) > T.$$

Indeed, this follows from (4.6) and the fact that $M > \tilde{C}_1 T$. Therefore, by Lemma 4.5 and the choice of T, it follows that

$$\int_J f(g_{(m+n)t}u(\varphi(s))x)\,ds \leqslant C_1 \int_J \int_{-1}^1 f(g_t u(r\dot{\varphi}(s))g_{(m+n-1)t}u(\varphi(s))x)\,drds$$
$$\leqslant 4C_1 \tilde{c} e^{-\beta\alpha(t)} \int_J f(g_{(m+n-1)t}u(\varphi(s))x)\,ds. \tag{4.7}$$

Now, consider an interval $J_0 \in \mathcal{P}_m$ satisfying $J_0 \cap B(M,n) \neq \emptyset$. Then, since $B(M,n)$ is contained in $B(M,n-1)$, we have that $J_0 \cap B(M,n-1) \neq \emptyset$. Next, note that the following inclusion holds.

$$B(M,n-1)\cap J_0 \subseteq \bigcup_{\substack{J\in\mathcal{P}_{m+n-1}\\ J\cap B(M,n-1)\cap J_0\neq\emptyset}} J.$$

In particular, by (4.7), we get

$$\int_{B(M,n-1)\cap J_0} f(g_{(m+n)t}u(\varphi(s))x)\,ds \leqslant \sum_{\substack{J\in\mathcal{P}_{m+n-1}\\ J\cap B(M,n-1)\cap J_0\neq\emptyset}} \int_J f(g_{(m+n)t}u(\varphi(s))x)\,ds$$
$$\leqslant 4C_1\tilde{c}e^{-\beta\alpha(t)} \sum_{\substack{J\in\mathcal{P}_{m+n-1}\\ J\cap B(M,n-1)\cap J_0\neq\emptyset}} \int_J f(g_{(m+n-1)t}u(\varphi(s))x)\,ds. \tag{4.8}$$

Since $e^{\alpha(t)} \in \mathbb{N}$, for each $1 \leq j \leq k$, the partition $\mathcal{P}_k$ is a refinement of $\mathcal{P}_j$. This implies the following inclusion.

$$\bigcup_{\substack{J\in\mathcal{P}_{m+n-1}\\ J\cap B(M,n-1)\cap J_0\neq\emptyset}} J \subseteq \bigcup_{\substack{J\in\mathcal{P}_{m+n-2}\\ J\cap B(M,n-1)\cap J_0\neq\emptyset}} J. \tag{4.9}$$

Hence, the following inequality follows from (4.8), (4.9), and the fact that f is non-negative:

$$\int_{B(M,n-1)\cap J_0} f(g_{(m+n)t}u(\varphi(s))x)\,ds \leqslant 4C_1\tilde{c}e^{-\beta\alpha(t)} \sum_{\substack{J\in\mathcal{P}_{m+n-2}\\ J\cap B(M,n-1)\cap J_0\neq\emptyset}} \int_J f(g_{(m+n-1)t}u(\varphi(s))x)\,ds. \tag{4.10}$$

Iterating (4.10), by induction, we obtain the following exponential decay integral estimate.

$$\int_{B(M,n-1)\cap J_0} f(g_{(m+n)t}u(\varphi(s))x)\,ds \leqslant (4C_1\tilde{c})^n e^{-\beta\alpha(nt)} \sum_{\substack{J\in\mathcal{P}_{m}\\ J\cap B(M,n-1)\cap J_0\neq\emptyset}} \int_J f(g_{mt}u(\varphi(s))x)\,ds$$
$$= (4C_1\tilde{c})^n e^{-\beta\alpha(nt)} \int_{J_0} f(g_{mt}u(\varphi(s))x)\,ds, \tag{4.11}$$

where on the second line, we used the following consequence of $\mathcal{P}_m$ being a partition.

$$J\in\mathcal{P}_m,\, J\cap J_0\neq\emptyset \Longrightarrow J = J_0.$$

Suppose $s_0 \in J_0 \cap B(M,n-1)$. Then, by definition of the set $B(M,n-1)$, we have $f(g_{mt}u(\varphi(s_0))x)$ is at most M. Thus, arguing as before, using (4.6), we obtain the following inequality for all $s \in J_0$,

$$f(g_{mt}u(\varphi(s))x) \leqslant \tilde{C}_1 M. \tag{4.12}$$

Combining this observation with (4.11), it follows that

$$\int_{B(M,n-1)\cap J_0} f(g_{(m+n)t}u(\varphi(s))x)\,ds \leqslant (4C_1\tilde{c})^n e^{-\beta\alpha(nt)} \tilde{C}_1 M |J_0|. \tag{4.13}$$

Hence, by Chebyshev's inequality, we obtain

$$|B(M,n)\cap J_0| \leqslant c_0^n e^{-\beta\alpha(nt)}|J_0|.$$

This completes the proof. □

The following corollary allows us to convert measure estimates into an estimate on covers.

Corollary 4.7. *There exists a constant $C_2 \geqslant 1$, depending only on the height function f and the curve φ, such that the following holds. Suppose M_0 and c_0 are as in Proposition 4.6. Then, for all $M > C_2M_0$, $t>0$, $m,n\in\mathbb{N}$ with $m \geq 1/\gamma$ and $x \in X\setminus\{f=\infty\}$, the number of elements of the partition $\mathcal{P}_{e^{\alpha((m+n)t)}}$ needed to cover the set $B_x(M,t,m;n)\cap J_0$, for any $J_0 \in \mathcal{P}_{e^{\alpha(mt)}}$, is at most $c_1^n e^{(1-\beta)\alpha(nt)}$, where $c_1 = C_2c_0$.*

Proof. Using (2) of Definition 4.2, one can find a constant $C_2 \geqslant 1$ so that the following holds. Let $J \in \mathcal{P}_{e^{\alpha((m+n)t)}}$ be such that $J \cap B_x(M,t,m;n)\cap J_0 \neq \emptyset$. Then, for all $s\in J$ and all $1\leq l\leq n$,

$$f(g_{mt}u(\varphi(s))x) < C_2M, \qquad f(g_{(m+l)t}u(\varphi(s))x) \geqslant C_2^{-1}M.$$

In particular, J is contained in the set:

$$B_x^{C_2}(M,t,m;n) = \left\{s : f(g_{mt}u(\varphi(s)x) < C_2M, f(g_{(m+l)t}u(\varphi(s))x) \geqslant C_2^{-1}M, \text{ for } 1\leq l\leq n\right\}.$$

Moreover, since $\mathcal{P}_{e^{\alpha((m+n)t)}}$ is a refinement of $\mathcal{P}_{e^{\alpha(mt)}}$, it follows that

$$J \subseteq B_x^{C_2}(M,t,m;n)\cap J_0. \tag{4.14}$$

The measure of the set $B_x^{C_2}(M,t,m;n)$ can be estimated as in the proof of Proposition 4.6, where in the last step of the proof, we use the estimate $f(g_{mt}u(\varphi(s))x) < C_2M$ in place of that in (4.12). We, thus, obtain that

$$|B_x^{C_2}(M,t,m;n)\cap J_0| \leqslant (C_2c_0)^n e^{-\beta\alpha(nt)}|J_0|, \tag{4.15}$$

where c_0 is the constant provided by Proposition 4.6. The corollary thus follows upon combining (4.14) and (4.15). □

4.4. **Integral estimates and coverings.** For $x\in X$, $Q\subseteq X$, $t,\delta>0$ and $N\in\mathbb{N}$, we define the following sets

$$Z_x(Q,N,t,\delta) = \{s\in[-1,1] : \#\{1\leq l\leq N : g_{lt}u(\varphi(s))x \notin Q\} > \delta N\}. \tag{4.16}$$

To simplify notation, we denote the sets $Z_x(X_{\leq M},N,t,\delta)$ by $Z_x(M,N,t,\delta)$ for all $M>0$. The following is the main covering result that will imply Theorem 4.3.

Proposition 4.8. *There exists a constant $C_3 \geqslant 1$ such that the following holds. For all $t>0$ with $e^{\alpha(t)}\in\mathbb{N}$ and $x\in X\setminus\{f=\infty\}$, there exists $M_1 = M_1(t,x) > 0$ so that for all $M > M_1$, $\delta > 0$ and $N\in\mathbb{N}$, the set $Z_x(M,N,t,\delta)$ can be covered by at most $C_3^N e^{(1-\delta\beta)\alpha(Nt)}$ intervals of radius $e^{-\alpha(Nt)}$.*

Proof. Using (2) of Definition 4.2, we have that

$$\tilde{M}_1 := \sup_{s\in[-1,1], l\in[0,1/\gamma]} f(g_{lt}u(\varphi(s))x) < \infty.$$

Let $C_2 \geqslant 1$ be the constant in Corollary 4.7 and let $M_0 > 0$ be as in Proposition 4.6. Define M_1 as follows

$$M_1 := \max\left\{C_2M_0, \tilde{M}_1\right\}.$$

Consider a set $\Phi \subseteq \{1,\dots,N\}$ containing at least δN elements. Define the following set of trajectories whose behavior is determined by Φ:

$$Z(\Phi) = \{s\in Z_x(M,N,t,\delta) : f(g_{lt}u(\varphi(s))x) > M \text{ iff } l\in\Phi\}.$$

Following [KKLM17], we decompose the set Φ into maximal connected intervals as follows:

$$\Phi = \bigsqcup_{i=1}^{q} B_i.$$

Thus, we may write the set $\{1, \ldots, N\}$ as disjoint union of maximal connected intervals in the following manner:

$$\{1, \ldots, N\} = \bigsqcup_{i=1}^{q} B_i \sqcup \bigsqcup_{j=1}^{p} G_j.$$

Let $c_1 \geq 1$ be the constant in Corollary 4.7. We claim that $Z(\Phi)$ can be covered by at most $c_1^N e^{\alpha(Nt)-\beta\alpha(|\Phi|t)}$ intervals of radius $e^{-\alpha(Nt)}$, where $|\Phi|$ denotes the cardinality of Φ. Since the set $Z_x(M, N, t, \delta)$ is a union of at most 2^N subsets of the form $Z(\Phi)$, the claim of the proposition follows by taking $C_3 = 2c_1$.

Order the intervals B_i and G_j in the way they appear in the sequence $1 \leq \cdots \leq N$. For $1 \leq r \leq p+q$, let R_r denote the cardinality of the union of the first r intervals in this sequence. In particular, $R_{p+q} = N$. We construct a cover by induction on r. In each step, we will show that if we write

$$\{1, \ldots, R_r\} = \bigsqcup_{i=1}^{r_1} B_i \sqcup \bigsqcup_{j=1}^{r_2} G_j,$$

then the set $Z(\Phi)$ can be covered by

$$c_1^{R_r} e^{\alpha(t)\left(R_r - \beta \sum_{i=1}^{r_1} |B_i|\right)}$$

intervals of radius $e^{-\alpha(R_r t)}$ coming from the partition $\mathcal{P}_{e^{\alpha(R_r t)}}$. Note that by definition of M_1, we have $1 \in G_1$. Hence, $R_1 = |G_1|$ and the first step of our induction is verified by taking all $e^{\alpha(R_1 t)}$ intervals of radius $e^{-\alpha(R_1 t)}$ which are needed to cover $[-1, 1]$.

Now, assume the claim holds for some $r < p+q$. Suppose that the $(r+1)$-st interval in the sequence of ordered intervals is of the form G_j for some $1 < j \leq p$. Let $J_0 \in \mathcal{P}_{e^{\alpha(R_r t)}}$ be an interval of radius $e^{-\alpha(R_r t)}$ in the cover constructed by the inductive hypothesis. Then, since $e^{\alpha(t)} \in \mathbb{N}$, J_0 contains $e^{(\alpha(R_{r+1})-\alpha(R_r))t} = e^{\alpha(|G_j|t)}$ intervals of radius $e^{-\alpha(R_{r+1}t)}$. Thus, by taking all such intervals contained in each such J_0, we get a new cover of the desired cardinality in step $r+1$.

Now, assume the $r+1$ interval in the sequence of ordered intervals is of the form B_i for some $1 \leq i \leq q$. We wish to apply Corollary 4.7. By definition of M_1, we have that $M > C_2 M_0$. Moreover, since $[1, 1/\gamma] \cap \mathbb{N}$ is contained in G_1, we have that $R_r \geq 1/\gamma$. Thus, by Corollary 4.7, we can cover the set $B_x(M, t, R_r; |B_i|) \cap J_0$ by

$$c_1^{|B_i|} e^{(1-\beta)\alpha(|B_i|t)}$$

intervals of radius $e^{-\alpha((R_r+|B_i|)t)}$. Moreover, we have that

$$Z(\Phi) \subseteq B_x(M, t, R_r; |B_i|).$$

Thus, the inductive step holds in this case as well by the inductive hypothesis on the number of the intervals $J_0 \in \mathcal{P}_{e^{\alpha(R_r t)}}$ needed to cover $Z(\Phi)$. ☐

4.5. **Proof of Theorem 4.3.** Having established Proposition 4.8, the proof of Theorem 4.3 follows the same lines as in [KKLM17]. Let $x \in X$ and let $Z_x \subseteq [-1, 1]$ denote the set of points s for which the trajectory $g_t u(\varphi(s))x$ diverges on average. To prove part (1) of the theorem, we first note that for all compact sets $Q \subset X$ and for all $0 < \delta < 1$,

$$Z_x \subseteq \liminf_{N \to \infty} Z_x(Q, N, t, \delta) = \bigcup_{N_0 \geq 1} \bigcap_{N \geq N_0} Z_x(Q, N, t, \delta), \tag{4.17}$$

where the sets $Z_x(Q, N, t, \delta)$ were defined in (4.16). We wish to apply Proposition 4.8 by taking $Q = X_{\leq M}$ for an appropriate M.

Fix some $t > 0$ and let $M_1 = M_1(t, x) > 0$ be as in Proposition 4.8. Suppose $M > M_1$ and $\delta \in (0, 1)$. Then, Proposition 4.8 says that we can cover $Z_x(X_{\leq M}, N, t, \delta)$ by at most $C_3^N e^{(1-\delta\beta)\alpha(Nt)}$ intervals of radius $e^{-\alpha(Nt)}$, where $C_3 \geq 1$ is a constant which is independent of x, t and N.

Then, we have

$$\overline{\dim}_{box}\left(\bigcap_{N\geq N_0} Z_x(Q, N, t, \delta)\right) \leq \lim_{N\to\infty} \frac{N\log(C_3) + (1-\delta\beta)\alpha(Nt)}{\alpha(Nt)} = \frac{\log(C_3) + (1-\delta\beta)\alpha(t)}{\alpha(t)},$$

where for a set $A \subseteq [-1, 1]$, $\overline{\dim}_{box}(A)$ denotes its upper box dimension.

Since Z_x is contained in countably many such sets by (4.17) and since the upper box dimension dominates the Hausdorff dimension (which is stable under countable unions), we get that

$$dim_H(Z_x) \leqslant \frac{\log(C_3)}{\alpha(t)} + 1 - \delta\beta,$$

where dim_H denotes the Hausdorff dimension. Taking the limit as $t \to \infty$ and $\delta \to 1$, we obtain the desired dimension bound.

Part (2) of Theorem 4.3 follows from Proposition 4.8 and the Borel-Cantelli Lemma. More precisely, it follows from the statement of the Proposition that the set $Z_x(M, N, t, \delta)$ has measure at most $C_3^N e^{-\delta\beta\alpha(Nt)}$. Choosing $t > 0$ (and hence M) to be large enough, depending on δ and C_3, we see that the measures of these sets are summable in N.

5. Bounded Orbits and Schmidt Games

We describe a version of Schmidt's games played on intervals of the real line. These games were introduced in [KW13, KW10] in the general setting of connected Lie groups building on earlier ideas of Schmidt [Sch66].

Fix a compact interval $I_0 \subset \mathbb{R}$ and a positive constant $\sigma > 0$. For each $t > 0$, consider the following contraction of $\mathbb{R}$:

$$\Phi_t(x) = e^{-\sigma t}x.$$

Denote by $\mathfrak{F} = \{\Phi_t : t > 0\}$ this one-parameter semigroup of contractions.

Now pick two real numbers $a, b > 0$ and, following [KW10, KW13], we define a game, played by two players Alice and Bob. First, Bob picks $t_0 > 0$ and $x_1 \in \mathbb{R}$ so that the set $B_1 = \Phi_{t_0}(I_0) + x_1$ is contained in I_0. Then, Alice picks a translate A_1 of $\Phi_a(B_1)$ which is contained in B_1, Bob picks a translate B_2 of $\Phi_b(A_1)$ which is contained in A_1, after that Alice picks a translate A_2 of $\Phi_a(B_2)$ which is contained in B_2, and so on. In other words, for $k \in \mathbb{N}$, we set

$$t_k = t_0 + (k-1)(a+b), \quad \text{and} \quad s_k = t_k + a. \tag{5.1}$$

Thus, at the k^{th} step of the game, Alice picks a translate A_k of $\Phi_{s_k}(I_0)$ which is contained inside B_k. Then, Bob picks a translate B_{k+1} of $\Phi_{t_{k+1}}(I_0)$ which is contained inside A_k. From compactness of I_0 and the definition of the sets A_k and B_k, we see that the following intersections

$$\bigcap_{k\geqslant 1} A_k = \bigcap_{k\geqslant 1} B_k \tag{5.2}$$

are non-empty and consist of a single point. Note also that

$$\operatorname{diam}(A_k) = e^{-\sigma s_k}\operatorname{diam}(I_0), \quad \operatorname{diam}(B_k) = e^{-\sigma t_k}\operatorname{diam}(I_0), \tag{5.3}$$

where the diameter of sets is with respect to the standard metric on $\mathbb{R}$. This game is referred to as the $(\mathbf{a}, \mathbf{b})-$**modified Schmidt game** on I_0.

A subset $S \subseteq \mathbb{R}$ is said to be $(\mathbf{a}, \mathbf{b})-$**winning** if Alice can always pick her translates A_k so that the point in the intersection (5.2) always belongs to S, no matter how Bob picks his translates B_k.

We say S is **a-winning** if it is (a,b)-winning for all $b>0$ and **winning** if it is a-winning for some a.

5.1. **Admissible Curves and Induced Games.** Suppose G is a connected Lie group with Lie algebra $\mathfrak{g}$ and g_t is a 1-parameter Ad-diagonalizable subgroup of G. Consider a g_t-admissible curve $\varphi: I_0 \to \mathfrak{g}$ as defined in 4.1, where I_0 is a compact interval in $\mathbb{R}$. Then, g_t induces a Schmidt game on I_0 in the sense described above as follows.

Suppose $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the eigenspace for the Adjoint action of g_t which contains the image of φ. The $\mathfrak{F}_\alpha$-induced game on I_0 is given by the action of the one parameter semigroup $\mathfrak{F}_\alpha = \{\Phi_t : t>0\}$ where for every $x\in\mathbb{R}$,

$$\Phi_t(x) = e^{-\alpha(t)}x = e^{-\alpha(1)t}x,$$

and $\alpha(t)$ is the eigenvalue of g_t corresponding to the eigenspace $\mathfrak{g}_\alpha$ as in (4.1).

The main result of this section states that the contraction hypothesis in addition to the following continuity property of the height function f along unipotent orbits imply the winning property of bounded orbits.

Assumption 5.1. There exists $N\in\mathbb{N}$ such that for every $T,R>0$, there exists $M_1>0$ such that for all $x\in X$, $Y\in\mathfrak{g}_\alpha$, $\|Y\|\leq R$ and $M>M_1$, the following holds.

$$\text{The set } \{|s|\leqslant T : f(u(sY)x) > M\} \text{ has at most } N \text{ connected components.} \tag{$*$}$$

The following is the main result of this section.

Theorem 5.2. *Let X be a metric space equipped with a proper G-action. Suppose g_t is an Ad-diagonalizable 1-parameter subgroup of G and $\varphi: I_0\to\mathfrak{g}$ is a g_t-admissible curve (Def. 4.1) satisfying the β-contraction hypothesis (Def. 4.2) on X for some $\beta>0$. Assume further that the height function f satisfies Assumption 5.1. Then, there exists $a_*>0$ such that for all $x\in X$ with $f(x)<\infty$, the set*

$$\left\{ s\in I_0 : \overline{\{g_t u(\varphi(s))x : t>0\}} \text{ is compact in } X\right\} \tag{5.4}$$

is a-winning for the $\mathfrak{F}_\alpha$-induced modified Schmidt game on I_0 for all $a>a_*$.

Corollary 5.3 (Corollary 3.4, [KW10])**.** *Under the same hypotheses of Theorem 5.2, the set in (5.4) is thick in I_0.*

Remark 5.4. The contraction hypothesis alone, without Assumption 5.1, can be used to show Corollary 5.3. This can be done by a straightforward adaptation of the argument in [KW04].

Proof of Theorem 5.2. Denote by f the height function in the definition of the β-contraction hypothesis. Suppose $\tilde{c}\geqslant 1$ is as in (4.3) and $C_1\geqslant$ is the constant in the conclusion of Lemma 4.5.

Next, let C_H denote the Hölder constant of $\dot{\varphi}$. Let $\mathcal{O}$ denote a compact neighborhood of identity in G containing the image under the exponential map of a ball of radius $C_H|I_0|$ around 0 in $\mathfrak{g}$. Denote by $C = C_{\mathcal{O}} \geq 1$ a constant so that (4.2) holds.

Let $N\in\mathbb{N}$ be as in Assumption 5.1. Choose a_* to be sufficiently large so that

$$\alpha(a_*) \geqslant \frac{\log(40\tilde{c}C_1C_{\mathcal{O}}^2)}{\beta} + \log\left(10(N+1)\right). \tag{5.5}$$

Fix some $a>a_*, b>0$ and $x\in X$. We show that there exists some $M\geqslant 1$ and a choice of subintervals A_k for Alice so that for all $k\geqslant 1$ and all $s\in A_k$, we have

$$f(g_{t_{k+1}}u(\varphi(s))x) \leqslant M, \tag{5.6}$$

where t_k is given by (5.1). Thus, by Definition 4.2, this shows that the point s_0 in the intersection (5.2) will have that $g_tu(\varphi(s_0))x$ is bounded in X for all $t>0$.

By Definition 4.2, there exists a constant $\tilde{b} > 0$ depending on a and b, so that the following holds for all $y \in X$ and all $s \in I_0$.

$$\frac{1}{2}\int_{-1}^{1} f(g_{a+b}u(r\dot{\varphi}(s))y)\; dr \leqslant \tilde{c}e^{-\beta\alpha(a+b)}f(y) + \tilde{b}. \tag{5.7}$$

Now, suppose that Bob chose some $t_0 > 0$ and a subinterval $B_1 \subset I_0$ to initialize the game. Let $\gamma > 0$ be the Hölder exponent of the derivative of φ and define M_0 as follows:

$$M_0 := \sup_{s \in I_0, 1 \leqslant j \leqslant 1/\gamma+1} f(g_{j(a+b)+t_0}u(\varphi(s))x).$$

By the properties of f in Definition 4.2 and the compactness of I_0, it follows that M_0 is finite.

Let $T = e^{\alpha(a+b)}|I_0|/2$ and $R = \sup_{s\in I_0} \|\dot{\varphi}(s)\|$. Let $M_1 > 0$ be as in Assumption 5.1 applied with T and R. Define M by

$$M = 40\tilde{b}C_1C_{\mathcal{O}}^2 + M_0 + M_1C_{\mathcal{O}}, \tag{5.8}$$

where α is such that $\mathfrak{g}_\alpha$ is the eigenspace inside $\mathfrak{g}$ containing the image of φ.

In the first $\lfloor 1/\gamma + 1\rfloor$ steps of the game, Alice may choose her intervals $A_k \subset B_k$ anyway she likes. By definition of M_0 and M, (5.6) is satisfied for $1 \leqslant k \leqslant 1/\gamma + 1$.

The rest of the proof consists of 2 steps. First, we show that no matter how Bob chooses his sets B_k, the following integral estimate will always be satisfied for all $k \geqslant 1/\gamma + 1$:

$$\frac{1}{|B_k|}\int_{B_k} f(g_{t_{k+1}}u(\varphi(s))x)\; ds \leqslant 2\tilde{c}C_1e^{-\beta\alpha(a+b)}\frac{1}{|B_k|}\int_{B_k} f(g_{t_k}u(\varphi(s))x)\; ds + 2\tilde{b}C_1. \tag{5.9}$$

Then, we show that the estimate (5.9) implies that Alice can choose her sets $A_k \subset B_k$ so that (5.6) is satisfied, completing the proof.

To show (5.9), let $k \geqslant 1/\gamma + 1$ and let $B_k \subset I_0$ be a subinterval of length $e^{-\alpha(t_k)}|I_0|$. By an argument identical to that of Lemma 4.5, it follows that

$$\int_{B_k} f(g_{t_{k+1}}u(\varphi(s))x)\; ds \leqslant C_1\int_{B_k}\int_{-1}^{1} f(g_{a+b}u(r\dot{\varphi}(s))g_{t_k}u(\varphi(s))x)\; drds.$$

Then, by (5.7), we get

$$\int_{B_k} f(g_{t_{k+1}}u(\varphi(s))x)\; ds \leqslant 2C_1\tilde{c}e^{-\beta\alpha(a+b)}\int_{B_k} f(g_{t_k}u(\varphi(s))x)\; ds + 2C_1\tilde{b}|B_k|.$$

This proves (5.9). We complete the proof by induction, noting that (5.6) is satisfied for all $1 \leqslant k \leqslant 1/\gamma+1$. Since $B_k \subset A_{k-1}$, by the induction hypothesis, we get that for all $s \in B_k$, $f(g_{t_k}u(\varphi(s))x) \leqslant M$. Thus, the estimate in (5.9) becomes

$$\frac{1}{|B_k|}\int_{B_k} f(g_{t_{k+1}}u(\varphi(s))x)\; ds \leqslant 2\tilde{c}C_1e^{-\beta\alpha(a+b)}M + 2\tilde{b}C_1.$$

By Chebyshev's inequality, the fact that $a > a_*$ chosen in (5.5), and the choice of M in (5.8), we obtain the following measure estimate:

$$\begin{aligned}\left|\left\{s \in B_k : f(g_{t_{k+1}}u(\varphi(s))x) > M/C_{\mathcal{O}}^2\right\}\right| &\leqslant \left[2\tilde{c}C_1C_{\mathcal{O}}^2e^{-\beta\alpha(a+b)} + \frac{2\tilde{b}C_1C_{\mathcal{O}}^2}{M}\right]|B_k| \\ &\leqslant |B_k|/10.\end{aligned} \tag{5.10}$$

Let s_0 be the center of the interval B_k and let $s \in B_k$ be any other point. Then, we have that

$$g_{t_{k+1}}u(\varphi(s)) = u(O\left(e^{\alpha(t_{k+1}-(1+\gamma)t_k)}\right))u(r\dot{\varphi}(s_0))g_{t_{k+1}}u(\varphi(s_0)),$$

where $r = (s - s_0)e^{\alpha(t_{k+1})}$ and γ is the Hölder exponent of $\dot{\varphi}$. Since $k \geq 1/\gamma + 1$, the element $u(O\left(e^{\alpha(t_{k+1}-(1+\gamma)t_k)}\right))$ belongs to our chosen bounded neighborhood $\mathcal{O}$ of identity which is independent of all the parameters. Hence, by the log Lipschitz property (4.2) of f, we obtain

$$f(u(r\dot{\varphi}(s_0))g_{t_{k+1}}u(\varphi(s_0))x) > M/C_{\mathcal{O}} \Longrightarrow f(g_{t_{k+1}}u(\varphi(s))x) > M/C_{\mathcal{O}}^2, \quad r = (s - s_0)e^{\alpha(t_{k+1})}. \tag{5.11}$$

Moreover, since $|B_k| = e^{-\alpha(t_k)}|I_0|$, we have $|r| \leq e^{\alpha(a+b)}|I_0|/2 = T$.

Thus, since $M/C_{\mathcal{O}} > M_1$, by Assumption 5.1, the set

$$\left\{|r| \leq T : f(u(r\dot{\varphi}(s_0))g_{t_{k+1}}u(\varphi(s_0))x) > M/C_{\mathcal{O}}\right\}$$

has at most N connected components. In particular, the complement of this set has at most $N+1$ connected components (intervals).

Moreover, the measure estimate in (5.10), combined with (5.11), imply that

$$\left|\left\{|r| \leq T : f(u(r\dot{\varphi}(s_0))g_{t_{k+1}}u(\varphi(s_0))x) > M/C_{\mathcal{O}}\right\}\right| \leqslant 2T/10. \tag{5.12}$$

Denote by Q the set on the left-hand side of (5.12). Suppose that each connected component of $[-T,T]\backslash Q$ has length at most $2e^{-\alpha(a)}T$. Then, since $[-T,T]\backslash Q$ has at most $N+1$ components, we get that

$$|[-T,T]\backslash Q| \leqslant 2(N+1)e^{-\alpha(a)}T < 2T/10,$$

by the choice of a. This contradicts the measure estimate in (5.12).

It follows that we can find a subinterval $\tilde{A}_k$ of $[-T,T]$ of length $2e^{-\alpha(a)}T$ which is disjoint from the set in (5.12). Let A_k be defined as follows:

$$A_k = e^{-\alpha(t_{k+1})}\tilde{A}_k + s_0.$$

Then, A_k is a subinterval of B_k of length $e^{-\alpha(a)}|B_k|$. Moreover, applying the the log Lipschitz property of f once more, we see that for all $s \in A_k$,

$$f(u(r\dot{\varphi}(s_0))g_{t_{k+1}}u(\varphi(s_0))x) \leqslant M/C_{\mathcal{O}} \Longrightarrow f(g_{t_{k+1}}u(\varphi(s))x) \leqslant M, \quad r = (s - s_0)e^{\alpha(t_{k+1})}.$$

This proves (5.6) and concludes the proof. □

6. The Contraction Hypothesis and Shrinking Curves

The purpose of this section is to demonstrate the link between the contraction hypothesis and the growth of orbits. In all the situations we consider, the height function f which satisfies the contraction hypothesis also has the property that the ratio of $1+\log f(\cdot)$ and $1+d(\cdot, x_0)$ is uniformly bounded from above and below for any fixed base point $x_0 \in G/\Gamma$, where $d(\cdot,\cdot)$ is the Riemannian metric on G/Γ.

In fact, we establish the much stronger statement on the quantitative non-divergence of expanding translates of shrinking segments of admissible curves. In particular, Proposition 6.1 below implies that orbits with linear growth have measure 0 using the Borel-Cantelli lemma along with Chebyshev's inequality. Throughout this section, we retain the same notation as in Section 4.

Proposition 6.1. *Let G be a real Lie group and X be a metric space equipped with a proper G-action. Suppose g_t is an* Ad*-diagonalizable one-parameter subgroup of G and φ is a g_t-admissible curve satisfying the β-contraction hypothesis on X. Suppose $\delta \in [0,\beta)$ is fixed. Then, for all $x_0 \in X$ with $f(x_0) < \infty$,*

$$\sup_{\substack{t \geqslant 0, s_0 \in [-1,1] \\ J_t + s_0 \subseteq [-1,1]}} \frac{1}{|J_t|} \int_{J_t + s_0} f(g_t u(\varphi(s))x_0)\, ds < \infty,$$

where $J_t := [-e^{-\delta\alpha(t)}, e^{-\delta\alpha(t)}]$. Moreover, the supremum can be taken to be uniform over base points $x_0 \in \{f \leqslant M\}$ for any $M > 0$.

Proof. Let a choice of $\delta \in [0,\beta)$ be fixed. Suppose $s_0 \in [-1,1]$ and $n \geq 0$ is an integer. Fix $t > 0$ so that (4.3) holds with constants $\tilde{c}$ and $\tilde{b}$. By Lemma 4.5, we have

$$\int_{J_{nt}+s_0} f(g_{(n+1)t}u(\varphi(s))x_0)\,ds \leqslant C_1 \int_{J_{nt}+s_0}\int_{-1}^{1} f(g_t u(r\dot{\varphi}(s))g_{nt}u(\varphi(s))x_0)\,dr\,ds. \tag{6.1}$$

Since C_1 and $\tilde{c}$ are independent of t, we may assume that $t > 0$ is sufficiently large so that

$$2C_1\tilde{c}e^{-(\beta-\delta)\alpha(t)} < 1.$$

Therefore, by (6.1) and (4.3), we get

$$\int_{J_{nt}+s_0} f(g_{(n+1)t}u(\varphi(s))x_0)\,ds \leqslant 2C_1\tilde{c}e^{-\beta\alpha(t)} \int_{J_{nt}+s_0} f(g_{nt}u(\varphi(s))x_0)\,ds + 2C_1\tilde{b}|J_{nt}|.$$

Next, for all $n \geq 1$, since $J_{nt} \subseteq J_{(n-1)t}$ and $f \geq 0$, we get

$$\int_{J_{nt}+s_0} f(g_{(n+1)t}u(\varphi(s))x_0)\,ds \leqslant 2C_1\tilde{c}e^{-\beta\alpha(t)} \int_{J_{(n-1)t}+s_0} f(g_{nt}u(\varphi(s))x_0)\,ds + 2C_1\tilde{b}|J_{nt}|.$$

Moreover, since $|J_{(n-1)t}|/|J_{nt}| = e^{\delta\alpha(t)}$, the above inequality implies

$$\begin{aligned}\frac{1}{|J_{nt}|}&\int_{J_{nt}+s_0} f(g_{(n+1)t}u(\varphi(s))x_0)\,ds \\ &\leqslant 2C_1\tilde{c}e^{-(\beta-\delta)\alpha(t)}\frac{1}{|J_{(n-1)t}|}\int_{J_{(n-1)t}+s_0} f(g_{nt}u(\varphi(s))x_0)\,ds + 2C_1\tilde{b}.\end{aligned} \tag{6.2}$$

Define $M_0 > 0$ and M by

$$\begin{aligned}M_0 &= \frac{1}{2}\int_{-1}^{1} f(u(\varphi(s))x_0)\,ds,\\ M &= \max\left\{M_0, 2C_1\tilde{c}e^{-(\beta-\delta)\alpha(t)}M_0 + 2C_1\tilde{b}, \frac{2C_1\tilde{b}}{\left(1-2C_1\tilde{c}e^{-(\beta-\delta)\alpha(t)}\right)}\right\}.\end{aligned}$$

We claim that

$$\sup_{n\geqslant 0, s_0}\frac{1}{|J_{nt}|}\int_{J_{nt}+s_0} f(g_{(n+1)t}u(\varphi(s)x_0)\,ds \leqslant M. \tag{6.3}$$

We proceed by induction on n. When $n = 0$, inequality (6.2), the definition of M_0 and the fact that $M_0 \leqslant M$ show that the integrand in (6.3) is bounded above by M. Inequality (6.2) and the definition of M finish the proof of the claim by induction.

The conclusion of the proposition follows from the log-smoothness of f. Furthermore, we note that M can be chosen to be uniform over the base point x_0 as it varies in sublevel sets of f as evident from the definition of M_0. □

7. Dynamics in Linear Representations

This section is dedicated to proving estimates on the average rate of expansion of vectors in linear representations of $\mathrm{SL}(2,\mathbb{R})$. The main result is Proposition 7.5. In subsection 7.3, we prove an important fact regarding the orbit of a highest weight vector which will allow us to obtain precise average expansion rates in the sequel.

7.1. **(C, α)-good functions.** We recall the notion of (C,α)-good functions introduced by Kleinbock and Margulis in [KM98] and used, in different form, in prior works of Dani, Margulis and Shah.

Definition 7.1. A function $f:\mathbb{R}^m\to\mathbb{R}$ is (C,α)-good on some subset $B\subset\mathbb{R}^m$ of finite Lebesgue measure if there exist constants $C,\alpha>0$ such that for any $\varepsilon>0$, one has

$$|\{x\in B:|f(x)|<\varepsilon\}|\leq C\left(\frac{\varepsilon}{\sup_{x\in B}|f(x)|}\right)^{\alpha}|B|,$$

where, for a Borel set $A\subseteq\mathbb{R}^m$, $|A|$ denotes its Lebesgue measure.

The following lemma summarizes some basic properties of (C,α)-good functions which will be useful for us. The proof follows directly from the definition.

Lemma 7.2. *Let $C,\alpha>0$. Then,*

(1) If f is a (C,α)-good function on B, then so is $|f|$.
(2) If $f_1,\dots,f_n$ is a collection of (C,α)-good function on B, then so is $\max_k|f_k|$.

An important class of (C,α)-good functions is polynomials. The exact exponent will be of importance to us and so we recall the following fact.

Proposition 7.3 (Proposition 3.2, [KM98]). *For any $k\in\mathbb{N}$, any polynomial in $\mathbb{R}[x]$ of degree at most k is $(2k(k+1)^{1/k},1/k)$-good on any interval in $\mathbb{R}$.*

The following elementary lemma concerning polynomials will be useful for us.

Lemma 7.4. *For each $k\in\mathbb{N}$, there exists some $\rho>0$, such that any polynomial $p\in\mathbb{R}[x]$ of degree at most k of the form $p(x)=\sum_{i=0}^k c_ix^i$ satisfies*

$$\sup_{x\in[-1,1]}|p(x)|\geq\rho\max_{0\leq i\leq k}|c_i|.$$

Proof. Let $k\in\mathbb{N}$ and suppose the lemma does not hold. Then, there exists a sequence of vectors $v_n\in\mathbb{R}^{k+1}$ with $\|v_n\|_\infty=1$ such that

$$\sup_{x\in[-1,1]}|p_n(x)|<\frac{1}{n},\tag{7.1}$$

where for each n,

$$p_n(x)=\sum_{0\leq i\leq k}v_n^{(i)}x^i.$$

By passing to a subsequence, we may assume that v_n converges to a vector $v_0\neq0$. Thus, p_n converges to p_0 on $[-1,1]$ in the uniform norm. But, then, by (7.1), we have $p_0\equiv0$ on $[-1,1]$. This necessarily implies that $v_0=0$ which is a contradiction. □

7.2. **Expansion in $\mathbf{SL(2,R)}$ Representations.** Throughout this section, we fix a one-parameter Ad-diagonalizable subgroup of $G=\mathrm{SL}(2,\mathbb{R})$ which we denote by g_t. Then, $\mathfrak{g}=\mathrm{Lie}(G)$ decomposes as a direct sum of eigenspaces of $\mathrm{Ad}(g_t)$ as follows:

$$\mathfrak{g}=\mathfrak{g}_{-\alpha}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_\alpha,\tag{7.2}$$

where α is a non-trivial character of the group $A=\{g_t:t\in\mathbb{R}\}$ such that $\alpha(g_t)>0$ for all $t>0$ and $\mathfrak{g}_0$ consists of fixed vectors of $\mathrm{Ad}(g_t)$. Let $H_0\in\mathfrak{g}_0$ be such that $g_t=\exp(tH_0)$. Let $X\in\mathfrak{g}_\alpha\setminus\{0\}$ and let u_s denote the following one-parameter horocyclic subgroup

$$u_s=\exp(sX).$$

Let P denote the set of all characters of A. Then, α induces a partial order $\leqslant$ on P as follows: $\lambda\leqslant\mu$ if and only if $\mu-\lambda$ is a positive multiple of α. Given any irreducible representation V of

G, we can decompose V into weight spaces for the A action. The set of restricted weights of V contains a unique maximal element for the partial order, called the highest weight. Denote the set of all the highest weights of G by P^+, i.e. P^+ consists of characters of A which occur as highest weights in some irreducible representation of G. From the representation theory of $\mathrm{SL}(2,\mathbb{R})$, we can identify P^+ with $\mathbb{N} \cup \{0\}$.

The following is the main result of this section.

Proposition 7.5. *Suppose V is a non-trivial representation of $G = \mathrm{SL}(2,\mathbb{R})$ and let $P^+(V)$ denote the set of highest weights appearing in the decomposition of V into irreducible representations. Define*

$$\lambda := \max P^+(V), \qquad \delta_\lambda := 2\lambda(H_0)/\alpha(H_0),$$

where α is as in (7.2). Then, for all $\beta \in (0,1)$, there exists a constant $D = D(\beta) \geq 1$ such that for all $t > 0$ and all $w \in V$,

$$\frac{1}{2}\int_{-1}^{1} \|g_t u_s w\|^{-\beta/\delta_\lambda} \, ds \leqslant D e^{-\beta\alpha(H_0)t/2} \|\pi_\lambda(w)\|^{-\beta/\delta_\lambda}, \tag{7.3}$$

where $\pi_\lambda : V \to V$ denotes the $\mathrm{SL}(2,\mathbb{R})$-equivariant projection onto the direct sum of irreducible sub-representations of V with highest weight λ.

Proof. Suppose $w \in V$ and write $v = \pi_\lambda(w)$. Then, we have that $\|g_t u_s w\| \geqslant \|g_t u_s v\|$, for all t and s. In particular, it suffices to prove (7.3) with v in place of w and we may assume that λ is the only highest weight appearing in V.

Since $\mathrm{SL}(2,\mathbb{R})$ is semisimple, V decomposes into irreducible representations as follows:

$$V = V_1 \oplus \cdots \oplus V_r.$$

For $1 \leq i \leq r$, let $\pi_i : V \to V_i$ denote the associated projections and note that u_s commutes with π_i for all i. Note that all the V_i have the same dimension since they have the same highest weight. Let $n \in \mathbb{N}$ be such that

$$\dim(V_i) = n + 1,$$

for all $1 \leq i \leq r$. From the the description of $\mathrm{SL}(2,\mathbb{R})$ representations, we get that

$$n = \delta_\lambda. \tag{7.4}$$

Let $1 \leq i \leq r$ be fixed. By the standard description of irreducible $\mathrm{SL}(2,\mathbb{R})$ representations, V_i decomposes into 1 dimensional eigenspaces for the action of g_t as follows:

$$V_i = W_0^{(i)} \oplus W_1^{(i)} \oplus \cdots \oplus W_n^{(i)},$$

where we assume that $W_0^{(i)}$ denotes the highest weight subspace of V_i. In particular, for each $w \in W_0^{(i)}$,

$$g_t w = e^{\lambda(H_0)t} w.$$

Let $q_l : V_i \to W_l^{(i)}$ denote the associated projections. Let $\left\{ w_l^{(i)} : 0 \leq l \leq n \right\}$ denote a basis of V_i consisting of eigenvectors of g_t and write

$$\pi_i(v) = \sum_{l=0}^{n} c_l^{(i)} w_l^{(i)}.$$

Note that for each l, we have that

$$u_s w_l^{(i)} = \sum_{k=0}^{l} \binom{l}{k} s^{l-k} w_k^{(i)}.$$

In particular, we get the following

$$q_0(\pi_i(u_s v)) = q_0(u_s \pi_i(v)) = \sum_{k=0}^{n} c_k^{(i)} s^k w_0^{(i)}. \tag{7.5}$$

Denote by $\|\cdot\|_\infty$ an ℓ^∞ norm on V with respect to the basis chosen above for each irreducible representation. Note that all coordinates of $\pi_i(v)$ appear in the polynomial in (7.5). In particular, this implies

$$\|g_t u_s \pi_i(v)\|_\infty \geqslant \|g_t q_0(\pi_i(u_s v))\|_\infty = e^{\lambda(H_0)t} \|q_0(\pi_i(u_s v))\|_\infty . \tag{7.6}$$

Denote by V_λ the direct sum of the highest weight subspaces of V. More precisely, let

$$V_\lambda = \bigoplus_{1\leq i\leq r} W_0^{(i)},$$

and let $\pi_+ : V \to V^+$ denote the associated projection. Hence, for all $w \in V$, by (7.6), we have that

$$\|g_t w\|_\infty \geqslant \|g_t \pi_+(w)\|_\infty \geqslant e^{\lambda(H_0)t} \|\pi_+(w)\|_\infty . \tag{7.7}$$

The polynomials in (7.5) have degree at most $n = \delta_\lambda$. Hence, by Lemma 7.2 and Proposition 7.3, we see that $\|\pi_+(u_s v)\|_\infty$ is (C, δ_λ)-good on $[-1, 1]$ for C as in Proposition 7.3. Now, by (7.5) and Lemma 7.4, there exists some $\rho > 0$ such that

$$\sup_{s\in[-1,1]} \|\pi_+(u_s v)\|_\infty \geqslant \rho \|v\|_\infty .$$

Thus, by definition of (C, α)-good functions, for any $\varepsilon > 0$, we have

$$|\{s \in [-1, 1] : \|\pi_+(u_s v)\|_\infty < \varepsilon \|v\|_\infty\}| \leqslant 2C \left(\frac{\varepsilon}{\rho}\right)^{1/\delta_\lambda} . \tag{7.8}$$

Denote by $E(v, \varepsilon)$ the set on the left-hand side of inequality (7.8). Let $\beta \in (0, 1)$.

Without loss of generality, we may assume $\|v\|_\infty = 1$. Then, for $n \in \mathbb{N}$, by (7.7) and (7.8), we get

$$\begin{aligned}\int_{E(v,2^{-n}\rho)\setminus E(v,2^{-(n+1)}\rho)} \|g_t u_s v\|_\infty^{-\beta/\delta_\lambda}\, ds &\leqslant e^{-\beta\lambda(H_0)t/\delta_\lambda} \int_{E(v,2^{-n}\rho)\setminus E(v,2^{-(n+1)}\rho)} \|\pi_+(u_s v)\|_\infty^{-\beta/\delta_\lambda}\, ds \\ &\leqslant e^{-\beta\alpha(H_0)t/2} 2^{\beta(n+1)/\delta_l} \rho^{-\beta/\delta_\lambda} 2C 2^{-n/\delta_\lambda} \\ &= \rho^{-\beta/\delta_\lambda} 2^{1+\beta/\delta_l} C 2^{-(1-\beta)n/\delta_\lambda} e^{-\beta\alpha(H_0)t/2}.\end{aligned}$$

Now, note that (7.8) implies that $|E(v, 0)| = 0$. Hence, since

$$[-1, 1] = E(v, 0) \sqcup \left(\bigsqcup_{n\geqslant 0} E(v, 2^{-n}\rho) \setminus E(v, 2^{-(n+1)}\rho) \right),$$

we get that

$$\frac{1}{2} \int_{-1}^{1} \|g_t u_s v\|_\infty^{-\beta/\delta_\lambda}\, ds \leqslant \frac{\rho^{-\beta/\delta_\lambda} 2^{\beta/\delta_l} C}{1 - 2^{(1-\beta)/\delta_\lambda}} e^{-\beta\alpha(H_0)t/2}.$$

Thus, the claim of the Proposition follows since all norms are equivalent. $\square$

7.3. **Avoidance of Non-Extremal Subspaces.** The purpose of this section is to prove a useful property of the orbit of a highest weight vector under a semisimple group. This property will allow us to obtain precise expansion rates in the situations we are interested in.

Suppose G is a semisimple Lie group with Lie algebra $\mathfrak{g}$, and $\mathbf{S}$ is a maximal split torus in G which we also identify with its Lie algebra. Denote by $\Delta \subset \mathbf{S}^*$ the set of roots on which we fix an order and denote by Δ^+ the subset of positive roots. Define the following subalgebras of $\mathfrak{g}$

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{n}^+,$$

where $\mathfrak{g}_\alpha$ denotes the root space corresponding to α. Denote by N^+ and B the subgroups of G whose Lie algebras are $\mathfrak{n}^+$ and $\mathfrak{b}$ respectively.

We let $\mathcal{W}$ denote the Weyl group of $(G, \mathbf{S}, \Delta)$ and recall that $\mathcal{W}$ acts naturally on $\mathbf{S}^*$. The Bruhat decomposition of G [Bou02, Section 3, Theorem 1] implies

$$G = \bigcup_{\mathrm{w}\in\mathcal{W}} B\mathrm{w}B. \tag{7.9}$$

Given a representation V of G and a linear functional $\mu \in \mathbf{S}^*$, we denote by V^μ the weight subspace of V with weight μ.

Proposition 7.6. *Suppose V is an irreducible representation of G with highest weight λ. Then, for all $0 \neq v \in V^\lambda$,*

$$G\cdot v \quad \bigcap \bigoplus_{\mu\in\mathbf{S}^*\setminus\mathcal{W}\cdot\lambda} V^\mu = \emptyset.$$

Proof. Let $0 \neq v \in V^\lambda$ and $g \in G$. Denote by $\pi : V \to \bigoplus_{\mathrm{w}\in\mathcal{W}} V^{\mathrm{w}\cdot\lambda}$ the projection parallel to the weight spaces of $\mathbf{S}$. It suffices to show that $\pi(gv) \neq 0$.

Using the Bruhat decomposition (7.9), we can write

$$g = b_1\mathrm{w}b_2,$$

for some $b_1, b_2 \in B$ and $\mathrm{w} \in \mathcal{W}$. The group B stabilizes the line $\mathbb{R}\cdot v$. In particular, we have that $gv \in b_1\mathrm{w}V^\lambda \subseteq b_1V^{\mathrm{w}\cdot\lambda}$.

We can further decompose b_1 as follows.

$$b_1 = n^+m,$$

where $n^+ \in N^+$ and $m \in C_G(\mathbf{S})$ commutes with $\mathbf{S}$. In particular, m preserves the eigenspaces of $\mathbf{S}$ and thus we have

$$gv \in b_1V^{\mathrm{w}\cdot\lambda} = n^+V^{\mathrm{w}\cdot\lambda}. \tag{7.10}$$

Let $Y \in \mathfrak{n}^+$ be such that $n^+ = \exp(Y)$. Denote by $\rho : G \to \mathrm{GL}(V)$ the representation of G on V and let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ denote its derivative. Then, since Y is nilpotent, so is $d\rho(Y)$. In particular, $\rho(n^+) = \exp(d\rho(Y))$ is a polynomial in $d\rho(Y)$ of the form

$$\rho(n^+) = \mathrm{I} + d\rho(Y) + \cdots + \frac{d\rho(Y)^k}{k!}, \tag{7.11}$$

for some $k \in \mathbb{N}$, where I is the identity map. From the standard representation theory of semisimple Lie groups, we have

$$d\rho(\mathfrak{g}_\alpha)V^\mu \subseteq V^{\alpha+\mu},$$

for any root $\alpha \in \Delta$ and any weight $\mu \in \mathbf{S}^*$. Thus, for each $1 \leq j \leq k$, we see that

$$d\rho(Y)^jV^{\mathrm{w}\cdot\lambda} \subseteq V^\kappa, \qquad \kappa = \mathrm{w}\cdot\lambda + \sum_{\alpha\in\Delta^+} k_\alpha\alpha,$$

for some non-negative integers k_α, at least one of which is non-zero, and, in particular, $V^\kappa \cap V^{\mathrm{w}\cdot\lambda} = \{0\}$. Hence, in view of (7.11), for all $w \in V^{\mathrm{w}\cdot\lambda}$, we have

$$\pi^{\mathrm{w}\cdot\lambda}(\rho(n^+)w) = w,$$

where $\pi^{\mathrm{w}\cdot\lambda} : V \to V^{\mathrm{w}\cdot\lambda}$ denotes the projection parallel to the eigenspaces of $\mathbf{S}$. Combined with (7.10), this shows that $\pi(gv) \neq 0$ as desired. □

8. The Contraction Hypothesis in Homogeneous Spaces of Rank One

Throughout this section, G is a simple Lie group of real rank 1 and Γ is a lattice in G. We let $X = G/\Gamma$. The goal of this section is to construct a height function on X and show that it satisfies the strong β-contraction hypothesis for admissible curves. The main result of this section, Theorem 8.5, combined with those of Sections 4, 5 and 6 complete the proof of Theorem A.

8.1. **Construction of a Height Function.** Following [EM04] and [BQ11], we construct a proper function $\tilde{\alpha} : G/\Gamma \to \mathbb{R}_+$ which will allow us to control recurrence of trajectories to compact sets.

By the work of Garland and Raghunathan in [GR70], there exist finitely many Γ-conjugacy classes of maximal unipotent subgroups $\{U_i : 1 \leq i \leq p\}$ of G such that $U_i \cap \Gamma$ is a lattice in U_i. Moreover, for any sequence $g_n \in G$ such that $g_n\Gamma$ tends to infinity in G/Γ, after passing to a subsequence, for each n, there exists $\gamma_n \in \Gamma$ and i such that

$$g_n\gamma_n u (g_n\gamma_n)^{-1} \xrightarrow{n\to\infty} e,$$

for all $u \in U_i$. In addition, γ_n and i are determined uniquely for all n sufficiently large.

Given any faithful irreducible normed representation V of G, for each i, we fix a non-zero vector v_i which is fixed by U_i. By the Iwasawa decomposition, for any i and any sequence g_n in G, one has that $g_n v_i \to 0$ if and only if $g_n u g_n^{-1} \to e$ for all $u \in U_i$. Moreover, the Γ orbit of the identity coset in G/U_i is discrete. In particular, the orbit $\Gamma \cdot v_i$ is discrete (and hence closed) for each i.

Thus, the function $\tilde{\alpha} : G/\Gamma \to \mathbb{R}_+$ defined by

$$\tilde{\alpha}(g\Gamma) := \max_{w \in \bigcup_{i=1}^p g\Gamma\cdot v_i} \|w\|^{-1} \tag{8.1}$$

is proper. The following Lemma provides us with other properties of the function $\tilde{\alpha}$.

Lemma 8.1. *Suppose $\tilde{\alpha}$ is as in (8.1). Then,*

1. *Given a bounded neighborhood $\mathcal{O}$ of identity in G, there exists a constant $C_{\mathcal{O}} > 1$, such that for all $g \in \mathcal{O}$ and all $x \in X$,*
$$C_{\mathcal{O}}^{-1}\tilde{\alpha}(x) \leq \tilde{\alpha}(gx) \leq C_{\mathcal{O}}\tilde{\alpha}(x).$$
2. *For all $M > 0$, the set $\overline{\tilde{\alpha}^{-1}([0,M])}$ is compact.*
3. *(cf. [GR70]) There exists a constant $\varepsilon_1 > 0$ such that for all $x = g\Gamma \in X$, there exists at most one vector $v \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\|v\| \leq \varepsilon_1$.*

8.2. **Rank One and Linear Expansion.** We retain the same notation as in the previous section. Suppose g_t is a one-parameter subgroup of G which is Ad-diagonalizable over $\mathbb{R}$. Since G has real rank equal to 1, we can decompose the Lie algebra $\mathfrak{g}$ of G into eigenspaces for the adjoint action of g_t as follows

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}. \tag{8.2}$$

Then, we can find $H_0 \in \mathfrak{g}_0$ so that

$$g_t = \exp(tH_0), \tag{8.3}$$

for all $t > 0$.

The following lemma is the form in which we use Proposition 7.5. The key point of the lemma is that vectors expand at a maximal rate.

Lemma 8.2. *Suppose V is an irreducible real representation of G with highest weight λ and $\mu \in \{\alpha, 2\alpha\}$ is such that $\mathfrak{g}_\mu \neq 0$. Let $\delta_\lambda = 2\lambda(H_0)/\mu(H_0)$ and suppose $0 \neq v \in V$ is a highest weight vector. Then, for all $\beta \in (0,1)$, there exists $\tilde{c} > 0$ such that for all $g \in G$, $Z \in \mathfrak{g}_\mu \setminus \{0\}$, and all $t > 0$, the following holds*

$$\frac{1}{2}\int_{-1}^{1} \|g_t u_s g v\|^{-\beta/\delta_\lambda} \, ds \leqslant \tilde{c} e^{-\beta\mu(H_0)/2} \|gv\|^{-\beta/\delta_\lambda},$$

where $u_s = \exp(sZ)$.

Proof. Let $v \in V$ be a highest weight vector. Suppose μ and $0 \neq Z \in \mathfrak{g}_\mu$ are given and let $u_s = \exp(sZ)$. Since u_s is normalized by g_t and G has rank 1, the Jacobson-Morozov theorem implies that we can find $Z^- \in \mathfrak{g}_{-\mu}$ so that $[Z, Z^-] = H_0$. In particular, the sub-algebra $\mathfrak{h}$ generated by Z and Z^- is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Denote by H the corresponding subgroup of G.

Note that since $H_0 \in \mathfrak{h}$, λ can be regarded as a weight for H in its induced representation on V. In particular, V decomposes as a direct sum

$$V = V_\lambda \oplus V_0,$$

where V_λ is a direct sum of irreducible representations of H with highest weight λ and V_0 is an H-invariant complement. Hence, $v \in V_\lambda$. Denote by $\pi_\lambda : V \to V_\lambda$ the H-equivariant projection.

Note that $\|g_t u_s g v\| \geqslant \|g_t u_s \pi_\lambda(gv)\|$ for all t and s. Hence, by Proposition 7.5, we get

$$\frac{1}{2}\int_{-1}^{1} \|g_t u_s g v\|^{-\beta/\delta_\lambda} \, ds \leqslant \frac{1}{2}\int_{-1}^{1} \|g_t u_s \pi_\lambda(gv)\|^{-\beta/\delta_\lambda} \, ds \leqslant c e^{-\beta\mu(H_0)/2} \|\pi_\lambda(gv)\|^{-\beta/\delta_\lambda}, \tag{8.4}$$

for some constant $c \geqslant 1$.

For a weight μ, denote by V^μ the corresponding weight space. Since G has rank 1, its Weyl group contains one non-trivial element sending λ to $-\lambda$. Thus, by Proposition 7.6, since $V^{-\lambda} \oplus V^\lambda \subseteq V_\lambda$, we get that

$$G \cdot v \cap V_0 = \emptyset. \tag{8.5}$$

Since the stabilizer of the line $\mathbb{R} \cdot v$ is a parabolic subgroup P and $G = KP$ for a compact group K, it follows from (8.5) that $G \cdot v$ projects to a compact subset of the projective space $P(V)$ which is disjoint from the closed image of V_0 in $P(V)$. In particular, there exists $\varepsilon' > 0$ such that for all $g \in G$,

$$\|\pi_\lambda(gv)\| \geqslant \varepsilon' \|gv\|.$$

Combining this estimate with (8.4), we obtain the desired conclusion with $\tilde{c} = c\varepsilon_1^{-\beta/\delta_\lambda}$. ∎

8.3. **The Main Integral Estimate.** The height function $\tilde{\alpha}$ constructed in the previous sections satisfies the following integral estimate.

Proposition 8.3. *Suppose λ is the highest weight for G in V and $\mu \in \{\alpha, 2\alpha\}$ is such that $\mathfrak{g}_\mu \neq 0$. Define the following exponent*

$$\delta_\lambda = 2\lambda(H_0)/\mu(H_0).$$

Then, for every $\beta \in (0,1)$, there exists $\tilde{c} \geq 1$ such that the following holds: for all $t > 0$, there exists $b = b(t) > 0$ such that for all $x \in X$ and all $Z \in \mathfrak{g}_\mu$ with $\|Z\| = 1$,

$$\frac{1}{2}\int_{-1}^{1} \tilde{\alpha}^{\beta/\delta_\lambda}(g_t \exp(rZ)x) \, dr \leqslant \tilde{c} e^{-\beta\mu(H_0)t/2} \tilde{\alpha}^{\beta/\delta_\lambda}(x) + b.$$

Proof. Let $t > 0$ be fixed and define

$$\omega := \sup_{\substack{r \in [-1,1] \\ Z \in \mathfrak{g}_\lambda, \|Z\|=1}} \max\left\{ \|g_t \exp(rZ)\|, \left\|(g_t \exp(rZ))^{-1}\right\| \right\}.$$

Now, fix some $Z \in \mathfrak{g}_\lambda$ with $\|Z\| = 1$. For simplicity, we use the following notation

$$u_r := \exp(rZ).$$

Then, for all $r \in [-1,1]$ and all $x \in X$, we have

$$\omega^{-1}\tilde{\alpha}(x) \leqslant \tilde{\alpha}(g_t u_r x) \leqslant \omega\tilde{\alpha}(x), \tag{8.6}$$

where $\|\cdot\|$ denotes the operator norm of the action of G on V. Let ε_1 be as in (3) of Lemma 8.1. Suppose $x \in X$ is such that $\tilde{\alpha}(x) \leq \omega/\varepsilon_1$. Then, by (8.6), for any $\beta > 0$, we have that

$$\frac{1}{2}\int_{-1}^{1} \tilde{\alpha}^{\beta/\delta_\lambda}(g_t u_r x)\, dr \leqslant (\omega^2 \varepsilon_1^{-1})^{\beta/\delta_\lambda}. \tag{8.7}$$

Now, suppose $x \in X$ is such that $\tilde{\alpha}(x) \geq \omega/\varepsilon_1$ and write $x = g\Gamma$ for some $g \in G$. Then, by (3) of Lemma 8.1, there exists a unique vector $v_0 \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\tilde{\alpha}(x) = \|v_0\|^{-1}$. Moreover, by (8.6), we have that $\tilde{\alpha}(g_t u_r x) \geq 1/\varepsilon_1$ for all $r \in [-1,1]$. And, by definition of ω, for all $r \in [-1,1]$, $\|g_t u_r v_0\| \leq \varepsilon_1$. Thus, applying (3) of Lemma 8.1 once more, we see that $g_t u_r v_0$ is the unique vector in $\bigcup_i g_t u_r g\Gamma \cdot v_i$ satisfying

$$\tilde{\alpha}(g_t u_r x) = \|g_t u_r v_0\|^{-1},$$

for all $r \in [-1,1]$. Moreover, since all the (minimal) parabolic subgroups of G are conjugate, we see that the vectors v_i all belong to the G-orbit of a highest weight vector $\tilde{v}$.

Thus, we may apply Lemma 8.2 as follows. Fix some $\beta \in (0,1)$ and let $\tilde{c} \geqslant 1$ be the constant in the conclusion of the lemma.

$$\frac{1}{2}\int_{-1}^{1} \tilde{\alpha}^{\beta/\delta_\lambda}(g_t u_r x)\, dr = \frac{1}{2}\int_{-1}^{1} \|g_t u_r v_0\|^{-\beta/\delta_\lambda}\, dr \leqslant \tilde{c} e^{\frac{-\beta\mu(H_0)t}{2}} \|v_0\|^{-\beta/\delta_\lambda} = \tilde{c} e^{\frac{-\beta\mu(H_0)t}{2}} \tilde{\alpha}^{\beta/\delta_\lambda}(x).$$

Combining this estimate with (8.7), we obtain the desired estimate. □

In order to obtain the winning property for bounded orbits, we need to show that the height function $\tilde{\alpha}$ satisfies Assumption 5.1. This is the content of the following lemma. Its proof is a combination of (3) of Lemma 8.1 and the fact that polynomial maps have finitely many zeros.

Lemma 8.4. *There exists $N \in \mathbb{N}$, depending only on the dimension of G, such that for every $T, R > 0$, there exists $M_0 > 0$ such that for all $x \in G/\Gamma$, $Y \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ with $\|Y\| \leq R$ and $M \geq M_0$, the following holds.*

$$\text{The set } \{|s| \leqslant T : \tilde{\alpha}(u(sY)x) > M\} \text{ has at most } N \text{ connected components.} \tag{8.8}$$

Proof. Let $T, R > 0$, $Y \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ with $\|Y\| \leq R$ and let $u_s = u(sY)$. Fix some $x = g\Gamma \in X$. Let $\varepsilon_1 > 0$ be the constant in (3) of Lemma 8.1. Define M_0 as follows.

$$M_0 = \varepsilon_1^{-1} \sup\{\|u(sZ)\| : Z \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}, \|Z\| \leq R, |s| \leq T\}.$$

Let $M \geq M_0$. If $\tilde{\alpha}(u_s x) \leqslant M$ for all $|s| \leq T$, then the set in (8.8) is empty and the claim follows. On the other hand, if $\tilde{\alpha}(u_{s_0} x) > M$ for some $|s_0| \leq T$, then, by definition of M, we see that $\tilde{\alpha}(u_s x) > \varepsilon_1^{-1}$ for all $|s| \leq T$. In particular, by (3) of Lemma 8.1, there exists a unique vector $w \in \bigcup_i g\Gamma \cdot v_i$ such that

$$\tilde{\alpha}(u_s x) = \|u_s w\|^{-1}, \text{ for all } |s| \leq T.$$

Note that for any vector $w \in V$, since u_s is a unipotent transformation, the map $s \mapsto \|u_s w\|^2$ is a polynomial of degree at most N, where N depends only on the dimension of V. Thus, since polynomials have finitely many zeros, for any $\epsilon > 0$, the set $\{|s| \leq T : \|u_s w\| < \epsilon\}$ has a number of connected components uniformly bounded above only in terms of N. This concludes the proof. □

Given a g_t-admissible curve φ (Def. 4.1), applying Proposition 8.3 to the derivative $\dot{\varphi}$ yields the following.

Theorem 8.5. *Suppose φ is a non-constant g_t-admissible curve. Then, φ satisfies the β-contraction hypothesis (Def. 4.2) for all $\beta \in (0, 1/2)$ with a height function satisfying Assumption 5.1.*

9. Height Functions and Reduction Theory

The purpose of this section is to construct a height function on arithmetic homogeneous spaces and establish its main properties. This construction will be used in Section 10 to verify the β-contraction hypothesis in the setting of Theorem B. The height function we use here was introduced in [EM04]. It generalizes the construction for $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ introduced in [EMM98] and builds on ideas which were used for the problem of quantitative recurrence of unipotent flows in [DM91]. However, we follow the approach of [KW13] which replaces the method of systems of integral inequalities with the notion of W-active Lie algebras.

Throughout this section, we assume G is a semisimple algebraic Lie group defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$ such that the real rank of G is at least 2. We fix a lattice $\Gamma \subset G(\mathbb{Q})$. In particular, the rational structure on $\mathfrak{g}$ is $\mathrm{Ad}(\Gamma)$-invariant. We let $\mathfrak{g}_{\mathbb{Z}}$ denote an integer lattice of $\mathfrak{g}$ with respect to this $\mathbb{Q}$-structure.

Suppose $\mathbf{S}$ is a maximal $\mathbb{Q}$-split torus in G. We identify $\mathbf{S}$ with its Lie algebra and denote by $\mathbf{S}^*$ its linear dual. Let $\mathcal{C} \subseteq \mathbf{S}$ be a closed Weyl chamber and fix an order on the roots of $\mathbf{S}$ making $\mathcal{C}$ positive. Denote by $\Pi = \{\alpha_1, \ldots, \alpha_r\} \in \mathbf{S}^*$ a set of simple positive roots. We assume that G/Γ is not compact. In particular, $r = \mathrm{rank}_{\mathbb{Q}} G \geq 1$. Let Δ^+ denote the set of positive $\mathbb{Q}$-roots. For each root β, denote by $\mathfrak{g}_\beta$ the corresponding root space. The reader is referred to [Bor91, Section 14] for standard facts regarding root systems over $\mathbb{Q}$.

For each $1 \leqslant k \leqslant r$, let P_k be the maximal standard parabolic $\mathbb{Q}$-subgroup obtained from $\Pi \setminus \{\alpha_k\}$. Then, each P_k is defined over $\mathbb{Q}$. We note that every maximal parabolic $\mathbb{Q}$-subgroup of G is conjugate over $G(\mathbb{Q})$ to P_k for some k.

We fix a maximal compact subgroup K of G which is fixed by a Cartan involution leaving $\mathbf{S}$ invariant. In particular, $G = K\mathrm{P}_k$ for all k.

9.1. Siegel Sets and Reduction Theory.

A subset $\Omega \subset G$ is said to be a *fundamental set* for Γ if the following hold.

(1) $G = \Omega\Gamma$, and
(2) The set of elements $\gamma \in \Gamma$ such that $\Omega\gamma \cap \Omega \neq \emptyset$ is finite.

Let $\mathrm{P}_0 = \cap_k \mathrm{P}_k$ be the standard minimal parabolic subgroup associated with $\mathbf{S}$ and Π and let U_0 be the unipotent radical of P_0. Denote by $\mathrm{M}_0 \subset \mathrm{P}_0$ the identity component of the maximal $\mathbb{Q}$-anisotropic subgroup of $Z_G(\mathbf{S})$.

A Siegel set $\mathfrak{S}$ (relative to K, P_0 and $\mathbf{S}$) is a set of the form $\mathfrak{S} = K\mathbf{S}_t W$, where W is a compact subset of $\mathrm{M}_0\mathrm{U}_0$, $t \geq 0$, and

$$\mathbf{S}_t = \{s \in \mathbf{S} : \alpha_k(s) \leq t, k = 1, \ldots, r\}. \tag{9.1}$$

The following classical result, due to Borel and Harish-Chandra, shows that Siegel sets give rise to fundamental sets for Γ.

Proposition 9.1 (Theorem 15.5, Proposition 15.6, [Bor69])**.** *The space $\Gamma\backslash G(\mathbb{Q})/\mathrm{P}_0(\mathbb{Q})$ has finitely many double cosets. Given a finite set of representatives $F \subset G(\mathbb{Q})$, there exists a Siegel set $\mathfrak{S}$ such that $\Omega = \mathfrak{S}F$ is a fundamental set for Γ.*

Through the remainder of this section, we fix a Siegel set $\mathfrak{S}$ and a finite set $F \subset G(\mathbb{Q})$ as in Proposition 9.1. We denote by F^{-1} the set of inverses of the elements of F.

9.2. **The functions $\tilde{\alpha}_k$.** Denote by U_k the unipotent radical of P_k and let $d_k = \dim \mathrm{U}_k$. Then, each U_k is defined over $\mathbb{Q}$. In particular, $\mathrm{U}_k\Gamma$ is closed in G and $\mathrm{U}_k/\mathrm{U}_k \cap \Gamma$ is compact.

Let $\mathfrak{u}_k$ be the Lie algebra of U_k and let $u_{k_1}, \ldots, u_{k_{d_k}} \in \mathfrak{u}_k \cap \mathfrak{g}_{\mathbb{Z}}$ be an integral basis for $\mathfrak{u}_k$. Define $\mathbf{p}_{\mathfrak{u}_k}$ as follows

$$\mathbf{p}_{\mathfrak{u}_k} = u_{i_1} \wedge \cdots \wedge u_{i_{d_k}} \in \bigwedge^{d_k} \mathfrak{g}. \tag{9.2}$$

Note that the stabilizer of the line $\mathrm{span}(\mathfrak{p}_{\mathfrak{u}_k})$ is P_k. For each $1 \leq k \leq r$, consider the following vector space

$$V_k = \mathrm{span}\left(\bigwedge^{d_k} (\mathrm{Ad}(G))\mathbf{p}_{\mathfrak{u}_k} \right). \tag{9.3}$$

Then, the representation of G on each V_k is irreducible. Indeed, the vector $\mathfrak{p}_{\mathfrak{u}_k}$ is fixed by $\oplus_{\beta \in \Delta^+} \mathfrak{g}_\beta$ and is thus a highest weight vector and so $V_k = \mathrm{span}(\bigwedge^{d_k} \mathrm{ad}(\mathfrak{g}) \cdot \mathfrak{p}_{\mathfrak{u}_k})$ is irreducible. Moreover, since $\mathfrak{p}_{\mathfrak{u}_k} \in \bigwedge^{d_k} \mathfrak{g}_{\mathbb{Z}}$, F is a finite subset of $G(\mathbb{Q})$, and $\mathrm{Ad}(\Gamma)(\mathfrak{g}_{\mathbb{Z}}) \subseteq \mathfrak{g}_{\mathbb{Z}}$, we see that $\Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k}$ is discrete since it is contained in $\bigwedge^{d_k} \mathfrak{g}_{\frac{1}{N}\mathbb{Z}}$, for some $N \in \mathbb{N}$ depending on F.

We use $\|\cdot\|$ to denote a K-invariant norm on V_k, where K is our fixed maximal compact subgroup. Define $\tilde{\alpha}_k : G \to \mathbb{R}_+$ as follows

$$\tilde{\alpha}_k(g) = \max\left\{ \left\| g\gamma f^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k} \right\|^{-1} : \gamma \in \Gamma, f \in F \right\}. \tag{9.4}$$

Note that the functions $\tilde{\alpha}_k$ are Γ-invariant and can be regarded as functions on G/Γ. In particular, we define a function $f : G/\Gamma \to \mathbb{R}^+$ as follows

$$f(x) = \max_{1 \leq k \leq r} \tilde{\alpha}_k^{1/d_k}(g), \tag{9.5}$$

for $x = g\Gamma \in G/\Gamma$. The following proposition shows that f encodes divergence in G/Γ.

Proposition 9.2. *A subset $L \subseteq G/\Gamma$ is bounded if and only if*

$$\max_{1 \leq k \leq r} \sup_{l \in L} \tilde{\alpha}_k(l) < \infty.$$

Proof. This result is well-known and is present in several places in the literature. See for example Steps 1 and 2 in the proof of Proposition 4.1 in [KW13]. We include a proof for completeness. The direction " $\Rightarrow$ " follows from the discreteness of the sets $\Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k}$. Conversely, suppose x_n is an unbounded sequence in G/Γ and let $g_n \in \mathfrak{S}F$ be a representative of x_n in the fundamental set for Γ. Hence, we can write

$$g_n = k_n w_n s_n f_n,$$

with $k_n \in K$, $w_n \in W$, $s_n \in \mathbf{S}_t$ and $f_n \in F$ such that, possibly after passing to a subsequence, there is some $1 \leq j \leq r$ satisfying

$$\alpha_j(s_n) \to -\infty.$$

By the K-invariance of $\|\cdot\|$ and compactness of W, we get that

$$\tilde{\alpha}_j(x_n) \geqslant \left\| k_n w_n s_n f_n f_n^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_j} \right\|^{-1} \gg \left\| s_n \cdot \mathfrak{p}_{\mathfrak{u}_j} \right\|^{-1}.$$

Now, observe that $\mathfrak{p}_{\mathfrak{u}_j}$ is a weight vector for $\mathbf{S}$ with weight $\chi_j$of the form

$$\chi_j = \sum n_\beta \beta,$$

where the sum is taken over all positive roots β which have α_j in their expansion in terms of simple roots and n_β denotes the dimension of the root space corresponding to β. Finally, note that since $s_n \in \mathbf{S}_t$, the values $\beta(s_n)$ are bounded above for all positive roots β. In particular, $s_n \cdot \mathfrak{p}_{\mathfrak{u}_j} = e^{\chi_j(s_n)} \mathfrak{p}_{\mathfrak{u}_j}$ and $\chi_j(s_n) \to -\infty$ which concludes the proof. $\square$

9.3. **W-Active Lie Algebras and The Contraction Hypothesis in G/Γ.** We recall here several facts concerning unipotent radicals of parabolic subgroups which will be useful for us. The first is the following observation due to Tomanov and Weiss [TW03].

Lemma 9.3 (Proposition 3.3 in [TW03])**.** *There exists a compact neighborhood W of 0 in $\mathfrak{g}$ such that for all $g \in G$, the Lie algebra generated by $\mathrm{Ad}(g)(\mathfrak{g}_{\mathbb{Z}}) \cap W$ is unipotent.*

Next, we record the following classical facts regarding intersections of parabolic groups.

Lemma 9.4. *Suppose that P_0 is a minimal parabolic subgroup and P is a parabolic subgroup of G. Then, the following hold.*

(i) If P_0 contains the unipotent radical of P, then $P_0 \subseteq P$.

(ii) [Bor91, Proposition 14.22(iii)] If Q is conjugate to P and Q contains the unipotent radical of P, then $Q = P$.

Proof. Item (i) in fact follows from (ii). Let $P_0' \subset P$ be a minimal parabolic subgroup (over $\mathbb{R}$) containing the unipotent radical of P. Then, since all minimal parabolic subgroups are conjugate [Bor91, Proposition 21.12], there exists $g \in G$ such that $gP_0'g^{-1} = P_0 \subseteq gPg^{-1}$. In particular, we can apply (ii) with $Q = gPg^{-1}$ to get that $P = Q$ and hence the claim follows. □

Following [KW13], we make the following key definition.

Definition 9.5. Given a neighborhood $W \subset \mathfrak{g}$ of 0 and $g \in G$, we say a Lie sub-algebra $\mathfrak{u}$ is **W-active** for g if

$$\mathrm{Ad}(g)(\mathfrak{u}) \subseteq \mathrm{span}\left(\mathrm{Ad}(g)(\mathfrak{g}_{\mathbb{Z}}) \cap W\right). \tag{9.6}$$

The following is a key result obtained in [KW13].

Proposition 9.6 (Proposition 4.1 in [KW13])**.** *For every compact neighborhood W of 0 in $\mathfrak{g}$ and every $\omega > 0$, there exists $M > 0$ such that for all $x = g\Gamma \in G/\Gamma$ with $f(x) > M$ and all k, the set*

$$\Psi_k(g) = \left\{ v \in \Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k} : \|g \cdot v\|^{-1/d_k} \geqslant f(x)/\omega \right\}$$

consists of W-active elements for g.

The above facts will be used in the form of the following corollary.

Corollary 9.7. *Suppose W is a compact neighborhood of 0 in $\mathfrak{g}$ as in the conclusion of Lemma 9.3. Then, for every $\omega > 0$, there exists $M > 0$ such that for all $x = g\Gamma \in G/\Gamma$ with $f(x) > M$ and all k, the span of the set*

$$\Psi_k(g) = \left\{ v \in \Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k} : \|g \cdot v\|^{-1/d_k} \geqslant f(x)/\omega \right\}$$

has dimension at most 1.

Proof. By Proposition 9.6, let M be chosen so that for each k, the set $\Psi_k(g)$ consists of W-active elements. For $i = 1, 2$, let $v_i = \gamma_i f_i^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k} \in \Psi_k(g)$.

By Lemma 9.3, the Lie algebra generated by $\mathrm{Ad}(g\gamma_1 f_1^{-1})\mathfrak{u}_k$ and $\mathrm{Ad}(g\gamma_2 f_2^{-1})\mathfrak{u}_k$ is unipotent. In particular, both $\mathrm{Ad}(g\gamma_1 f_1^{-1})(\mathrm{U}_k)$ and $\mathrm{Ad}(g\gamma_2 f_2^{-1})(\mathrm{U}_k)$ are contained in the same minimal parabolic subgroup P_0. By (i) of Lemma 9.4, for $i = 1, 2$, since $\mathrm{Ad}(g\gamma_i f_i^{-1})(\mathrm{U}_k)$ is the unipotent radical of $\mathrm{Ad}(g\gamma_i f_i^{-1})(\mathrm{P}_k)$, it follows that $P_0 \subseteq \mathrm{Ad}(g\gamma_i f_i^{-1})(\mathrm{P}_k)$.

In particular, $\mathrm{Ad}(g\gamma_2 f_2^{-1})(\mathrm{P}_k)$ contains the unipotent radical of $\mathrm{Ad}(g\gamma_1 f_1^{-1})(\mathrm{P}_k)$. By (ii) of Lemma 9.4, $\mathrm{Ad}(g\gamma_2 f_2^{-1})(\mathrm{P}_k) = \mathrm{Ad}(g\gamma_1 f_1^{-1})(\mathrm{P}_k)$. Hence, since P_k is its own normalizer, we get

$$f_2 \gamma_2^{-1} \gamma_1 f_1^{-1} \in \mathrm{P}_k.$$

In particular, since P_k normalizes $\mathfrak{u}_k$,

$$v_1 = c v_2,$$

for some $c \neq 0$ (in fact, $c \in \frac{1}{N}\mathbb{Z}$ for some $N \in \mathbb{N}$ depending on F). □

10. The Contraction Hypothesis in Arithmetic Homogeneous Spaces

In this section, we establish the contraction hypothesis for certain curves on arithmetic homogeneous spaces using the height function constructed in the previous section. The main result is Theorem 10.7. Combined with the results in Sections 4, 5, and 6, we obtain, for a wide class of curves on arithmetic homogeneous spaces, an explicit bound on the dimension of divergent on average orbits, thickness of the set of bounded orbits, and quantitative non-divergence of translates of shrinking curve segments. We retain the same notation as in the previous section.

10.1. **Deformations of Maximal Representations and Linear Expansion.** We introduce the notion of deformations of a maximal representation of an $\mathfrak{sl}_2$-triple to abstract the exact properties we require from the derivative of our curves which imply that they satisfy the β-contraction hypothesis.

Definition 10.1. Given a bounded interval $B \subset \mathbb{R}$ and an $\mathfrak{sl}_2$-triple (Y, h, X), we say a map $\rho : \mathfrak{sl}(2,\mathbb{R}) \times B \to \mathfrak{g}$ is a **deformation of a maximal representation** if the following conditions hold.

(1) ρ is continuous and for each $s \in B$, $\rho_s := \rho|_{\mathfrak{sl}(2,\mathbb{R})\times\{s\}}$ is a faithful Lie algebra homomorphism. In particular, $\rho_s(X) \neq 0$ for all $s \in B$.
(2) $H_\rho := \rho_s(h)$ belongs to (closure of) the positive Weyl chamber $\mathcal{C} \subset \mathbf{S}$ and is independent of s.
(3) For each $s \in B$, $\rho_s(X) \in \bigoplus_{\beta\in\Delta^+} \mathfrak{g}_\beta$ and $\rho_s(\mathfrak{sl}(2,\mathbb{R}))$ is not contained in any conjugate of a proper parabolic $\mathbb{Q}$-subalgebra of $\mathfrak{g}$.

In the examples we study, the curves φ satisfy $\dot{\varphi}(s) = \rho_s(X)$ for some such ρ. In the remainder of this section, we fix $\rho : \mathfrak{sl}(2,\mathbb{R}) \times B \to \mathfrak{g}$ a deformation of a maximal representation.

The simple roots Π induce a partial order on $\mathbf{S}^*$ in the following natural way.

$$\mu \leqslant \nu \Leftrightarrow \nu - \mu = \sum_{\Delta^+} k_\alpha \alpha \text{ for some } k_\alpha \in \mathbb{N} \cup \{0\}\,.$$

In particular, given any irreducible representation V of G, defined over $\mathbb{Q}$, the set of $\mathbb{Q}$-weights of $\mathbf{S}$ admits a maximal element which we call the highest $\mathbb{Q}$-weight.

10.2. **Maximal Representations and Linear Expansion.** The following lemma is a direct analogue of Lemma 8.2 in the setting of Lie groups of real rank equal to 1.

Lemma 10.2. *Suppose V is an irreducible representation of G defined over $\mathbb{Q}$ with highest $\mathbb{Q}$-weight λ. Let $\delta_\lambda = \lambda(H_\rho)$ and suppose $0 \neq v \in V(\mathbb{Q})$ is a highest weight vector. Then, for all $\beta \in (0,1)$, there exists $\tilde{c} > 0$ and $0 < \beta' \leqslant \beta$ such that for all $g \in G$, $s \in B$, and all $t > 0$,*

$$\frac{1}{2}\int_{-1}^{1} \|g_t u(r\rho_s(X))gv\|^{-\beta/\delta_\lambda}\, dr \leqslant \tilde{c}e^{-\beta' t}\|gv\|^{-\beta/\delta_\lambda}\,,$$

where $g_t = \exp(tH_\rho)$ and $u(r\rho_s(X)) = \exp(r\rho_s(X))$. Moreover, if $\mathrm{rank}_{\mathbb{Q}} G = 1$, we can take $\beta' = \beta$.

Proof. The proof of Lemma 10.2 in the case $\mathrm{rank}_{\mathbb{Q}} G = 1$ is identical to that of Lemma 8.2. Indeed, the key ingredients in the proof of Lemma 8.2 are Proposition 7.6 and the fact that the only non-trivial Weyl group element sends the highest weight λ to $-\lambda$.

In the higher rank case, fix some $s \in B$ and let $\mathfrak{h} = (\mathfrak{sl}(2,\mathbb{R}) \times \{s\})$. Then, we can decompose $V = V_1 \oplus V^{\mathfrak{h}}$, where $\mathfrak{h}$ acts trivially on $V^{\mathfrak{h}}$ and V_1 is the $\mathfrak{h}$-invariant complement of $V^{\mathfrak{h}}$ and contains no trivial sub-representations. Let π_1 denote the $\mathfrak{h}$-equivariant projection onto V_1.

Note that the stabilizer of $\mathbb{R}\cdot v$ is a parabolic $\mathbb{Q}$-subgroup of G. Thus, since ρ is a maximal representation, we have that

$$G\cdot v \cap V^{\mathfrak{h}} = \emptyset.$$

In particular, arguing as in the proof of Lemma 8.2, this implies that there exists some $\varepsilon > 0$ such that for all $g \in G$,

$$\|\pi_1(gv)\| \geqslant \varepsilon \|gv\|. \tag{10.1}$$

Since ρ is continuous in s and B is compact, we note further that ε may be chosen uniformly over $s \in B$ since the spaces $V^{\mathfrak{h}}$ vary continuously.

Denote by $P^+(V)$ the set of highest weights for $\mathfrak{h}$ appearing in the decomposition of V into irreducible representations. For each $\mu \in P^+(V)$, we let $\delta_\mu = \mu(H_\rho)$ and V_μ be the direct sum of irreducible sub-representations of V with highest weight μ. Let $\pi_\mu : V \to V_\mu$ denote the associated projection. Inequality (10.1) implies that there exists $\mu \in P^+(V)\setminus\{0\}$ such that

$$\|\pi_\mu(gv)\| \asymp \|\pi_1(gv)\| \geqslant \varepsilon \|gv\|. \tag{10.2}$$

Define β' as follows

$$\beta' = \frac{\beta}{\delta_\lambda} \min\left\{\delta_\mu : \mu \in P^+(V)\setminus\{0\}\right\}. \tag{10.3}$$

The Lemma now follows immediately from Proposition 7.5 applied to the projection of gv onto V_μ with $0 \neq \mu \in P^+(V)$ and satisfying (10.2).

$\square$

Remark 10.3. It is natural to ask whether the constant of proportionality between β' and β in (10.3) is optimal. When $\operatorname{rank}_{\mathbb{Q}}(G) > 1$, the Weyl group typically contains more than one non-trivial element. This fact played a key role in the (real and rational) rank 1 cases in showing that $\beta' = \beta$, allowing us to obtain the fastest possible contraction rate. In particular, it is not clear whether it is possible to modify the argument in the proof of Lemma 10.2 to show that in the case $\operatorname{rank}_{\mathbb{Q}}(G) > 1$, the G-orbit of a highest weight vector avoids sub-representations of V with non-maximal, non-zero highest weights for $\mathfrak{h}$. More precisely, it is not clear whether the analogue of equation (8.5) holds in the setting of Lemma 10.2 when $\operatorname{rank}_{\mathbb{Q}}(G) > 1$.

10.3. **The Main Integral Estimate.** For each $1 \leqslant k \leqslant r$, let $\chi_k \in \mathbf{S}^*$ denote the highest $\mathbb{Q}$-weight with respect to Π in the representation V_k defined in (9.3). Then, since ρ_s is a maximal representation, for each $1 \leq k \leq r$,

$$\delta_k := \chi_k(H_\rho) \neq 0.$$

Indeed, otherwise, if $\delta_k = 0$, this implies that $\operatorname{Ad}\left(\rho\left(\mathfrak{sl}(2,\mathbb{R}) \times \{s\}\right)\right)$ annihilates $\mathfrak{p}_{\mathfrak{u}_k}$. But, since P_k is the normalizer of U_k, this implies that $\rho\left(\mathfrak{sl}(2,\mathbb{R}) \times \{s\}\right)$ is contained in the Lie algebra of P_k, contradicting the fact that ρ_s is maximal.

The following proposition is the main result of this section.

Proposition 10.4. *For all $0 < \beta < \min_k d_k/\delta_k$, there exists $c_0 \geqslant 1$ and $0 < \beta' \leqslant \min_k \beta\delta_k/d_k$, depending on β, so that the following holds. For every $t > 0$, there exists a positive constant b such that for all $x \in G/\Gamma$ and all $s \in B$,*

$$\frac{1}{2}\int_{-1}^{1} f^\beta(g_t u(r\rho_s(X))x)\, dr \leqslant c_0 e^{-\beta' t} f^\beta(x) + b,$$

where $g_t = \exp(tH_\rho)$ and $u(r\rho_s(X)) = \exp(r\rho_s(X))$. Moreover, if $\operatorname{rank}_{\mathbb{Q}} G = 1$, we can take $\beta' = \beta\delta_1/d_1$.

Proof. Let W be a compact neighborhood of 0 for which Lemma 9.3 holds. Fix some $t > 0$ and define ω as follows.

$$\omega = \sup_{|r|\leqslant 1, s\in B} \max\left\{\|g_t u(r\rho_s(X))\|, \left\|(g_t u(r\rho_s(X)))^{-1}\right\|\right\}.$$

Here $\|\cdot\|$ refers to the operator norm for the G action on $\bigoplus_k \bigwedge^k \mathfrak{g}$. Then, for all $s \in B$ and $r \in [-1,1]$ and all $x \in G/\Gamma$, we have

$$\omega^{-1/d_0} f(x) \leqslant f(g_t u(r\rho_s(X))x) \leqslant \omega^{1/d_0} f(x), \tag{10.4}$$

where

$$d_0 = \min_{1 \le k \le r} d_k. \tag{10.5}$$

Let $M > 0$ be as in Corollary 9.7 applied to our chosen W and with ω^{2/d_0} in place of ω. Suppose that $x_0 \in G/\Gamma$ is such that $f(x) \leqslant M$. Fix $\beta \in (0,1)$. Then, we have

$$\frac{1}{2}\int_{-1}^{1} f^{\beta}(g_t u(r\rho_s(X))x_0)\, dr \leqslant b, \tag{10.6}$$

for $b = \omega^{\beta/d_0} M^{\beta}$.

Now, suppose $f(x_0) > M$ and write $x_0 = g\Gamma$ for some $g \in G$. For each $1 \le k \le r$, consider the sets

$$\Psi_k(g) = \left\{ v \in \Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k} : \|g \cdot v\|^{-1/d_k} \geqslant f(x_0)/\omega^{2/d_0} \right\}.$$

We claim that for all $s \in B$ and $r \in [-1,1]$, one has

$$f\left(g_t u(r\rho_s(X))x_0\right) = \max\left\{ \|g_t u(r\rho_s(X))g \cdot v\|^{-1/d_k} : v \in \Psi_k(g), 1 \le k \le r \right\}.$$

Indeed, suppose $v \in \Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k}$ satisfies $f\left(g_t u(r\rho_s(X))x_0\right) = \|g_t u(r\rho_s(X))g \cdot v\|^{-1/d_k}$ for some k and some $(s,r) \in B \times [-1,1]$. Then, by definition of ω and (10.4), we obtain

$$\omega^{1/d_k} \|g \cdot v\|^{-1/d_k} \geqslant \|g_t u(r\rho_s(X))g \cdot v\|^{-1/d_k} = f\left(g_t u(r\rho_s(X))x_0\right) \geqslant f(x_0)/\omega^{1/d_0}.$$

Hence, $v \in \Psi_k(g)$ as desired. Say a vector $v \in \Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k}$ is primitive if v has minimal norm in $\mathbb{R} \cdot v \cap \Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k}$. Next, we note that Corollary 9.7 implies that for each k, the set $\Psi_k(g)$ contains at most one primitive vector up to a sign. Denote by $\Psi_k^0(g)$ the following set.

$$\Psi_k^0(g) = \{ v \in \Psi_k(g) : v \text{ is primitive} \}.$$

In order to apply Lemma 10.2, let $\delta_k = \chi_k(H_\rho)$ and $\gamma_k = \beta\delta_k/d_k$. The choice of β implies that $0 < \gamma_k < 1$ for all $1 \le k \le r$. Thus, by Lemma 10.2 applied with γ_k in place of β, there exists $0 < \beta' \leqslant \min_k \gamma_k$ such that the following inequalities hold:

$$\begin{aligned}\frac{1}{2}\int_{-1}^{1} f^{\beta}(g_t u(r\rho_s(X))x_0)\, dr &\leqslant \sum_{\substack{v \in \Psi_k^0(g)\\ 1 \le k \le r}} \frac{1}{2}\int_{-1}^{1} \|g_t u(r\rho_s(X))g \cdot v\|^{-\beta/d_k} \\ &\leqslant \tilde{c} e^{-\beta' t} \sum_{\substack{v \in \Psi_k^0(g)\\ 1 \le k \le \mathrm{rank}_{\mathbb{Q}} G}} \|g \cdot v\|^{-\beta/d_k} \leqslant 2r\tilde{c}e^{-\beta' t} f^{\beta}(x_0),\end{aligned} \tag{10.7}$$

where $\tilde{c}$ is as in the conclusion of Lemma 10.2. Combining (10.6) and (10.7) completes the proof. ☐

Remark 10.5. An analogue of Proposition 10.4 was obtained in [EM04, Section 3.2] in the context of random walks on homogeneous spaces. It was assumed in [EM04] that the Zariski closure of the group generated by the support of the measure generating the random walk is a semisimple group which is not contained in any proper parabolic $\mathbb{Q}$-subgroup of G. This assumption is replaced here with the notion of a deformation of a maximal representation. Lemma 10.2 acts as a substitute for the positivity of the top Lyapunov exponent in the context of random walks. In the case when the rational rank of G is equal to 1, we also observe that we can obtain a precise contraction rate which allows us to obtain a sharp dimension upper bound for divergent on average orbits.

In the following lemma, we show that the height function f satisfies Assumption 5.1. Its proof is very similar to the analogous Lemma 8.4 in rank 1.

Lemma 10.6. *There exists $N \in \mathbb{N}$, depending only on G, such that for every $T, R > 0$, there exists $M_0 > 0$ such that for all $x \in G/\Gamma$, $Y \in \bigoplus_{\beta\in\Delta^+} \mathfrak{g}_\beta$ with $\|Y\| \leq R$ and $M_1 \geq M_0$, the following holds.*

$$\text{The set } \{|s| \leqslant T : f(u(sY)x) > M_1\} \text{ has at most } N \text{ connected components.} \tag{10.8}$$

Proof. Let $T, R > 0$, $Y \in \bigoplus_{\beta\in\Delta^+} \mathfrak{g}_\beta$ with $\|Y\| \leq R$ and let $u_s = u(sY)$. Fix some $x = g\Gamma \in X$ and define ω as follows:

$$\omega = \sup\left\{ \|u(sZ)\| : Z \in \bigoplus_{\beta\in\Delta^+} \mathfrak{g}_\beta, \|Z\| \leq R, |s| \leq T \right\},$$

where $\|\cdot\|$ refers to the operator norm on $\bigoplus_k \bigwedge^k \mathfrak{g}$. Arguing as in the proof of Proposition 10.4, let W be a compact neighborhood of 0 for which Lemma 9.3 holds. Let $M > 0$ be as in Corollary 9.7 applied to W and ω^{2/d_0}, where d_0 is defined in (10.5). Now, define M_0 as follows.

$$M_0 = \omega^{2/d_0} M.$$

Let $M_1 \geq M_0$. If $f(u_s x) \leqslant M_1$ for all $|s| \leq T$, then the set in (10.8) is empty and the claim follows. On the other hand, if $f(u_{s_0} x) > M_1$ for some s_0 with $|s_0| \leq T$, then, by definition of M_1 and ω, we see that $f(u_s x) > M$ for all $|s| \leq T$. For each $1 \leq k \leq r = \mathrm{rank}_{\mathbb{Q}}(G)$, define the following sets.

$$\Psi_k^0(g) = \left\{ v \in \Gamma F^{-1} \cdot \mathfrak{p}_{\mathfrak{u}_k} : \|g \cdot v\|^{-1/d_k} \geqslant f(x)/\omega^{2/d_0}, v \text{ is primitive} \right\}.$$

By an argument identical to that in the proof of Proposition 10.4, it follows that

$$f(u_s x) = \max\left\{ \|u_s g \cdot v\|^{-1/d_k} : v \in \Psi_k^0(g), 1 \leq k \leq r \right\},$$

for all $|s| \leq T$ and the sets $\Psi_k^0(g)$ contain at most one vector up to a sign for each k. In particular, for each $|s| \leq T$, $f(u_s x)$ is a maximum over functions of the form $\|u_s w\|^{-1/d_k}$ for at most $2r$ vectors w.

Finally, for any vector $w \in V = \bigoplus_k \bigwedge^k \mathfrak{g}$, the map $s \mapsto \|u_s w\|^2$ is a polynomial of degree at most d, where d depends only on the dimension of V. Thus, since polynomials have finitely many zeros, for any $\epsilon > 0$, the set $\{|s| \leq T : \|u_s w\| < \epsilon\}$ has a number of connected components uniformly bounded above only in terms of d. Moreover, each connected component of the set $\{s : f(u_s x) > M_1\}$ is a union of connected components of sets of the form $\{s : \|u_s g \cdot v\| < \epsilon\}$ for an appropriate $\epsilon > 0$. The claim now follows by taking $N = 2rd$.

□

Proposition 10.4 establishes the main contraction property of the function f while the other properties in Definition 4.2 follow easily from the definition and Proposition 9.2. Thus, we have established the following.

Theorem 10.7. *Suppose G is a semisimple algebraic real Lie group defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$ and Γ is a lattice in G. Let $\rho : \mathfrak{sl}(2, \mathbb{R}) \times B \to \mathfrak{g}$ be a deformation of a maximal representation (Def. 10.1) and let $g_t = \exp(tH_\rho)$. Suppose $\varphi : B \to \mathfrak{g}$ is a differentiable curve satisfying $\dot{\varphi}(s) = \rho_s(X)$ for each $s \in B$. Then, there exists $0 < \beta_0 < 1$ such that φ satisfies the β-contraction hypothesis for all $\beta \in (0, \beta_0)$ with a height function satisfying Assumption 5.1. Moreover, if $\mathrm{rank}_{\mathbb{Q}}(G) = 1$, then $\beta_0 = 1/2$.*

Remark 10.8. An explicit estimate for β_0 is given in (10.3) when $\mathrm{rank}_{\mathbb{Q}}(G) > 1$.

10.4. **Proof of Theorem A.** In light of Lemma 4.4, it suffices to prove the result when Γ is an irreducible, non-uniform lattice in G. If $\mathrm{rank}_{\mathbb{R}} G = 1$, i.e. G is a simple real Lie group of real rank 1 and finite center, then Theorem A follows from Theorem 8.5 which establishes the β-contraction hypothesis for all $\beta \in (0, 1/2)$ with a height function satisfying Assumption 5.1. One can thus apply Theorems 4.3 and 5.2, and Proposition 6.1 to conclude.

When $\mathrm{rank}_{\mathbb{R}} G > 1$, we wish to apply Theorem 10.7 in place of Theorem 8.5. Thanks to Margulis' arithmeticity theorem, Γ is arithmetic, i.e. Γ is commensurable with $G_{\mathbb{Z}}$ in some $\mathbb{Q}$-structure on G. It follows from [Mor15, 5.5.12] that Γ arises via a restriction of scalars construction[1]. The reader is referred to [Mor15, Section 5.5] for more details. In particular, since G is a product of simple Lie groups of real rank 1, we necessarily have that $\mathrm{rank}_{\mathbb{Q}} \Gamma \leq 1$. Since we are assuming that G/Γ is not compact, we thus have that $\mathrm{rank}_{\mathbb{Q}} \Gamma = 1$.

It remains to show that the curves considered in Theorem A arise as deformations of a maximal representation. To this end, we only need to show that g_t and $u(\dot{\varphi}(s))$ are part of a maximal SL(2)-triple for every $s \in B$.

For each $1 \leq i \leq k$, write $g_t^{(i)} = \exp(tH_i)$ for $H_i \in \mathfrak{g}_i$. Using the fact that each simple factor of G is a rank 1 group, it follows from the Jacobson-Morozov Lemma that for each $1 \leq i \leq k$, H_i and $\dot{\varphi}_i(s)$ can be completed to an $\mathfrak{sl}(2)$-triple $\mathfrak{h}_i = \langle Y_i(s), H_i, \dot{\varphi}_i(s) \rangle$. One can then check that $\mathfrak{h} = \langle \oplus_{i=1}^k Y_i(s), \oplus_{i=1}^k H_i, \dot{\varphi}(s) \rangle$ is the desired $\mathfrak{sl}(2)$-triple.

The maximality of $\mathfrak{h}$ follows from the fact that the only proper parabolic $\mathbb{Q}$-subgroups in G are minimal and have an abelian Levi part in this case. In particular, $\mathfrak{h}$ cannot be contained in any proper parabolic $\mathbb{Q}$-subalgebra of $\mathfrak{g}$ as desired.

10.5. **Examples of Maximal Representations.** The goal of this subsection is to produce more examples of deformations of maximal representations. In Section 11, we discuss the case G is a product of SL(2)'s.

Observe that if a reductive subgroup $H < G$ is contained in some proper parabolic $\mathbb{R}$-subgroup $P < G$, then H must be contained inside a Levi subgroup $L < P$. The centralizer $Z_P(L)$ of L in P is a non-trivial $\mathbb{R}$-split torus and is, thus, non-compact. This proves the following simple criterion for checking whether an $\mathfrak{sl}_2$-triple is maximal in the sense of Definition 10.1.

Lemma 10.9. *If the centralizer $Z_G(H)$ of a reductive real Lie subgroup $H < G$ is compact, then H is not contained in any proper parabolic $\mathbb{R}$-subgroup of G.*

Note that if $Z_G(H)$ is compact, then $Z_{G\times G}(\Delta(H))$ is also compact, where $\Delta(H)$ denotes the diagonal embedding of H inside $G \times G$. We can use Lemma 10.9 to construct other examples as follows.

Example 10.10. Let $G = \mathrm{SO}(p, 2)$ with $p \geq 1$. Let H be a $\mathbb{Q}$-subgroup isomorphic to $\mathrm{SO}(1, 2)$. Then, $Z_G(H) \cong \mathrm{SO}(p-1)$ is compact. Let A denote a $\mathbb{Q}$-split torus inside H. Suppose $B \subset \mathbb{R}$ is an interval and let

$$z : B \to Z_G(A)$$

be an arbitrary continuous map. Then, one can check that the map $\rho : \mathfrak{sl}(2, \mathbb{R}) \times B \to \mathrm{Lie}(G)$ defined by setting

$$\rho\left(\mathfrak{sl}(2, \mathbb{R}) \times \{s\}\right) = \mathrm{Ad}(z(s))\left(\mathrm{Lie}(H)\right)$$

is indeed a deformation of a maximal representation.

11. Specializing to Products of SL(2)

In this section, we specialize the results of the previous sections to the case $G = \mathrm{SL}(2, \mathbb{R})^r \times \mathrm{SL}(2, \mathbb{C})^s$, in order to complete the proof of Theorem B. Moreover, we consider curves in this setting

[1] In fact, the complexifications of each simple factor of G must be isogenous, but we do not need this fact.

which do not fit within the notion of maximal representations as defined in 10.1. The main result of this section is Theorem 11.5.

Suppose Γ is a lattice in G. Then, up to finite index, and thanks to Lemma 4.4, we may assume $\Gamma = \Gamma_1 \times \cdots \times \Gamma_l$, where each Γ_i is irreducible in a sub-product of G. In light of Lemma 4.4, it suffices to establish the contraction hypothesis in each irreducible factor and thus we may assume Γ is irreducible. If $r+s=1$, then G has real rank 1 and this result was established in Section 8. Thus, we may also assume that $r+s>1$ and in particular that $\mathrm{rank}_{\mathbb{R}}(G)>1$. Define the following elements of G:

$$g_t = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right)_{1\leqslant i\leqslant r+s}, \qquad u(\mathbf{x}) = \left(\begin{pmatrix} 1 & \mathbf{x}_i \\ 0 & 1 \end{pmatrix} \right)_{1\leqslant i\leqslant r+s}. \tag{11.1}$$

By Margulis' arithmeticity theorem, there exists a rational structure on G so that Γ is commensurable with $G(\mathbb{Z})$. In this section, we assume that the $\mathbb{Q}$-rank of G is equal to 1 so that G/Γ is not compact. Without loss of generality and to simplify notation, we will assume that g_t is $\mathbb{Q}$-split. Hence, the group $\mathrm{U} = \{u(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^r \times \mathbb{C}^s\}$ is the unipotent radical of the minimal parabolic group P_0 associated with the $\mathbb{Q}$-torus $\mathbf{S} = \{g_t : t \in \mathbb{R}\}$. The group P_0 has the following form.

$$\mathrm{P}_0 = \left\{ \left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right) \right\}.$$

For each i, let G_i denote the i^{th} factor of G. Let $\mathfrak{g} = \oplus_{i=1}^{r+s} \mathfrak{g}_i$ denote the Lie algebra of G, where $\mathfrak{g}_i$ is the Lie algebra of G_i. For $1 \leq i \leq r+s$, we let $H_i \in \mathfrak{g}_i$ denote the following element.

$$H_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that any irreducible representation V of G is isomorphic to a tensor product $\bigotimes_i W_i$, where each W_i is an irreducible representation of G_i. In particular, if $\lambda \in \mathbf{S}^*$ is a highest weight for G in V, then

$$\lambda = \sum_i \lambda_i, \tag{11.2}$$

where each $\lambda_i \in (\mathbb{R} \cdot H_i)^*$ is a highest weight for G_i in W_i. Given any such representation V with highest weight λ and $0 \neq \mathbf{x} = (\mathbf{x}_i) \in \mathbb{R}^r \times \mathbb{C}^s$, we define the following exponents:

$$\delta_i = \lambda_i(H_i), \qquad \delta_{\mathbf{x}} = \sum_{i:\mathbf{x}_i \neq 0} \delta_i, \qquad \zeta_{\mathbf{x}} = \sum_{i:\mathbf{x}_i = 0} \delta_i. \tag{11.3}$$

The following Lemma acts as a substitute for Lemma 10.2 in this setting.

Lemma 11.1. *Suppose V is a non-trivial irreducible representation for G and $0 \neq v \in V$ is a highest weight vector. Then, for all $0<\beta<1$ and $0 \neq \mathbf{x} \in \mathbb{R}^r \times \mathbb{C}^s$, there exists $c \geq 1$ such that for all $t>0$ and all $g \in G$, the following holds:*

$$\frac{1}{2}\int_{-1}^{1} \|g_t u(r\mathbf{x}) \cdot gv\|^{-\beta/\delta_{\mathbf{x}}} \, dr \leqslant c e^{-\beta' t} \|gv\|^{-\beta/\delta_{\mathbf{x}}},$$

where β' is given by

$$\beta' = \beta \left[1 - \frac{\zeta_{\mathbf{x}}}{\delta_{\mathbf{x}}} \right]. \tag{11.4}$$

Moreover, the constant c can be chosen uniformly as $\mathbf{x}$ varies in a fixed compact set.

Proof. Let $0 \neq \mathbf{x} = (\mathbf{x}_i) \in \mathbb{R}^r \times \mathbb{C}^s$ be given and define $\mathbf{y} = (\mathbf{y}_i) \in \mathbb{R}^r \times \mathbb{C}^s$ by

$$\mathbf{y}_i = \begin{cases} 1/\mathbf{x}_i, & \mathbf{x}_i \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\mathfrak{h} = \mathfrak{h}(\mathbf{x})$ the subalgebra of $\mathfrak{g}$ generated by

$$\underline{\mathbf{x}} = \left(\begin{pmatrix} 0 & \mathbf{x}_i \\ 0 & 0 \end{pmatrix}\right)_{1\leqslant i\leqslant r+s}, \qquad \underline{\mathbf{y}} = \left(\begin{pmatrix} 0 & 0 \\ \mathbf{y}_i & 0 \end{pmatrix}\right)_{1\leqslant i\leqslant r+s}.$$

Thus, in particular, $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{R})$ with the following distinguished positive diagonalizable element.

$$\underline{\mathbf{h}} := [\underline{\mathbf{x}}, \underline{\mathbf{y}}] = \sum_{i:\mathbf{x}_i\neq 0} H_i.$$

Consider the following elements of G.

$$a_t = \exp\left(t\underline{\mathbf{h}}\right), \qquad b_t = g_t a_{-t} = \exp\left(t \sum_{i:\mathbf{x}_i=0} H_i\right).$$

Since the smallest eigenvalue of b_t in V is $\exp\left(t\zeta_{\mathbf{x}}\right)$, we get that

$$\frac{1}{2}\int_{-1}^{1} \|g_t u(r\mathbf{x})\cdot gv\|^{-\beta/\delta_{\mathbf{x}}}\ dr \leqslant e^{\beta t\zeta_{\mathbf{x}}/\delta_{\mathbf{x}}}\frac{1}{2}\int_{-1}^{1} \|a_t u(r\mathbf{x})\cdot gv\|^{-\beta/\delta_{\mathbf{x}}}\ dr. \tag{11.5}$$

Denote by $P^+(V)$ the set of highest weights for $\mathfrak{h}$ appearing in the decomposition of V into irreducible representations and denote by χ the maximal element in $P^+(V)$. For each $\mu \in P^+(V)$, we let V_μ be the direct sum of irreducible sub-representations of V with highest weight μ. Let $\pi_\mu : V \to V_\mu$ denote the associated projection.

Note that $\delta_{\mathbf{x}}$ is the largest eigenvalue of $\underline{\mathbf{h}}$ in V. In particular, we can apply Proposition 7.5 to get the following estimate.

$$\frac{1}{2}\int_{-1}^{1} \|a_t u(r\mathbf{x})\cdot gv\|^{-\beta/\delta_{\mathbf{x}}}\ dr \leqslant c_1 e^{-\beta t}\left\|\pi_\chi(gv)\right\|^{-\beta/\delta_{\mathbf{x}}}, \tag{11.6}$$

for some constant $c_1 \geq 1$ depending only on β. From (11.5) and (11.6), to conclude the proof, it remains to show the existence of a constant $\varepsilon > 0$ so that for all $g \in G$,

$$\|\pi_\chi(gv)\| \geqslant \varepsilon\,\|gv\|\,. \tag{11.7}$$

To do so, we wish to apply Proposition 7.6. For a weight $\eta \in \mathbf{S}^*$, we denote by V^η the weight space for $\mathbf{S}$ with weight η. Note that $V^{-\lambda}\oplus V^{\lambda} \subseteq V_\chi$, where $\lambda \in \mathbf{S}^*$ denote the highest weight for G in V. This follows from (11.2) and the definition of $\underline{\mathbf{h}}$. In particular, by Proposition 7.6, we get that

$$G\cdot v \quad \bigcap \bigoplus_{\mu\in P^+(V)\setminus\{\chi\}} V_\mu = \emptyset. \tag{11.8}$$

Now, observe that no conjugate of $\mathfrak{h}$ is contained in the Lie algebra of the group P_0 since the Levi part of P_0 is abelian while $\mathfrak{h}$ is semisimple. Moreover, the group P_0 stabilizes the line $\mathbb{R}\cdot v$. Arguing as in the proof of Lemma 8.2, we see that the image of $G\cdot v$ is compact in projective space and disjoint from the closed image of $\bigoplus_{\mu\in P^+(V)\setminus\{\chi\}} V_\mu$. Thus, (11.7) follows.

∎

Remark 11.2. Consider the case $V = V_1$ in Lemma 11.1, where V_1 is the representation defined in 9.3 and used to define the height function on G/Γ. Then, we have

$$\delta_i = \begin{cases} 2, & \text{if } G_i = \mathrm{SL}(2,\mathbb{R}). \\ 4, & \text{if } G_i = \mathrm{SL}(2,\mathbb{C}). \end{cases}$$

and $\delta_V = 2r + 4s$. In particular, the exponent β' defined in (11.4) is positive if $\mathbf{x} = (\mathbf{x}_i) \in \mathbb{R}^r \times \mathbb{C}^s$ is such that

$$\#\left\{1 \leqslant i \leqslant r : \mathbf{x}_i \neq 0\right\} + 2\cdot\#\left\{r < i \leqslant r+s : \mathbf{x}_i \neq 0\right\} > \frac{r+2s}{2}.$$

Given $0 \neq \mathbf{x} \in \mathbb{R}^r \times \mathbb{C}^s$, define a height function $f_{\mathbf{x}} : G/\Gamma \to \mathbb{R}_+$ by

$$f_{\mathbf{x}}(x_0) = \tilde{\alpha}_1^{1/\delta_{\mathbf{x}}}(x_0), \tag{11.9}$$

where $\tilde{\alpha}_1$ is defined in (9.4).

Thus, we can apply Lemma 11.1 in place of Lemma 10.2 and obtain the following direct analogue of Proposition 10.4.

Proposition 11.3. *For all $0 < \beta < 1$, there exists $c_0 \geqslant 1$, depending on β, so that the following holds. For every $t > 0$, there exists a positive constant b such that for all $x_0 \in G/\Gamma$ and all $0 \neq \mathbf{x} \in \mathbb{R}^r \times \mathbb{C}^s$,*

$$\frac{1}{2}\int_{-1}^{1} f_{\mathbf{x}}^{\beta}(g_t u(r\mathbf{x})x_0)\, dr \leqslant c_0 e^{-\beta' t} f_{\mathbf{x}}^{\beta}(x_0) + b,$$

where β' is given by (11.4).

Remark 11.4. The proof of Proposition 10.4 in the $\mathbb{Q}$-rank 1 case used the function $f = \tilde{\alpha}_1^{1/d_1}$, where $d_1 = r + 2s$ is the dimension of the group U. However, this different exponent does not change the main properties of f. In particular, the key ingredient, Corollary 9.7 still holds for our definition of $f_{\mathbf{x}}$.

Given a non-constant differentiable map $\varphi = (\varphi_i) : B \to \mathrm{Lie}(U^+(g_1)) \cong \mathbb{R}^r \oplus \mathbb{C}^s$ such that $\dot{\varphi}_i$ is either identically 0 or does not vanish on B for $1 \leq i \leq r + s$, we observe that $\delta_{\dot{\varphi}(s)}$ (eqn. (11.3)) is independent of s. Thus, by taking our height function to be $f_{\dot{\varphi}(s)}^{\beta}$ for any s, Proposition 11.3 along with the results of Section 9 and 10 imply the following.

Theorem 11.5. *Suppose $G = \mathrm{SL}(2,\mathbb{R})^r \times \mathrm{SL}(2,\mathbb{C})^s$, Γ is an irreducible lattice in G and g_t is a split 1-parameter subgroup. Let $\varphi = (\varphi_i) : B \to \mathrm{Lie}(U^+(g_1)) \cong \mathbb{R}^r \oplus \mathbb{C}^s$ be a non-constant $C^{1+\varepsilon}$-map for some $\varepsilon > 0$ such that $\dot{\varphi}_i$ is either identically 0 or does not vanish on B for $1 \leq i \leq r + s$. Define β_φ as follows:*

$$\beta_\varphi := \frac{1}{2}\left[1 - \frac{\#\{1 \leqslant i \leqslant r : \dot{\varphi}_i \equiv 0\} + 2 \cdot \#\{r < i \leqslant r+s : \dot{\varphi}_i \equiv 0\}}{\#\{1 \leqslant i \leqslant r : \dot{\varphi}_i \not\equiv 0\} + 2 \cdot \#\{r < i \leqslant r+s : \dot{\varphi}_i \not\equiv 0\}}\right].$$

If $\beta_\varphi > 0$, then φ is a g_t-admissible curve satisfying the β-contraction hypothesis for the G action on G/Γ for all $0 < \beta < \beta_\varphi$. Moreover, the β-contraction hypothesis holds with a height function satisfying Assumption 5.1.

11.1. **Proof of Theorem B.** When Γ is irreducible, the result follows by combining Theorem 11.5 with Theorems 4.3 and 5.2, and Proposition 6.1. In particular, the dimension of divergent on average orbits is at most $1 - \beta_\varphi$, where β_φ is as in Theorem 11.5. Note that this upper bound is less than 1 if and only if $\beta_\varphi > 0$.

11.2. **Non-maximal Curves on Products of SO(d,1).** The methods of this section can be used with minor modifications to obtain an analogous result to Theorem B when G is a product of of copies of $\mathrm{SO}(n,1)$.

Theorem 11.6. *Suppose $G = G_1 \times \cdots \times G_k$ is such that for each $1 \leq i \leq k$, $G_i \cong \mathrm{SO}(d_i, 1)$ for some $d_i \geq 2$. Let Γ be an irreducible lattice in G. For each $1 \leq i \leq k$, let $g_t^{(i)}$ be a 1-parameter subgroup of G_i which is Ad-diagonalizable over $\mathbb{R}$, and suppose $\varphi_i : B \to \mathrm{Lie}(U^+(g_1^{(i)}))$ is a $C^{1+\varepsilon}$-map for some $\varepsilon > 0$. Assume that for each i, $\dot{\varphi}_i$ is either non-vanishing or vanishes identically on B. Let $g_t = (g_t^{(i)})_{1 \leq i \leq k}$ and $\varphi = \oplus_{i=1}^k \varphi_i$. Assume that g_t is split and φ is g_t-admissible and non-constant. Then, for every $x_0 \in X = G/\Gamma$, the Hausdorff dimension of the set of points $s \in B$ for which the* *forward orbit* $(g_t u(\varphi(s))x_0)_{t\geqslant 0}$ *is divergent on average is at most*

$$\frac{1}{2}+\frac{1}{2}\frac{\displaystyle\sum_{1\le i\le k,\dot{\varphi}_i\equiv 0}(d_i-1)}{\displaystyle\sum_{1\le i\le k,\dot{\varphi}_i\not\equiv 0}(d_i-1)}.$$

Moreover, if the above quantity is strictly less than 1*, then parts* $(ii)-(iv)$ *of Theorem A also hold in this setting.*

12. The Contraction Hypothesis for SL(2,R) Actions

In this section, we construct a family of functions that will allow us to control recurrence to compact sets in $\mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. This construction was introduced in [EMM98] and generalized in [BQ11]. Here, we follow the approach of [BQ11]. The main result of this section, Theorem 12.5, establishes the contraction hypothesis in the context of $\mathrm{SL}(2,\mathbb{R})$ actions completing the proof of Theorem C.

We recall the set up and notation of Theorem C. Let L be a semisimple algebraic Lie group defined over $\mathbb{Q}$ and let Γ be an arithmetic lattice in L. We let $\rho : \mathrm{SL}(2,\mathbb{R}) \to L$ be a non-trivial homomorphism and let G denote the image of ρ. Let g_t and u_s be as in the statement of Theorem C.

The aim of this section is to show that the "curve" u_s satisfies the β-contraction hypothesis for the G-action on L/Γ. In light of Lemma 4.4, we have the freedom of replacing Γ by a commensurable lattice without loss of generality.

In particular, we may regard L as a subgroup of $S = \mathrm{SL}(d,\mathbb{R})$ for some $d \geq 1$ so that $\Gamma = L\cap\Lambda$, for $\Lambda = \mathrm{SL}(d,\mathbb{Z})$. Since L is defined over $\mathbb{Q}$, we have that $L\Lambda$ is closed in S and the homogeneous space $X = L/\Gamma \cong L/L\cap\Lambda$ can be regarded a closed subspace of $X' = S/\Lambda$. As a result, the contraction hypothesis for the G-action on L/Γ will follow from that of the G-action on S/Λ. Therefore, without loss of generality, we will assume through the remainder of this section that

$$L = \mathrm{SL}(d,\mathbb{R}), \qquad \Gamma = \mathrm{SL}(d,\mathbb{Z}), \qquad X = L/\Gamma.$$

Using the results in [BQ11], Shi showed in [Shi14] the β-contraction hypothesis for the G action on X for some $\beta > 0$. We reproduce the proof in this section with some modifications to obtain a more precise range for the exponent β.

12.1. **The Contraction Hypothesis in Vector Spaces.**

As before, we first construct functions in linear representations encoding divergence in X and then convert our linear estimates into a height function on X. The relevant representation in this case is $\bigoplus_i \bigwedge^i \mathbb{R}^d$.

Let $H_0, Z \in \mathrm{Lie}(G) \cong \mathfrak{sl}(2,\mathbb{R})$ be such that

$$g_t = \exp(tH_0), \qquad u_s \exp(sZ). \tag{12.1}$$

In particular, by definition of g_t and u_s, we have

$$[H_0, Z] = 2Z.$$

Denote by P^+ the set of all possible highest weights appearing in linear representations of G. From the representation theory of $\mathrm{SL}(2,\mathbb{R})$, the set P^+ of highest weights can be identified with $\mathbb{N}\cup\{0\}$. Given an arbitrary finite dimensional representation V of G and $\lambda \in P^+$, we use V^λ to denote the direct sum of all irreducible subrepresentations of V whose highest weight is λ. We denote by $\pi_\lambda : V \to V^\lambda$ the associated G-equivariant projection.

Following [BQ11], we define two sets of exponents. For $i \in \{1,\dots,d-1\}$ and $\lambda \in P^+$, define

$$\delta_i = i(d-i), \qquad \delta_\lambda = \lambda(H_0). \tag{12.2}$$

In particular, we have

$$\lambda(H_0) = 0 \Leftrightarrow \lambda = 0.$$

For every $\epsilon > 0$ and $0 < i < d$, we define a function φ_ϵ on $\bigwedge^i \mathbb{R}^d$ as follows. For $v \in \bigwedge^i \mathbb{R}^d$, let

$$\varphi_\epsilon(v) = \begin{cases} \min_{\lambda \in P^+ \setminus \{0\}} \epsilon^{\frac{\delta_i}{\delta_\lambda}} \|\pi_\lambda(v)\|^{-1/\delta_\lambda} & \text{if } \|\pi_0(v)\| < \epsilon^{\delta_i}, \\ 0 & \text{otherwise.} \end{cases} \tag{12.3}$$

Here, we use the convention $1/0 = \infty$.

The following Lemma is the form in which we use Proposition 7.5 in our setting.

Lemma 12.1. *For every $\beta \in (0,1)$, there exists $D \geqslant 1$ such that for all $t, \epsilon > 0$ and all $v \in \bigwedge^i \mathbb{R}^d$ with $0 < i < d$,*

$$\frac{1}{2} \int_{-1}^{1} \varphi_\epsilon^\beta(g_t u_s v)\, ds \leqslant D e^{-\beta t} \varphi_\epsilon^\beta(v).$$

Proof. First, we note that for all $g \in G$, $\pi_0(gv) = g\pi_0(v) = \pi_0(v)$. In particular, if $\varphi_\epsilon(v) = 0$, then $\|\pi_0(v)\| = \|\pi_0(g_t u_s v)\| \geqslant \epsilon^{\delta_i}$ for all s and t and the statement follows in this case. Hence, we may assume $\varphi_\epsilon(v) \neq 0$.

Moreover, since the integral of the minimum of finitely many functions is bounded by the minimum of their integrals, it suffices to prove

$$\frac{1}{2} \int_{-1}^{1} \|\pi_\lambda(g_t u_s v)\|^{-\beta/\delta_\lambda}\, ds = \frac{1}{2} \int_{-1}^{1} \|g_t u_s \pi_\lambda(v)\|^{-\beta/\delta_\lambda}\, ds \leqslant D e^{-\beta t} \|\pi_\lambda(v)\|^{-\beta/\delta_\lambda}, \tag{12.4}$$

for each $\lambda \in P^+ \setminus \{0\}$ and for some constant D depending only on β.

If $\pi_\lambda(v) = 0$ for some λ, then the right-hand side of (12.4) is ∞ and the claim is proved in this case. Now, suppose that $\pi_\lambda(v) \neq 0$ for some $\lambda \in P^+ \setminus \{0\}$.

Denote by $V = \left(\bigwedge^i \mathbb{R}^d\right)_\lambda$, i.e. the image of $\bigwedge^i \mathbb{R}^d$ under the projection π_λ. From the representation theory of $\mathrm{SL}(2,\mathbb{R})$, we see that the dimension of an irreducible representation with weight λ is equal to $\delta_\lambda + 1$. In particular, choosing $\beta \in (0,1)$ allows us to apply Proposition 7.5 to get that

$$\frac{1}{2} \int_{-1}^{1} \|g_t u_s \pi_\lambda(v)\|^{-\beta/\delta_\lambda}\, ds \leqslant D e^{-\beta t} \|\pi_\lambda(v)\|^{-\beta/\delta_\lambda}.$$

This proves (12.4) and completes the proof. □

12.2. **The Contraction Hypothesis on X.** The space $X = S/\Lambda$ may be identified with the space of unimodular lattices in $\mathbb{R}^d$ via the map $g\mathrm{SL}(d,\mathbb{Z}) \mapsto g\mathbb{Z}^d$. For $x \in X$, let $P(x)$ denote the set of all primitive subgroups of the lattice x. Recall that a subgroup of a lattice in $\mathbb{R}^d$ is primitive if its $\mathbb{Z}$ basis can be completed to a basis of $\mathbb{R}^d$ as an $\mathbb{R}$-vector space.

We say a monomial $v_1 \wedge \cdots \wedge v_i \in \bigwedge^i \mathbb{R}^d$ is x-integral if the abelian subgroup of $\mathbb{R}^d$ generated by $\{v_1, \ldots, v_i\}$ belongs to $P(x)$.

Now, we define the function $f_\epsilon : X \to [0, \infty]$ by

$$f_\epsilon(x) = \max \varphi_\epsilon(v), \tag{12.5}$$

where the maximum is taken over all non-zero x-integral monomials $v \in \bigwedge^i \mathbb{R}^d$ and all $0 < i < d$.

Remark 12.2. The function f_ϵ can assume the value ∞. However, for any $x \in X$, one can choose ϵ to be small enough so that $f_\epsilon(x) < \infty$. In fact, one can choose such ϵ uniformly for compact subsets of X by Mahler's criterion.

Following the same lines as Proposition 5.3 in [BQ11], we obtain the following result.

Proposition 12.3. *For all $\beta \in (0,1)$, there exists $c_0 \geqslant 1$ such that for all $t > 0$, there exist constants $\varepsilon_0, b > 0$, depending on t, satisfying*

$$\frac{1}{2} \int_{-1}^{1} f_{\varepsilon_0}^\beta(g_t u_s x)\, ds \leqslant c_0 e^{-\beta t} f_{\varepsilon_0}^\beta(x) + b,$$

for all $x \in X$.

Proof. Suppose $\beta \in (0,1)$ and let $D \geqslant 1$ be the constant in the conclusion of Lemma 12.1. For a compact set $Q \subset G$, define

$$\omega(Q) = \sup_{g \in Q} \max \left\{ \|g\|, \left\|g^{-1}\right\| \right\}, \tag{12.6}$$

where $\|\cdot\|$ is the operator norm induced by the euclidean norm on $V = \bigoplus_{i=1}^{d-1} \bigwedge^i \mathbb{R}^d$.

Fix some $t > 0$ and denote by ω the following constant.

$$\omega = \omega\left(\{g_t u_s : s \in [-1,1]\}\right). \tag{12.7}$$

Let $\varepsilon_0 > 0$ be a constant to be determined later. Note that the exponents δ_λ in the definition of φ_{ε_0} satisfy

$$\delta_\lambda \geqslant 1/\alpha(H_0),$$

for all $0 \neq \lambda \in P^+$. Thus, by definition of φ_{ε_0}, for all $s \in [-1,1]$ and all $v \in V$,

$$\omega^{-\alpha(H_0)} \varphi_{\varepsilon_0}(v) \leqslant \varphi_{\varepsilon_0}(g_t u_s v) \leqslant \omega^{\alpha(H_0)} \varphi_{\varepsilon_0}(v). \tag{12.8}$$

It is shown in [BQ11, Claim 5.9] that given a compact subset Q of G, there exists constants $C_1 \geqslant 1$ and $\varepsilon_0 > 0$, depending on Q, such that whenever $f_{\varepsilon_0}(x) > C_1$, the set of x-integral monomials v, satisfying

$$\varphi_{\varepsilon_0}(v) \geqslant f_{\varepsilon_0}(x)/\omega(Q)^{2\alpha(H_0)}, \tag{12.9}$$

contains at most one vector up to a sign in each of $\bigwedge^i \mathbb{R}^d$ with $0 < i < d$. In particular, we may apply this result to the compact set $Q = \{g_t u_s : s \in [-1,1]\}$.

Suppose $x \in X$ satisfies $f_{\varepsilon_0}(x) > C_1$ and let Ψ denote the set of x-integral monomials satisfying (12.9). Then, Lemma 12.1 implies

$$\frac{1}{2}\int_{-1}^{1} f_{\varepsilon_0}^{\beta}(g_t u_s x)\, ds \leqslant \sum_{v \in \Psi} \frac{1}{2}\int_{-1}^{1} \varphi_{\varepsilon_0}^{\beta}(g_t u_s v)\, ds \leqslant 4dDe^{-\beta t} f_{\varepsilon_0}^{\beta}(x).$$

Finally, if $f_{\varepsilon_0}(x) < C_1$ for some $x \in X$, then (12.8) implies that

$$\frac{1}{2}\int_{-1}^{1} f_{\varepsilon_0}^{\beta}(g_t u_s x)\, ds \leqslant \omega^{\alpha(H_0)} C_1.$$

Thus, the statement of the Proposition follows by taking $c_0 = 4dD$ and $b = \omega^{\alpha(H_0)} C_1$. □

Proposition 8.3 establishes that the functions f_ε^β satisfy the main property in the β-contraction hypothesis (Def. 4.2) for the $G = \mathrm{SL}(2,\mathbb{R})$ action on homogeneous spaces X. Properties (1), (2) and (4) follow from Mahler's compactness criterion and the lower semi-continuity of f_{ε_0}. Finally, Assumption 5.1 follows from the following lemma.

Lemma 12.4. *There exists $N \in \mathbb{N}$, depending only on G and Γ, such that for every $T > 0$, there exists $M_0 > 0$ such that for all $x \in G/\Gamma$, and $M_1 \geq M_0$, the following holds.*

$$\textit{The set } \{|s| \leqslant T : f(u_s x) > M_1\} \textit{ has at most } N \textit{ connected components.} \tag{12.10}$$

Proof. The proof is identical to that of Lemma 10.6 and relies on the bounded cardinality of a set of vectors analogous to the set Ψ in the proof of Proposition 12.3. □

Thus, we have established the following statement.

Theorem 12.5. *Let $B \subset \mathbb{R}$ be an interval and suppose L is a semisimple algebraic Lie group defined over $\mathbb{Q}$, Γ an arithmetic lattice in L, and $\rho : \mathrm{SL}(2,\mathbb{R}) \to L$ a non-trivial representation. Let*

$$g_t = \rho\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right), \qquad u(\varphi(s)) = \rho\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right), s \in B.$$

Then, $\varphi(s)$ is a g_t-admissible curve satisfying the β-contraction hypothesis for the action of $G = \rho(\mathrm{SL}(2,\mathbb{R}))$ on L/Γ for all $\beta \in (0, 1/2)$, with a height function satisfying Assumption 5.1.

12.3. **Proof of Theorem C.** The result follows by combining Theorem 12.5 with Theorems 4.3 and 5.2, and Proposition 6.1.

13. Conclusions and Open Problems

The results of this article leave open several natural questions, which we now discuss.

13.1. **Lower Bounds.** It is known (cf. [KP17]) that when $G = \mathrm{SL}(2,\mathbb{R})$, then the upper bound obtained in Theorem A on the dimension of divergent on average orbits coincides with the lower bound. This fact can be used to deduce a lower bound on the dimension of divergent on average orbits in a special case of Theorem A as follows.

Proof of Corollary 2.4. By Theorem A, we only need to establish the lower bound. First, we observe[2] that if the orbit $(g_t^{(1)} u(\varphi_1(s))x_0)_{t\geqslant 0}$ is divergent on average in $\mathrm{SL}(2,\mathbb{R})/\Gamma_1$, then the orbit $(g_t u(\varphi(s))x_0)_{t\geqslant 0}$ is divergent on average in G/Γ. This follows from the fact that every compact subset $\mathcal{K} \subset G/\Gamma$ is contained in a set of the form $\mathcal{K}_1 \times \mathcal{K}_2$, where $\mathcal{K}_1 \subset \mathrm{SL}(2,\mathbb{R})/\Gamma_1$ and $\mathcal{K}_2 \subset G'/\Gamma'$ are compact sets.

The assumption that φ_1 is non-constant implies that $\varphi_1(B)$ is a compact non-trivial interval. It follows from [KP17, Theorem 1.3] that the set of points $r \in \varphi_1(B)$ for which the orbit $(g_t^{(1)} u(r)x_0)_{t\geqslant 0}$ is divergent on average has Hausdorff dimension $1/2$. Since φ_1 is Lipschitz, and Lipschitz maps do not increase Hausdorff dimension, we obtain the desired lower bound. □

Corollary 2.4 leaves open the question of whether $1/2$ is in fact a lower bound on the dimension of divergent orbits (not divergent on average) when Γ is reducible. However, this corollary motivates the following natural question.

Question 13.1. *In the settings of Theorems A, B, and C: If G/Γ is not compact, is the Hausdorff dimension of the divergent on average orbits of g_t equal to the minimum of 1 and the upper bounds obtained in loc. cit.?*

13.1.1. *A Counter Example.* As noted in the introduction, Theorem B gives a meaningful upper bound on the dimension of divergent on average orbits only when

$$\#\{1 \leqslant i \leqslant r_k : (\dot{\varphi}_k)_i \not\equiv 0\} + 2 \cdot \#\{r_k < i \leqslant r_k + s_k : (\dot{\varphi}_k)_i \not\equiv 0\} > \frac{r_k + 2s_k}{2}, \tag{13.1}$$

for all $1 \leq k \leq l$. We now show that this condition cannot be relaxed in general.

Suppose $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{C})$ and Γ is an irreducible, non-uniform lattice in G. One may construct such a lattice using the Galois embedding of $\mathrm{SL}(2,\mathcal{O}_K)$ into G, where $\mathcal{O}_K$ is the ring of integers in $K = \mathbb{Q}(\sqrt[3]{2})$ for example. Let $\varphi : B \to \mathbb{R} \times \mathbb{C}$ be given by $\varphi(s) = (s, 0)$ and let g_t be as in (3.3). Let $x_0 = g\Gamma$ for $g \in G$ the Weyl “element” given by

$$g = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

One can then check directly using the definition of the proper function f in (9.5) that $f(g_t u(\varphi(s))x_0)$ tends to ∞ as $t \to \infty$ for every $s \in B$. Roughly the amount of expansion provided by the first factor (an eigenvalue of 2) is negated by the contraction in the second factor (an eigenvalue of -4).

This, however, leaves open the question of whether the dimension of divergent on average orbits is strictly less than 1 in the critical case when the 2 sides of inequality (13.1) are equal.

[2]Note that the converse is not true in general.

13.2. **More General Diagonal Flows.** The notion of a deformation of a maximal representation a priori restricts our results to curves whose tangents form an $\mathfrak{sl}(2,\mathbb{R})$-triple with the diagonal flow in question. However, the proof of Theorem B (particularly Lemma 11.1) shows that our methods apply to a more general class of curves which we now describe.

Suppose ρ is a deformation of a maximal representation and let $g_t = \exp(tH_\rho)$ and $\varphi(s) = \rho_s(X)$. For each k, let V_k^+ denote the g_t-expanding subspace of the vector space V_k defined in 9.3. Suppose A is a maximal $\mathbb{R}$-split torus containing g_t, which we identify with its Lie algebra. Denote by $A^+(H_\rho)$ the cone inside A of semisimple elements H' which have positive eigenvalues on V_k^+ for each k. Note that $H_\rho \in A^+(H_\rho)$ by definition. Given any $H' \in A^+(H_\rho)$ such that φ is $\exp(tH')$-admissible, one can check that the proofs of the integral estimates obtained in Section 10 go through with $\exp(tH')$ in place of g_t, with a contraction rate depending the eigenvalues of H'. However, it is not clear whether the upper bounds on the dimension of divergent orbits for the flows obtained in this manner are sharp.

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