POLYNOMIAL FOURIER DECAY FOR PATTERSON-SULLIVAN MEASURES

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Abstract. We show that the Fourier transform of Patterson-Sullivan measures associated to convex cocompact groups of isometries of real hyperbolic space decays polynomially quickly at infinity. The proof is based on the L^2-flattening theorem obtained in [17] combined with a method based on dynamical self-similarity for ruling out the sparse set of potential frequencies where the Fourier transform can be large.

1. Introduction

1.1. **Background.** The Fourier transform of a Borel probability measure μ on $\mathbb{R}^d$ is defined as follows:

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{2\pi i\langle \xi, x\rangle}\, d\mu(x), \qquad \xi \in \mathbb{R}^d. \tag{1.1}$$

We say μ has *polynomial Fourier decay* if $|\hat{\mu}(\xi)| = O(\|\xi\|^{-\kappa})$ for some $\kappa > 0$ as $\|\xi\| \to \infty$.

Rates of decay of Fourier transforms of dynamically defined measures have been extensively studied in recent years. Beyond its intrinsic interest, this question has found many applications in other areas of mathematics; e.g. essential spectral gaps on hyperbolic manifolds [14, 8, 21], the uniqueness problem [23], quantum chaos and fractal uncertainty principles [13], Diophantine approximation [12], and geometric measure theory [35, 24] to name a few.

Moreover, the problem has motivated the development of many methods drawing on a wide varying tools ranging from spectral gaps of the underlying dynamics [4, 7], to renewal theory [20], sum-product phenomena [8, 21, 19], large deviation estimates for Fourier transforms [25, 1, 6], as well as many related developments; cf. [3, 2, 16, 34, 23, 22, 37, 10] for a non-exhaustive list. We refer the reader to the survey [33] for a comprehensive account of the history and recent developments in the subject.

A common strategy that is implicit in many of the aforementioned results proceeds as follows:

(1) Find a mechanism to show that the Fourier transform has the desired rate of decay for a large set of frequencies ξ.
(2) Use the dynamics (or the multiscale/convolution structure of μ) to express the Fourier transform of μ at frequency ξ as an *average* of Fourier transforms of (scaled copies of) μ at images of ξ by the dynamics.
(3) Show (through non-linearity of/Diophantine conditions on the dynamics) that images of ξ by the dynamics are reasonably well-distributed in the space in such a way that they avoid the potential exceptional set of frequencies arising in Step 1.

To demonstrate this strategy, consider the following basic estimate towards Step 1: if μ satisfies the Frostman condition $\mu(B(x,r)) \lesssim r^\alpha$ for some $\alpha > 0$ and all balls of radius $r \geq 0$, then the Fourier transform decays like $\|\xi\|^{-\alpha/2}$ *on average*, i.e.

$$\int_{\|\xi\|\leq R} |\hat{\mu}(\xi)|^2 \, d\xi \lesssim R^{d-\alpha}, \qquad \forall R \geq 1. \tag{1.2}$$

This estimate roughly means that the exceptional set of potentially problematic frequencies have (box) dimension at most $d-\alpha$; cf. [24, Section 3.8]. Hence, we can obtain Fourier decay as soon as we can show that the frequencies produced in Step 2 have dimension $> d-\alpha$. Since the image of ξ

under the dynamics tends to have a similar dimension to the support of μ itself, this procedure is sufficient for establishing Fourier decay in many situations when the dimension of the support of μ is $> d/2$.

However, in general, the estimate (1.2) is rather weak when $\alpha \leq d/2$. In that case, more involved methods are necessary to either produce stronger estimates in Step 1 (e.g. large deviations methods) or to produce better averaging and well-distribution schemes in Steps 2 and 3 (e.g. spectral gap and renewal theory methods).

Recently, a very general estimate towards Step 1 was obtained in [17, Corollary 11.5] under natural non-concentration hypotheses on the (not necessarily dynamically defined) measure μ. Namely, it is shown[1] that if μ does not concentrate near proper affine subsapces of $\mathbb{R}^d$ at many scales, then its Fourier transform decays polynomially outside of a very sparse set of frequencies, i.e. for all $\varepsilon > 0$, there is $\delta > 0$ such that:

$$\left| \left\{ \|\xi\| \leq R : |\hat{\mu}(\xi)| > R^{-\delta} \right\} \right| = O\left(R^{\varepsilon}\right). \tag{1.3}$$

The goal of this article to show that (1.3) can be used in conjunction with the strategy outlined above to give efficient proofs of quantitative Fourier decay of dynamically defined measures. We apply our method to a particular class of interest in applications, namely that of Patterson-Sullivan measures for convex cocompact groups of isometries of real hyperbolic space. Since the non-concentration conditions implying (1.3) are known to hold for large classes of dynamically-defined measures[2], we hope the simplicity of the method presented here will allow it to be extended to yield Fourier decay results in much broader contexts.

1.2. **Main result.** Let Γ be a discrete, Zariski-dense, convex cocompact, group of isometries of real hyperbolic space $\mathbb{H}^{d+1}$, $d \geq 1$. Let Λ_Γ be the limit set of Γ on $\partial\mathbb{H}^{d+1}$ and μ be the Patterson-Sullivan probability measure on Λ_Γ associated to Γ; cf. Section 2 for detailed definitions. The following is the main result of this article.

Theorem 1.1. *There exists $\kappa > 0$ such that the following holds for all $\varphi \in C^2$, $\psi \in C^1$ satisfying*

$$\|\varphi\|_{C^2} + \|\psi\|_{C^1} \leq A, \qquad \inf_{x\in\Lambda_\Gamma} \|\nabla_x \varphi\| > a,$$

for some constants $a > 0$ and $A \geq 1$. There exists a constant $C = C(A, a, \mu) \geq 1$, so that for all $\lambda \neq 0$, we have

$$\left| \int_{\Lambda_\Gamma} e^{2\pi i \lambda \varphi(x)} \psi(x)\, d\mu(x) \right| \leq C|\lambda|^{-\kappa}.$$

Remark 1.2. Our proof shows that the rate κ provided by Theorem 1.1 depends only on non-concentration parameters of μ; cf. Section 2.6 for the precise definition of non-concentration. In particular, the rate of decay does not change upon replacing Γ by a finite index subgroup since the measure μ remains the same in this case [31].

Theorem 1.1 generalizes prior work of Bourgain and Dyatlov in the case of hyperbolic surfaces [8] and of Li, Naud, and Pan in the case of Schottky hyperbolic 3-manifolds [21]. These prior results are based on Bourgain's sum-product theorem, while the proof of Theorem 1.1 is based on the estimate (1.3), which was obtained using purely additive methods.

To keep the presentation clear, we restricted our setup to the case of convex cocompact groups. Using the recurrence results obtained in [17], the proof of Theorem 1.1 can be adapted to handle the general case of geometrically finite manifolds.

[1] Cf. Definition 2.2 and Theorem 2.3 for precise statements.

[2] E.g. Self-conformal measures [11], and Patterson-Sullivan measures for (cusped) geometrically finite manifolds [17].

By the work of Dyatlov and Zahl [14], Theorem 1.1 is known to imply spectral bounds on the resolvent of the Laplace operator which yield an *essential spectral gap*[3] for the resolvent as well as Selberg's zeta function. Moreover, the size of the essential spectral gap obtained this way depends explicitly on the decay rate κ in Theorem 1.1. In particular, Theorem 1.1 implies that the resolvent admits a uniform essential spectral gap over all finite covers of $\mathbb{H}^{d+1}/\Gamma$; cf. Remark 1.2. We note that this essential spectral gap result was obtained independently in [5] by different methods and an essential gap of size depending on Γ was obtained previously in [26, 29].

Acknowledgements. This author is partially supported under NSF grant DMS-2247713.

2. Preliminaries

2.1. **Convex cocompact manifolds.** The standard reference for the material in this section is [9]. Let G denote the group of orientation preserving isometries of real hyperbolic space, denoted $\mathbb{H}^{d+1}$, of dimension $d \geq 1$. In particular, $G \cong \mathrm{SO}(d+1,1)^0$.

Fix a basepoint $o \in \mathbb{H}^{d+1}$. Then, G acts transitively on $\mathbb{H}^{d+1}$ and the stabilizer K of o is a maximal compact subgroup of G. We shall identify $\mathbb{H}^{d+1}$ with $K\backslash G$. Denote by $A = \{g_t : t \in \mathbb{R}\}$ a one parameter subgroup of G inducing the geodesic flow on the unit tangent bundle of $\mathbb{H}^{d+1}$. Let $M < K$ denote the centralizer of A inside K so that the unit tangent bundle $\mathrm{T}^1\mathbb{H}^{d+1}$ may be identified with $M\backslash G$. In Hopf coordinates, we can identify $\mathrm{T}^1\mathbb{H}^{d+1}$ with $\mathbb{R} \times (\partial\mathbb{H}^{d+1} \times \partial\mathbb{H}^{d+1} \setminus \Delta)$, where $\partial\mathbb{H}^{d+1}$ denotes the boundary at infinity and Δ denotes the diagonal.

Let $\Gamma < G$ be an infinite discrete subgroup of G. The limit set of Γ, denoted Λ_Γ, is the set of limit points of the orbit $\Gamma \cdot o$ on $\partial\mathbb{H}^{d+1}$. Note that the discreteness of Γ implies that all such limit points belong to the boundary. Moreover, this definition is independent of the choice of o in view of the negative curvature of $\mathbb{H}^{d+1}$. We often use Λ to denote Λ_Γ when Γ is understood from context. We say Γ is *non-elementary* if Λ_Γ is infinite.

The *non-wandering set* for the geodesic flow is the closure of the set of vectors in the unit tangent bundle whose orbit accumulates on itself. In Hopf coordinates, this set, denoted Ω, coincides with the projection of $\mathbb{R} \times (\Lambda_\Gamma \times \Lambda_\Gamma - \Delta) \bmod \Gamma$. We say $\mathbb{H}^{d+1}/\Gamma$ is *convex cocompact* if Ω is compact, cf. [9]. Denote by N^+ the expanding horospherical subgroup of G associated to g_t, $t \geq 0$.

Given $g \in G$, we denote by g^+ the coset of P^-g in the quotient $P^-\backslash G$, where $P^- = N^-AM$ is the stable parabolic group associated to $\{g_t : t \geq 0\}$. Similarly, g^- denotes the coset P^+g in $P^+\backslash G$. Since M is contained in $P^\pm$, such a definition makes sense for vectors in the unit tangent bundle $M\backslash G$. Geometrically, for $v \in M\backslash G$, v^+ (resp. v^-) is the forward (resp. backward) endpoint of the geodesic determined by v on the boundary of $\mathbb{H}^{d+1}$. Given $x \in G/\Gamma$, we say $x^\pm$ belongs to Λ if the same holds for any representative of x in G; this notion being well-defined since Λ is Γ invariant.

Notation. Throughout the remainder of the article, we fix a discrete, Zariski-dense, convex cocompact group Γ of isometries of $\mathbb{H}^{d+1}$.

2.2. **Patterson-Sullivan measures.** The *critical exponent*, denoted δ_Γ, is defined to be the infimum over all real number $s \geq 0$ such that the Poincaré series

$$P_\Gamma(s,o) := \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma\cdot o)} \tag{2.1}$$

converges. This exponent coincides with the Hausdorff dimension of the limit set as well as the topological entropy of the geodesic flow on the quotient orbifold $\mathbb{H}^{d+1}/\Gamma$. We shall simply write δ for δ_Γ when Γ is understood from context.

[3]That is to say a strip to the left of the critical line containing at most finitely many poles. The interested reader is referred to the survey [13] for more on this topic.

The *Busemann function* is defined as follows: given $x, y \in \mathbb{H}^{d+1}$ and $\xi \in \partial\mathbb{H}^{d+1}$, let $\gamma : [0,\infty) \to \mathbb{H}^{d+1}$ denote a geodesic ray terminating at ξ and define

$$\beta_\xi(x,y) = \lim_{t\to\infty} \mathrm{dist}(x,\gamma(t)) - \mathrm{dist}(y,\gamma(t)).$$

A Γ-invariant conformal density of dimension s is a collection of Radon measures $\{\nu_x\}$ on the boundary indexed by $x \in \mathbb{H}^{d+1}$ which satisfy the following equivariance property:

$$\gamma_*\nu_x = \nu_{\gamma x}, \qquad \text{and} \qquad \frac{d\nu_y}{d\nu_x}(\xi) = e^{-s\beta_\xi(x,y)}, \qquad \forall x,y \in \mathbb{H}^{d+1}, \xi \in \partial\mathbb{H}^{d+1}, \gamma \in \Gamma.$$

Patterson [27] and Sullivan [36] showed the existence of a unique (up to scaling) Γ-invariant conformal density of dimension δ_Γ, denoted $\left\{\mu_x^{\mathrm{PS}} : x \in \mathbb{H}^{d+1}\right\}$. These measures are known as the *Patterson-Sullivan measures*. We refer the reader to [30] and [28] and references therein for details of the construction in much greater generality.

2.3. **Stable and unstable foliations and leafwise measures.** Recall that we fixed a basepoint $o \in \mathbb{H}^{d+1}$. In what follows, we use the following notation for pullbacks of the Patterson-Sullivan measures to orbits of N^+ under the visual map:

$$d\mu_x^u(n) = e^{\delta_\Gamma \beta_{(nx)^+}(o,nx)} d\mu_o^{\mathrm{PS}}((nx)^+). \tag{2.2}$$

These measures have simpler transformation formulas under the action of the geodesic flow and N^+ which makes them relatively easier to analyze than the Patterson-Sullivan measures directly. In particular, they satisfy the following equivariance property under the geodesic flow:

$$\mu_{g_t x}^u = e^{\delta t} \mathrm{Ad}(g_t)_* \mu_x^u. \tag{2.3}$$

Moreover, it follows readily from the definitions that for all $n \in N^+$,

$$(n)_* \mu_{nx}^u = \mu_x^u, \tag{2.4}$$

where $(n)_*\mu_{nz}^u$ is the pushforward of μ_{nz}^u under the map $u \mapsto un$ from N^+ to itself. Finally, since M normalizes N^+, these conditionals are $\mathrm{Ad}(M)$-invariant in the sense that for all $m \in M$,

$$\mu_{mx}^u = \mathrm{Ad}(m)_* \mu_x^u. \tag{2.5}$$

2.4. **Local stable holonomy.** In this Section, we recall the definition of (stable) holonomy maps which are essential for our arguments. We give a simplified discussion of this topic which is sufficient in our homogeneous setting homogeneous. Let $x = u^- y$ for some $y \in \Omega$ and $u^- \in N_2^-$. Since the product map $N^- \times A \times M \times N^+ \to G$ is a diffeomorphism near identity, we can choose the norm on the Lie algebra so that the following holds. We can find maps $p^- : N_1^+ \to P^- = N^-AM$ and $u^+ : N_2^+ \to N^+$ so that

$$nu^- = p^-(n)u^+(n), \qquad \forall n \in N_2^+. \tag{2.6}$$

Then, it follows by (2.2) that for all $n \in N_2^+$, we have

$$d\mu_y^u(u^+(n)) = e^{\delta\beta_{(nx)^+}(u^+(n)y,nx)} d\mu_x^u(n).$$

Moreover, by further scaling the metrics if necessary, we can ensure that these maps are diffeomorphisms onto their images. In particular, writing $\Phi(nx) = u^+(n)y$, we obtain the following change of variables formula: for all $f \in C(N_2^+)$,

$$\int f(n)\, d\mu_x^u(n) = \int f((u^+)^{-1}(n)) e^{-\delta\beta_{\Phi^{-1}(ny)}(ny,\Phi^{-1}(ny))}\, d\mu_y^u(n). \tag{2.7}$$

Remark 2.1. To avoid cluttering the notation with auxiliary constants, we shall assume that the N^- component of $p^-(n)$ belongs to N_2^- for all $n \in N_2^+$ whenever u^- belongs to N_1^-.

2.5. **Notational convention.** Throughout the article, given two quantities A and B, we use the Vinogradov notation $A \ll B$ to mean that there exists a constant $C \geq 1$, possibly depending on Γ and the dimension of G, such that $|A| \leq CB$. In particular, this dependence on Γ is suppressed in all of our implicit constants, except when we wish to emphasize it. The dependence on Γ may include for instance the diameter of the complement of our choice of cusp neighborhoods inside Ω and the volume of the unit neighborhood of Ω. We write $A \ll_{x,y} B$ to indicate that the implicit constant depends parameters x and y. We also write $A = O_x(B)$ to mean $A \ll_x B$.

2.6. **The L^2-flattening theorem.** In light of the formula (2.2), Theorem 1.1 amounts to studying the Fourier transform of the measures μ_x^u. Moreover, the isomorphism $N^+ \cong \mathbb{R}^d$ allows us to view these measures as living on Euclidean space.

The key ingredient in the proof of Theorem 1.1 is [17, Corollary 1.8] which relates Fourier decay properties of measures on Euclidean space to the non-concentration properties of such measures near proper affine subspaces. This result in particular implies that PS measures enjoy polynomial Fourier decay outside of a very sparse set of frequencies.

We formulate here a special case of the aforementioned result which suffices for convex cocompact manifolds and refer the reader to [17, Theorem 11.5] for a more general result that holds in the presence of cusps.

Definition 2.2. We say that Borel measure μ on $\mathbb{R}^d$ is *uniformly affinely non-concentrated* if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ so that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and for all $x \in \mathrm{supp}(\mu)$, $0 < r \leq 1$, and every affine hyperplane $W < \mathbb{R}^d$, we have

$$\mu(W^{(\varepsilon r)} \cap B(x,r)) \leq \delta(\varepsilon)\mu(B(x,r)), \tag{2.8}$$

where $W^{(r)}$ and $B(x,r)$ denote the r-neighborhood of V and the r-ball around x respectively. We refer to $\delta(\varepsilon)$ as *the non-concentration parameters* of μ.

Theorem 2.3 ([17, Corollary 1.8])**.** *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$ which is uniformly affinely non-concentrated and denote by $\hat{\mu}$ its Fourier transform. Then, for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $T \geq 1$,*

$$\left|\left\{\|\xi\| \leq T : |\hat{\mu}(\xi)| > T^{-\delta}\right\}\right| = O_\varepsilon(T^\varepsilon),$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^d$. The implicit constant depends only on the non-concentration parameters of μ and the diameter of its support.

We note that Theorem 2.3 was obtained by different methods for measures on the real line in [32].

We will be able to apply Theorem 1.1 to PS measures (or, more precisely, their shadows μ_x^u) thanks to the following proposition.

Proposition 2.4 ([17, Corollary 12.2])**.** *For every $x \in N_1^-\Omega$, the measure $\mu_x^u\Big|_{N_1^+}$ is uniformly affinely non-concentrated in the sense of Definition 2.2, with uniform parameters in x.*

Remark 2.5. It is shown in [17, Corollary 12.2] that the non-concentration parameters of μ_x^u depend only on the injectivity radius at x, which is in turn uniformly bounded above and below on a neighborhood of the non-wandering set Ω due to convex cocompactness of Γ.

3. Proof of Theorem 1.1

The goal of this section is to provide the proof of Theorem 1.1. In light of the formula (2.2), it suffices to prove polynomial Fourier decay for the measures μ_x^u for x in the non-wandering set Ω.

Norms and Lie algebras. In what follows, we denote by $\mathfrak{n}^+$ and $\mathfrak{n}^-$ the Lie algebras of N^+ and N^- respectively. We fix an isomorphism of $\mathfrak{n}^+$ and $\mathfrak{n}^-$ using a Cartan involution sending g_t to g_{-t}. Moreover, we fix an isomorphism of $\mathfrak{n}^+$ (and hence of $\mathfrak{n}^-$) with $\mathbb{R}^d$. Finally, we fix a Euclidean inner product on $\mathbb{R}^d \cong \mathfrak{n}^+ \cong \mathfrak{n}^-$ denoted with $\langle \cdot, \cdot \rangle$ which is invariant by the Adjoint action of the group $M \cong \mathrm{SO}_d(\mathbb{R})$.

Reduction to linear phases. We begin with the following elementary lemma which reduces the proof to the study of linear phase functions.

Lemma 3.1. *To prove Theorem 1.1, it suffices to show that there exists $\kappa > 0$ so that for all $0 \neq \xi \in \mathbb{R}^d$, $x \in \Omega$, and $\psi \in C_c^1(N_1^+)$, we have*

$$\int_{N_1^+} e^{i\langle \xi, n\rangle} \psi(n)\, d\mu_x^u(n) \ll_\Gamma \|\psi\|_{C^1} \|\xi\|^{-\kappa}, \tag{3.1}$$

where, by abuse of notation, if $n = \exp(v)$ for some $v \in \mathfrak{n}^+ \cong \mathbb{R}^d$, we let $\langle \xi, n\rangle := \langle \xi, v\rangle$.

Remark 3.2. In the remainder of this section, we fix $\xi \in \mathbb{R}^d$ and $\psi \in C_c^1(N_1^+)$. Our goal is to prove the estimate (3.1).

Proof of Lemma 3.1. The proof is based on the uniformity of the estimate (3.1) as the basepoint x varies in Ω which roughly translates to uniform Fourier decay (with linear phases) over pieces of the measure of size $|\lambda|^{-1/2-\varepsilon}$. We include a sketch of the argument for completeness.

Recall the notation of Theorem 1.1. Let $\{\rho_j\}$ be a partition of unity of $\mu_x^u|_{N^+|}$. with bounded multiplicity and such that each ρ_j is supported in a ball B_j of radius

$$r = |\lambda|^{-(1+\kappa)/(2+\kappa)}$$

around a $n_j \in N_1^+$. Here, κ is the exponent in (3.1). In view of [17, Prop. 9.9], we can choose such partition of unity so that each ρ_j has first derivatives with norm $O(r^{-1})$ and

$$\sum_j \mu_x^u(B_j) \ll \mu_x^u(N_1^+). \tag{3.2}$$

Then, Taylor expanding φ to the second order around each n_j, we obtain

$$\begin{aligned}\int_{N_1^+} e^{i\lambda\varphi(n)}\psi(n)\, d\mu_x^u \leq \sum_j & \left| \int_{B_j} \exp(i\langle \lambda\nabla\varphi(n_j), nn_j^{-1}\rangle)(\psi\rho_j)(n)\, d\mu_x^u \right| \\ & + O(\|\psi\|_{C^0} \|\varphi\|_{C^2} |\lambda| r^2 \mu_x^u(N_1^+)).\end{aligned}$$

Next, we use a change of variables sending B_j to N_1^+ and apply (2.3) and (2.4). More precisely, let $t = -\log r, x_j = g_t n_j x$, $\xi_j = r\lambda\nabla\varphi(n_j)$, and $\psi_j(n) := (\psi\rho_j)(\mathrm{Ad}(g_{-t})(n)n_j)$. Then, the j^{th} term in the above sum can be rewritten as follows:

$$\int_{B_j} \exp(i\langle \lambda\nabla\varphi(n_j), nn_j^{-1}\rangle)(\psi\rho_j)(n)\, d\mu_x^u = r^\delta \int_{N_1^+} \exp(i\langle \xi_j, n\rangle)\psi_j(n)\, d\mu_{x_j}^u.$$

Note that, since the geodesic flow scales the first derivatives of ρ_j by a factor of e^{-t}, each ψ_j has C^1-norm $O(\|\psi\|_{C^1})$. Hence, since each x_j belongs to Ω, (3.1) implies that

$$\int_{N_1^+} e^{i\lambda\varphi(n)}\psi(n)\, d\mu_x^u \ll_A |r\lambda|^{-\kappa} a^{-\kappa} \sum_j r^\delta + |\lambda| r^2.$$

Finally, in view of Sullivan's shadow lemma (cf. Proposition 3.3 below), we have that $\mu_x^u(B_j) \asymp r^\delta$. This concludes the proof in light of (3.2). The decay exponent obtained in this manner is $\kappa/(2+\kappa)$, with κ as in (3.1). $\square$

Polynomial non-concentration estimates. We recall here well-known non-concentration estimates for PS measures. The first estimate is a direct consequence of Sullivan's shadow lemma.

Proposition 3.3 (Sullivan's Shadow Lemma). *For all $y \in \Omega$ and all $r > 0$, we have*

$$\mu_y^u(N_r^+) \asymp_\Gamma r^\delta,$$

where δ is the critical exponent of Γ.

We also recall the following quantitative decay property of the measure of hyperplane neighborhoods with respect to PS measures from [11]. Recall that N^+ is an abelian group which we identify with its Lie algebra $\mathfrak{n}^+ \cong \mathbb{R}^d$ via the exponential map.

Theorem 3.4 ([17, Theorem 11.17]). *There exist constants $C \geq 1$ and $\alpha > 0$ such that for all $\varepsilon, r > 0$, $x \in \Omega$, and all affine hyperplanes $L < N^+$, we have that*

$$\mu_x^u(N_r^+ \cap L^{(\varepsilon r)}) \leq C(\varepsilon/r)^\alpha \mu_x^u(N_r^+),$$

where $L^{(\varepsilon r)}$ denotes the εr-neighborhood of L in N^+.

Remark 3.5. A suitable form of Theorem 3.4 was derived in [17] from the flattening theorem in the case Γ is geometrically finite. However, the proof is much simpler in the case Γ is convex cocompact and can be deduced from the fact that PS measures give 0 mass to proper subvarieties of the boundary ([15, Corollary 9.4]) using the argument in [18, Section 8].

Partitions of unity and flow prisms. Given $r > 0$, we let N_r^+ (resp. P_r^-) the neighborhood of identity of radius r inside N^+ (resp. $P^- = MAN^-$). We refer to sets of the form $P_r^- N_s^+ \cdot x$ for $r, s > 0$ and $x \in G/\Gamma$ as *flow boxes*. We say that a collection of sets $\{S_i\}$ has multiplicity bounded by a constant $C \geq 1$ if for all x:

$$\sum_i \mathbb{1}_{S_i}(x) \leq C \mathbb{1}_{\cup_i S_i}(x).$$

Let ι denote the smaller of $1/2$ and the injectivity radius of G/Γ and set

$$\iota_\xi := \iota / \|\xi\|^{1/3}. \tag{3.3}$$

The following lemma provides us with an efficient cover of Ω by "thin flow boxes" in the unstable direction.

Lemma 3.6. *The collection $\left\{P_\iota^- N_{\iota_\xi}^+ \cdot x : x \in \Omega\right\}$ admits a finite subcover $\mathcal{B}_\xi$ such that*

$$\#\mathcal{B}_\xi \ll_\Gamma \|\xi\|^{\delta/3},$$

where δ is the critical exponent of Γ. Moreover, $\mathcal{B}_\xi$ has uniformly bounded multiplicity on Ω; i.e. for all $x \in \Omega$, $\sum_{B \in \mathcal{B}_\xi} \mathbb{1}_B(x) \ll_\Gamma 1$.

Proof. Let $\mathcal{Q}$ denote a cover of G/Γ by flow boxes of the form $N_\iota^+ P_\iota^- \cdot x$, where ι is a fixed lower bound on the injectivity radius of G/Γ as above. With the help of the Vitali covering lemma, such cover can be chosen to have multiplicity $C_G \geq 1$ depending only on the dimension of G. We will build our collection of boxes $\mathcal{B}_\xi$ by refining this cover as follows.

Let $\mathcal{Q}^0$ denote the collection of boxes $Q \in \mathcal{Q}$ such that $Q \cap \Omega \neq \emptyset$. By convex cocompactness, we have that $\#\mathcal{Q}^0 \asymp_\Gamma 1$. For each $Q \in \mathcal{Q}^0$, we fix some $x_Q \in Q \cap \Omega$. Then, we can find a finite set of points $\{u_i : i \in I_Q\} \subset N_{2\iota}^+$ such that the points $x_i := u_i x_Q$ belong to Ω. Moreover, these points be chosen so that the balls $N_{\iota_\xi}^+ \cdot x_i$ provide a cover of $\Omega \cap N_\iota^+ \cdot x_Q$ with multiplicity bounded by a constant $C_\Gamma \geq 1$ depending only Γ (i.e. C_Γ is independent of ξ). This is again possible thanks to the Vitali covering lemma and the fact that the measure $\mu_{x_Q}^u \big|_{N_{3\iota}^+}$ is doubling by Proposition 3.3.

With this notation, we define $\mathcal{B}_\xi$ as follows:

$$\mathcal{B}_\xi := \left\{ P_\iota^- N_{\iota_\xi}^+ \cdot u_i x_Q : i \in I_Q, Q \in \mathcal{Q}^0 \right\}.$$

This bounded multiplicity in particular implies that

$$\iota_\xi^\delta \times \# I_Q \asymp \sum_{i \in I_Q} \mu_{x_i}^u(N_{\iota_\xi}^+) \asymp \mu_{x_Q}^u(N_\iota^+) \asymp 1.$$

This estimate the desired the bound on the cardinality of $\mathcal{B}_\xi$. To bound the multiplicity of $\mathcal{B}_\xi$, let $x \in \Omega$ be arbitrary, and note that

$$\sum_{B \in \mathcal{B}_\xi} \mathbb{1}_B(x) = \sum_{Q \in \mathcal{Q}^0} \sum_{i \in I_Q} \mathbb{1}_{P_\iota^- N_{\iota_\xi}^+ \cdot u_i x_Q}(x) \leq C_\Gamma \sum_{Q \in \mathcal{Q}^0} \mathbb{1}_{\cup_{i \in I_Q} P_\iota^- N_{\iota_\xi}^+ \cdot u_i x_Q}(x) \leq C_\Gamma \# \mathcal{Q}^0 \ll_\Gamma 1.$$

This concludes the proof. □

Let $\mathcal{B}_\xi$ be the finite cover provided by Lemma 3.6 and let $\mathcal{P}_\xi$ denote a partition of unity subordinate to it. For each $\rho \in \mathcal{P}_\xi$, we denote by B_ρ the element of $\mathcal{B}_\xi$ containing the support of ρ. In particular, such partition of unity can be chosen so that for all $\rho \in \mathcal{P}_\xi$, we have

$$\|\rho\|_{C^1} \ll \|\xi\|^{1/3}. \tag{3.4}$$

Moreover, by Lemma 3.6, we have

$$\# \mathcal{P}_\xi \leq \# \mathcal{B}_\xi \ll_\Gamma \|\xi\|^{\delta/3}. \tag{3.5}$$

Transversals. We fix a system of transversals $\{T_\rho\}$ to the strong unstable foliation inside the boxes B_ρ. Since B_ρ meets Ω for all $\rho \in \mathcal{P}_\xi$, we fix some y_ρ in the intersection $B_\rho \cap \Omega$. In this notation, we have that

$$B_\rho = P_\iota^- N_{\iota_\xi}^+ \cdot y_\rho, \qquad T_\rho = P_\iota^- \cdot y_\rho. \tag{3.6}$$

We also let M_ρ, A_ρ, and N_ρ^- be neighborhoods of identity in M, A and N^- respectively so that $P_\rho^- = M_\rho A_\rho N_\rho^-$.

Saturation. Fix $t > 0$ to be chosen so that e^t is a small positive power of $\|\xi\|$; cf. (3.38). Using our partition of unity, we can write

$$\int_{N_1^+} e^{i\langle \xi, n\rangle} \psi(n) \, d\mu_x^u(n) = \sum_{\rho \in \mathcal{P}_\xi} \int_{N_1^+} e^{i\langle \xi, n\rangle} \psi(n) \rho(g_t n x) \, d\mu_x^u(n). \tag{3.7}$$

Here, we are using the fact that, since $x \in \Omega$, then the restriction of the support of μ_x^u to N_1^+ consists of points $n \in N_1^+$ with $nx \in \Omega$ (or equivalently, that $g_t n x \in \Omega$) and that $\sum_\rho \rho(y) = 1$ for all $y \in \Omega$.

Our first step is to partition the integrals on the right side of (3.7) over N_1^+ into pieces according to the flow box they land in under flowing by g_t. To simplify notation, we write

$$x_t := g_t x. \tag{3.8}$$

We denote by $N_1^+(t)$ a neighborhood of N_1^+ defined by the property that the intersection

$$B_\rho \cap (\mathrm{Ad}(g_t)(N_1^+(t)) \cdot x_t)$$

consists entirely of full local strong unstable leaves in B_ρ. We note that since $\mathrm{Ad}(g_t)$ expands N^+ and B_ρ has radius < 1, $N_1^+(t)$ is contained inside N_2^+. Since ψ is compactly supported inside N_1^+, we have

$$\chi_{N_1^+}(n)\psi(n) = \chi_{N_1^+(t)}(n)\psi(n), \qquad \forall n \in N^+. \tag{3.9}$$

For simplicity, we set

$$\xi_t := e^{-t}\xi, \qquad \psi_t(n) := \psi(\mathrm{Ad}(g_t)^{-1} n), \qquad \mathcal{A}_t := \mathrm{Ad}(g_t)(N_1^+(t)).$$

For $\rho \in \mathcal{P}_\xi$, we let $\mathcal{W}_{\rho,t}$ denote the collection of connected components of the set

$$\{n \in \mathcal{A}_t : nx_t \in B_\rho\}.$$

In view of (3.9), changing variables using (2.3) yields

$$\sum_{\rho \in \mathcal{P}_\xi} \int_{N_1^+} e^{i\langle \xi, n\rangle} \psi(n)\rho(g_t nx)\, d\mu_x^u = e^{-\delta t} \sum_{\rho \in \mathcal{P}_\xi, W \in \mathcal{W}_{\rho,t}} \int_{n \in W} e^{i\langle \xi_t, n\rangle} \psi_t(n)\rho(nx_t)\, d\mu_{x_t}^u. \tag{3.10}$$

Centering the integrals. It will be convenient to center all the integrals in (3.10) so that their basepoints belong to the transversals T_ρ of the respective flow box B_ρ; cf. (3.6).

Let $I_{\rho,t}$ denote an index set for $\mathcal{W}_{\rho,t}$. For $W \in \mathcal{W}_{\rho,t}$ with index $\ell \in I_{\rho,t}$, let $n_{\rho,\ell} \in W$, $m_{\rho,\ell} \in M_\rho$, $n_{\rho,\ell}^- \in N_\rho^-$, and $t_{\rho,\ell}$ with $|t_{\rho,\ell}| \ll \iota$ be such that

$$x_{\rho,\ell} := n_{\rho,\ell} \cdot x_t = n_{\rho,\ell}^- m_{\rho,\ell} g_{t_{\rho,\ell}} \cdot y_\rho \in T_\rho. \tag{3.11}$$

Note that since x and y_ρ both belong to Ω, we have that

$$x_{\rho,\ell} \in \Omega, \qquad n_{\rho,\ell}^- y_\rho \in \Omega. \tag{3.12}$$

For each such ℓ and W, let us denote $W_\ell = W n_{\rho,\ell}^{-1}$ and set

$$\tilde{\chi}_{\rho,\ell}(t,n) := \exp(i\langle \xi_t, n n_{\rho,\ell}\rangle). \tag{3.13}$$

Changing variables using (2.3) and (2.4), we can rewrite the right side of (3.10) as follows:

$$\begin{aligned} e^{-\delta t} &\sum_{\rho \in \mathcal{P}_\xi, W \in \mathcal{W}_{\rho,t}} \int_{n \in W} e^{i\langle \xi_t, n\rangle} \psi_t(n)\rho(nx_t)\, d\mu_{x_t}^u(n) \\ &= e^{-\delta t} \sum_{\rho \in \mathcal{P}_\xi} \sum_{\ell \in I_{\rho,t}} \int_{n \in W_\ell} \tilde{\chi}_{\rho,\ell}(t,n) \psi_t(n n_{\rho,\ell}) \rho(n x_{\rho,\ell})\, d\mu_{x_{\rho,\ell}}^u(n). \end{aligned} \tag{3.14}$$

Mass estimates. We record here certain counting estimates which will allow us to sum error terms in later estimates over $\mathcal{P}_\xi$. Note that by definition of $N_1^+(j)$, we have $\bigcup_{\rho \in \mathcal{P}_\xi, W \in \mathcal{W}_{\rho,t}} W \subseteq \mathcal{A}_t$. Thus, it follows that

$$\sum_{\rho \in \mathcal{P}_\xi, \ell \in I_{\rho,t}} \mu_{x_{\rho,\ell}}^u(W_\ell) \ll \mu_{x_t}^u(\mathcal{A}_t) = e^{\delta t}\mu_x^u(N_1^+(t)) \ll e^{\delta t}\mu_x^u(N_1^+), \tag{3.15}$$

where the last inequality follows since $N_1^+(j) \subseteq N_2^+$ using the doubling property of PS measures [17, Proposition 3.1]. We also used the fact that the partition of unity $\mathcal{P}_\xi$ has uniformly bounded multiplicity.

Weak-stable holonomy. Fix some $\rho \in \mathcal{P}_\xi$. Recall the points $y_\rho \in T_\rho$ and $n_{\rho,\ell}^- \in N_\rho^-$ satisfying (3.11). Let

$$p_{\rho,\ell}^- := n_{\rho,\ell}^- m_{\rho,\ell} g_{t_{\rho,\ell}}. \tag{3.16}$$

The product map $N^- \times A \times M \times N^+ \to G$ is a diffeomorphism on a ball of radius 1 around identity; cf. Section 2.4. Hence, given $\ell \in I_{\rho,t}$, we can define maps ϕ_ℓ and $\tilde{p}_\ell^-$ from W_ℓ to N^+ and P^- respectively by the following formula

$$n p_{\rho,\ell}^- = \tilde{p}_\ell^-(n)\phi_\ell(n). \tag{3.17}$$

We suppress the dependence on ρ and t to ease notation. Then, ϕ_ℓ induces a map between the strong unstable manifolds of $x_{\rho,\ell}$ and y_ρ, also denoted ϕ_ℓ, and defined by

$$\phi_\ell(n x_{\rho,\ell}) = \phi_\ell(n) y_\rho.$$

In particular, this induced map coincides with the local weak stable holonomy map inside B_ρ.

Note that we can find a neighborhood $W_\rho \subset N^+$ of identity of radius $\asymp \iota_\xi$ such that

$$\phi_\ell(W_\ell) \subseteq W_\rho, \tag{3.18}$$

for all $\ell \in I_{\rho,t}$. Moreover, by shrinking the radius ι_ξ of the flow boxes by an absolute amount (depending only on the metric on G) if necessary, we may assume that all the maps ϕ_ℓ are invertible on W_ρ. Hence, we can define the following:

$$\begin{aligned} p_\ell^-(n) &:= \tilde{p}_\ell^-(\phi_\ell^{-1}(n)) \in P^-, \qquad \widetilde{\psi}_{\rho,\ell}(t,n) := J\phi_\ell(n) \times \psi_t(\phi_\ell^{-1}(n) n_{\rho,\ell}), \\ \chi_{\rho,\ell}(t,n) &:= \widetilde{\chi}_{\rho,\ell}(t, \phi_\ell^{-1}(n)), \qquad \rho_\ell(n) := \rho(p_\ell^-(n) n y_\rho), \end{aligned} \tag{3.19}$$

where $J\phi_\ell$ denotes the Jacobian of the change of variable ϕ_ℓ; cf. (2.7).

Changing variables in the right side of (3.14), we obtain

$$\sum_{\ell \in I_{\rho,t}} \int_{n \in W_\ell} \widetilde{\chi}_{\rho,\ell}(t,n) \widetilde{\psi}_{\rho,\ell}(t,n) \rho(n x_{\rho,\ell}) \, d\mu^u_{x_{\rho,\ell}} = \sum_{\ell \in I_{\rho,t}} \int_{W_\rho} \chi_{\rho,\ell}(t,n) \widetilde{\psi}_{\rho,\ell}(t,n) \rho_\ell(n) \, d\mu^u_{y_\rho}. \tag{3.20}$$

Phase formula. The following lemma provides a formula for the stable holonomy maps ϕ_ℓ defined above (3.18) which are responsible for the oscillation of $\chi_{\rho,\ell}$ along N^+. The elementary proof of this lemma is given in Section 4.

Lemma 3.7. *Let $p^-_{\rho,\ell}$ be as in (3.16) and let $w_{\rho,\ell} \in \mathfrak{n}^-$ be such that $n^-_{\rho,\ell} = \exp(w_{\rho,\ell})$. Define vectors $z_{\rho,\ell} \in \mathfrak{n}^-$ by*

$$z_{\rho,\ell} := -e^{t_{\rho,\ell}} m_{\rho,\ell}^{-1} \cdot w_{\rho,\ell}. \tag{3.21}$$

Then, for every $n = \exp(v) \in N^+_{1/2}$, we have

$$\log \phi_\ell^{-1}(n) = e^{t_{\rho,\ell} - \tilde{\tau}_\ell(v)} m_{\rho,\ell} \cdot \left(v + \frac{\|v\|^2}{2} z_{\rho,\ell} \right),$$

where $\log \phi_\ell^{-1}(n)$ is viewed as an element of $\mathfrak{n}^+$ and $\tilde{\tau}_\ell : N^+_{1/2} \to \mathbb{R}_+$ is given by

$$\tilde{\tau}_\ell(v) = \log \left(1 + \langle v, z_{\rho,\ell} \rangle + \frac{\|v\|^2 \|z_{\rho,\ell}\|^2}{4} \right).$$

It will be convenient for our estimates to simplify the expression for $\tilde{\tau}_\ell$ by removing the quadratic term. This is the reason for our choice of flow boxes of width $\asymp \|\xi\|^{-1/3}$ along the strong unstable manifold. In what follows, to simplify notation, we set

$$\tau_\ell(v) = -t_{\rho,\ell} + \log\left(1 + \langle v, z_{\rho,\ell} \rangle\right), \qquad \Gamma_\ell(v) := e^{-\tau_\ell(v)} m_{\rho,\ell} \cdot \left(v + \frac{\|v\|^2}{2} z_{\rho,\ell} \right). \tag{3.22}$$

Recall the points centering points $n_{\rho,\ell}$ defined in (3.11). The following corollary provides a first step towards linearizing the phase in the oscillatory functions $\chi_{\rho,\ell}$ by replacing $\tilde{\tau}_\ell$ in Lemma 3.7 with τ_ℓ in (3.22).

Corollary 3.8. *With the same notation as in Lemma 3.7, we have for all $n = \exp(v) \in W_\rho$ that*

$$\chi_{\rho,\ell}(t,n) = \alpha_{\rho,\ell}(t,n) + O(e^{-t}),$$

where

$$\alpha_{\rho,\ell}(t,n) := \exp(i \langle \xi_t, n_{\rho,\ell} \rangle) \times \exp\left(i \langle \xi_t, \Gamma_\ell(v) \rangle\right). \tag{3.23}$$

Proof. Recall from (3.19) and (3.13) that $\chi_{\rho,\ell}(t,n) = \exp(i\langle \xi_t, \phi_\ell^{-1}(n) n_{\rho,\ell}\rangle)$. We also recall that $\xi_t = e^{-t}\xi$. Then, for all $n = \exp(v) \in W_\rho$, we have that

$$|\chi_{\rho,\ell}(t,n) - \alpha_{\rho,\ell}(t,n)| \ll \|v\|^2 \|\xi_t\| \|\Gamma_\ell(v)\|$$

Since W_ρ has radius $\asymp \iota_\xi \asymp \|\xi\|^{-1/3}$ (cf. (3.3)), we have that both v and $\Gamma_\ell(v)$ have norm $\ll \|\xi\|^{-1/3}$. In particular, the upper bound above is $O(e^{-t})$ as desired. $\square$

Let us summarize our progress so far. To simplify notation, set

$$\psi_{\rho,\ell}(t,n) := \widetilde{\psi}_{\rho,\ell}(t,n) \times \rho_\ell(n). \tag{3.24}$$

Then, in light of (3.7), (3.10), (3.14), (3.20), and Corollary 3.8, we find that

$$\int_{N_1^+} e^{i\langle \xi, n\rangle} \psi(n)\, d\mu_x^u = e^{-\delta t} \sum_{\rho\in\mathcal{P}_\xi} \int_{W_\rho} \sum_{\ell\in I_{\rho,t}} \alpha_{\rho,\ell}(t,n)\psi_{\rho,\ell}(t,n)\, d\mu_{y_\rho}^u + O\left(e^{-t}\right). \tag{3.25}$$

Cauchy-Schwarz. We are left with estimating integrals of the form:

$$\int_{W_\rho} \Psi_\rho(t,n)\, d\mu_{y_\rho}^u, \qquad \Psi_\rho(t,n) := \sum_{\ell\in I_{\rho,t}} \alpha_{\rho,\ell}(t,n)\psi_{\rho,\ell}(t,n). \tag{3.26}$$

By Cauchy-Schwarz, we get

$$\left|\int_{W_\rho} \Psi_\rho(t,n)\, d\mu_{y_\rho}^u\right|^2 \le \mu_{y_\rho}^u(W_\rho) \int_{W_\rho} |\Psi_\rho(t,n)|^2\, d\mu_{y_\rho}^u \tag{3.27}$$

We begin by noting the following apriori bounds on Ψ_ρ:

$$\|\psi_{\rho,\ell}\|_{L^\infty(W_\rho)} \ll 1, \qquad \|\Psi_\rho\|_{L^\infty(W_\rho)} \ll \# I_{\rho,t}. \tag{3.28}$$

Partitioning the support. Using [17, Proposition 9.9], we can find a cover $\{A_j\}$ of W_ρ with balls of radius

$$r = \|\xi\|^{-1/2} \tag{3.29}$$

centered around $u_j \in W_\rho \cap \operatorname{supp}(\mu_{y_\rho}^u)$ and satisfying $\sum_j \mu_{y_\rho}^u(A_j) \ll \mu_{y_\rho}^u(W_\rho)$. By the triangle inequality[4] we have

$$\int_{W_\rho} |\Psi_\rho(t,n)|^2\, d\mu_{y_\rho}^u \le \sum_j \int_{A_j} |\Psi_\rho(t,n)|^2\, d\mu_{y_\rho}^u. \tag{3.30}$$

For $k,\ell \in I_{\rho,t}$, we let

$$\psi_{k,\ell}(t,n) := \psi_{\rho,k}(t,n)\overline{\psi_{\rho,\ell}(t,n)}, \qquad \alpha_{k,\ell}(t,n) := \alpha_{\rho,k}(t,n)\overline{\alpha_{\rho,\ell}(t,n)}.$$

Expanding the square, we get

$$\sum_j \int_{A_j} |\Psi_\rho(t,n)|^2\, d\mu_{y_\rho}^u \le \sum_j \sum_{k,\ell\in I_{\rho,t}} \left|\int_{A_j} \alpha_{k,\ell}(t,n)\psi_{k,\ell}(t,n)\, d\mu_{y_\rho}^u\right|.$$

Using (2.3) and (2.4), we change variables in the integrals using the maps taking each A_j onto N_1^+. More precisely, recall that A_j is a ball of radius r around u_j such that $u_j y_\rho \in \Omega$. Letting

$$\begin{aligned} \tau = -\log r, \qquad y_\rho^j = g_\tau u_j y_\rho, \qquad \alpha_{k,\ell}^j(t,n) = \alpha_{k,\ell}(t, \mathrm{Ad}(g_{-\tau})(n)u_j),\\ \psi_{k,\ell}^j(t,n) = \psi_{k,\ell}(t, \mathrm{Ad}(g_{-\tau})(n)u_j), \end{aligned} \tag{3.31}$$

[4]Cauchy-Schwarz allows us to have a non-negative integrand which in turn enables this step.

we can rewrite the above sum as

$$\sum_j \sum_{k,\ell \in I_{\rho,t}} \left| \int_{A_j} \alpha_{k,\ell}(t,n) \psi_{k,\ell}(t,n) \; d\mu^u_{y_\rho} \right| \leq r^\delta \sum_j \sum_{k,\ell \in I_{\rho,t}} \left| \int_{N_1^+} \alpha^j_{k,\ell}(t,n) \psi^j_{k,\ell}(t,n) d\mu^u_{y^j_\rho} \right|. \tag{3.32}$$

One advantage of flowing forward by g_τ is that it provides smoothing of the amplitude functions $\psi_{k,\ell}$. In particular, it follows by (3.4) that

$$\left\| \psi^j_{k,\ell} \right\|_{C^1} \ll \|\psi\|_{C^1} \times r \times \|\xi\|^{1/3} \ll \|\psi\|_{C^1} \|\xi\|^{-1/6}.$$

Applied to the right side of (3.32), we obtain

$$\int_{W_\rho} |\Psi_\rho(t,n)|^2 \; d\mu^u_{y_\rho} = r^\delta \sum_j \sum_{k,\ell \in I_{\rho,t}} \left| \int_{N_1^+} \alpha^j_{k,\ell}(t,n) d\mu^u_{y^j_\rho} \right| + O(\|\psi\|_{C^1} \|\xi\|^{-1/6} \# I^2_{\rho,t} \mu^u_{y_\rho}(W_\rho)). \tag{3.33}$$

Linearizing the phase. We now turn to estimating the sum of oscillatory integrals in (3.33). Recall that u_j denotes the center of the ball A_j for each j and let $v_j \in \mathfrak{n}^+$ be such that

$$u_j = \exp(v_j).$$

Then, given $n = \exp(v) \in A_j$, and recalling the maps Γ_ℓ in (3.22), we get

$$\Gamma_\ell(v) = \Gamma_\ell(v_j) + D(\Gamma_\ell(v_j))(v - v_j) + O(r^2),$$

where $D(\Gamma_\ell)$ denotes the derivative of Γ_ℓ.

The following elementary lemma uses the explicit expression for Γ_ℓ in (3.22) to simplify the form of $D\Gamma_\ell(v_j)$.

Lemma 3.9. *For all ℓ and j, we have*

$$D\Gamma_\ell(v_j) = e^{-\tau_\ell(v_j)} m_{\rho,\ell} + O(\|\xi\|^{-2/3}).$$

Let

$$\beta^j_{k,\ell} := r\xi_t \cdot \left(e^{-\tau_k(v_j)} m_{\rho,k} - e^{-\tau_\ell(v_j)} m_{\rho,\ell} \right). \tag{3.34}$$

Recall that $\xi_t = e^{-t}\xi$ so that $\|\xi_t\| r^2 = e^{-t}$. Hence, by absorbing the constant terms into the absolute value, we obtain from (3.33) and Lemma 3.9 that

$$\begin{aligned} &\int_{W_\rho} |\Psi_\rho(t,n)|^2 \; d\mu^u_{y_\rho} \\ &= r^\delta \sum_j \sum_{k,\ell \in I_{\rho,t}} \left| \int_{N_1^+} \exp(i\langle \beta^j_{k,\ell}, v\rangle) d\mu^u_{y^j_\rho} \right| + O((e^{-t} + e^{-t}\|\xi\|^{-1/6} + \|\psi\|_{C^1} \|\xi\|^{-1/6}) \# I^2_{\rho,t} \mu^u_{y_\rho}(W_\rho)). \end{aligned} \tag{3.35}$$

Proof of Lemma 3.9. Recall the definition of the vectors $z_{\rho,\ell} \in \mathfrak{n}^-$. To simplify notation, set

$$\lambda_\ell(v_j) = \frac{1}{1 + \langle v_j, z_{\rho,\ell}\rangle}.$$

In particular, $e^{-\tau_\ell(v_j)} = e^{t_{\rho,\ell}} \lambda_\ell(v_j)$. Then, using the formula for Γ_ℓ in (3.22), we obtain

$$D\Gamma_\ell(v_j) = e^{t_{\rho,\ell}} \lambda_\ell(v_j) m_{\rho,\ell} \left[-\lambda_\ell(v_j) \left(v_j \cdot z^t_{\rho,\ell} + \frac{\|v_j\|^2}{2} z_{\rho,\ell} \cdot z^t_{\rho,\ell} \right) + \mathrm{Id} + z_{\rho,\ell} \cdot v^t_j \right].$$

Here, we are viewing v_j and $z_{\rho,\ell}$ as $(d\times 1)$-column vectors and use v_j^t and $z_{\rho,\ell}^t$ to denote the transpose of v_j and $z_{\rho,\ell}$ respectively. Now, observe that

$$\lambda_\ell(v_j)v_j = v_j - \frac{\langle v_j, z_{\rho,\ell}\rangle}{1+\langle v_j, z_{\rho,\ell}\rangle} v_j = v_j + O(\|v_j\|^2).$$

The lemma now follows upon recalling that $\exp(v_j)$ belongs to W_ρ so that $\|v_j\| \ll \|\xi\|^{-1/3}$ in view of our choice of flow boxes; cf. (3.3) and the discussion around it. ☐

Separation of frequencies. To apply the flattening theorem, it will be important to understand the distribution of the frequencies $\beta^j_{k,\ell}$. To this end, we have the following lemma which allows us to avoid studying the separation of the holonomy matrices $m_{\rho,\ell}$.

Lemma 3.10. *For all j,k,ℓ, we have*

$$\left\|\beta^j_{k,\ell}\right\| \gg \|r\xi_t\|\, |e^{-\tau_\ell(v_j)} - e^{-\tau_k(v_j)}|,$$

where $\tau_\ell(v_j)$ and $\tau_k(v_j)$ are defined in (3.22).

Proof. In what follows, to simplify notation, we let

$$m_k := m_{\rho,k}, \qquad c_k := e^{-\tau_k(v_j)}, \qquad Q_k := c_k m_k,$$

with the similar notation for the index ℓ in place of k defined analogously. The lemma is evident when $c_k = c_\ell$. Hence, we may assume without loss of generality that $c_k > c_\ell$, and recall that these functions are non-negative by definition; cf. (3.23).

Recall the elementary estimate $\|g\cdot v\| \geq \|v\| / \left\|g^{-1}\right\|$ for any invertible linear map g and any vector $v \in \mathbb{R}^d$. This estimate implies the following lower bound for $\left\|\beta^j_{k,\ell}\right\|$:

$$\left\|\beta^j_{k,\ell}\right\| \geq \frac{r\,\|\xi_t\|}{\|(Q_k - Q_\ell)^{-1}\|} = \frac{r\,\|\xi_t\|\, c_k}{\left\|(\mathrm{Id} - \frac{c_\ell}{c_k} m_\ell m_k^{-1})^{-1}\right\|}.$$

That $\mathrm{Id} - \frac{c_\ell}{c_k} m_\ell m_k^{-1}$ (and hence $Q_k - Q_\ell$) is invertible follows at once from the following estimate on the norm of its inverse. Recall that the rotation matrices m_k and m_ℓ have spectral radius 1. In particular, since $c_\ell < c_k$, we may use the power series expansion of $\mathrm{Id} - Q$, for matrices Q with spectral radius < 1, get that

$$\left\|(\mathrm{Id} - \frac{c_\ell}{c_k} m_\ell m_k^{-1})^{-1}\right\| \ll \sum_{n\geq 0} \left(\frac{c_\ell}{c_k}\right)^n = \frac{c_k}{c_k - c_\ell}.$$

The lemma follows by combining the above two estimates. ☐

To proceed, we recall that $t_{\rho,\ell}, m_{\rho,\ell}$, and $n^-_{\rho,\ell} = \exp(w_{\rho,\ell})$ parametrize respectively the geodesic flow, M, and strong stable coordinates of the transverse intersections of the expanded horospherical disk $g_t N_1^+ x$ with a fixed transversal T_ρ of the flow box B_ρ. We also recall that $z_{\rho,\ell} = e^{t_{\rho,\ell}} m_{\rho,\ell}^{-1} w_{\rho,\ell}$ and $\tau_\ell(v_j) = e^{t_{\rho,\ell}}/(1+\langle v_j, z_{\rho,\ell}\rangle)$.

Lemma 3.10 motivates the definition of the following subset of $I^2_{\rho,t}$ parametrizing pairs (k,ℓ) for which the frequencies $\beta^j_{k,\ell}$ are too small. Namely, we set

$$\mathrm{Small} := \left\{(k,\ell,j) : \left\|\beta^j_{k,\ell}\right\| < 1\right\}. \tag{3.36}$$

Roughly speaking, elements of $C_{k,\ell}$ correspond to points v_j lying in a small neighborhood of a hyperplane orthogonal to $m_{\rho,\ell}^{-1} w_{\rho,\ell} - m_{\rho,k}^{-1} w_{\rho,k}$. Theorem 3.4 will then provide us with a counting estimate on $C_{k,\ell}$. This is done in the following lemma.

Lemma 3.11. *Let $\alpha > 0$ be the exponent provided by Proposition 2.4. Then, for every fixed $k, \ell \in I_{\rho,t}$, we have*

$$\sum_{j:(k,\ell,j)\in\mathrm{Small}} r^\delta \ll \left(\frac{\|\xi\|^{-1/6} e^t}{\left\| m_{\rho,\ell}^{-1} w_{\rho,\ell} - m_{\rho,k}^{-1} w_{\rho,k} \right\|} \right)^\alpha \mu^u_{y_\rho}(W_\rho).$$

Proof. Let j be such that $(k,\ell,j) \in \mathrm{Small}$ and recall that $\xi_t = e^{-t}\xi$ and $r = \|\xi\|^{-1/2}$. To simplify notation, we also let

$$u_{k,\ell} := m_{\rho,\ell}^{-1} w_{\rho,\ell} - m_{\rho,k}^{-1} w_{\rho,k}.$$

Then, Lemma 3.10 and a direct calculation show that

$$\left| e^{t_{\rho,\ell}} - e^{t_{\rho,k}} + e^{t_{\rho,\ell}+t_{\rho,k}} \langle v_j, u_{k,\ell}\rangle \right| \ll \|\xi\|^{-1/2} e^t.$$

Let $\epsilon_1 = \|\xi\|^{-1/2} e^t / \|u_{k,\ell}\|$. It follows that v_j belongs to a neighborhood of radius $O(\epsilon_1)$ around an affine hyperplane L parallel to the kernel of the linear functional $v \mapsto \langle v, u_{k,\ell}\rangle$.

Recall that A_j denotes the ball of radius r around $\exp(v_j) \in W_\rho$ and that W_ρ has radius $\asymp \|\xi\|^{-1/3}$. It follows we can find a radius $\epsilon_2 \asymp \|\xi\|^{-1/3}$ such that

$$\bigcup_{j:(k,\ell,j)\in\mathrm{Small}} A_j \subseteq L^{(\epsilon_1+r)} \cap N^+_{\epsilon_2},$$

where $L^{(\epsilon_1+r)}$ denotes the (ϵ_1+r)-neighborhood of L. Furthermore, by the bounded multiplicity of the cover $\{A_j\}$ of W_ρ and the fact that each A_j has measure $\asymp r^\delta$ (cf. Proposition 3.3), we get that

$$\sum_{j:(k,\ell,j)\in\mathrm{Small}} r^\delta \ll \mu^u_{y_\rho}\left(\bigcup_{j:(k,\ell,j)\in\mathrm{Small}} A_j \right).$$

Hence, Theorem 3.4 implies that the above sum is $O(\mu^u_{y_\rho}(W_\rho)(\varepsilon_1 + r)^\alpha/\varepsilon_2^\alpha)$, which concludes the proof since $r \ll \epsilon_1$. $\square$

To apply Lemma 3.11, we need the following counting estimate on close by vectors of the form $m_{\rho,\ell}^{-1} w_{\rho,\ell}$. It is a consequence of Theorem 3.4.

Lemma 3.12. *For every $k \in I_{\rho,t}$ and $\eta > 0$, we have*

$$\#\left\{ \ell \in I_{\rho,t} : \left\| m_{\rho,\ell}^{-1} w_{\rho,\ell} - m_{\rho,k}^{-1} w_{\rho,k} \right\| < \|\xi\|^{-\eta} \right\} \ll \left(e^{-t} + \|\xi\|^{-\eta} \right)^\alpha e^{\delta t}, \tag{3.37}$$

where $\alpha > 0$ is the exponent provided by Theorem 3.4.

Proof. Let B_k denote the set on the left side of (3.37) and let ℓ be some element of B_k. Then, by M-invariance of the norm, we have[5]

$$\left| \|w_{\rho,\ell}\| - \|w_{\rho,k}\| \right| \ll \|\xi\|^{-\eta}.$$

In particular, the vectors $w_{\rho,\ell}$ with $\ell \in B_k$ all belong to a neighborhood of width $\ll \|\xi\|^{-\eta}$ of the sphere S of radius $\|w_{\rho,k}\|_\infty$ around the origin in the norm metric.

The next ingredient is to note that the points $w_{\rho,\ell}$ are separated by an amount $\gg e^{-t}$. This follows by a similar argument to the proof of [17, Proposition 9.13][6]. In particular, there is $\epsilon_1 \asymp (\|\xi\|^{-\eta} + e^{-t})$

[5]This estimate is again done to bypass studying the separation of the rotation matrices $w_{\rho,\ell}$.

[6]This proof is based on injectivity radius considerations along with the fact that g_t expands the stable manifold by e^t in backward time.

and $\epsilon_2 \asymp e^{-t}$ such that

$$\bigsqcup_{\ell \in B_k} N^-_{\epsilon_2} \cdot \exp(w_{\rho,\ell}) y_\rho \subseteq N^-_{\epsilon_1} \cdot S,$$

where $N^-_{\epsilon_1} \cdot S$ is the ϵ_1-neighborhood of S.

To conclude the proof, let $\mu^s_{y_\rho}$ denote the shadow of the PS measure on $N^- \cdot y_\rho$ defined analogously to the measures $\mu^u_{y_\rho}$ in (2.2). The above discussion implies that

$$\# B_k \ll \frac{\mu^s_{y_\rho}\left(N^-_{\epsilon_1} \cdot S\right)}{\min \mu^s_{y_\rho}(N^-_{\epsilon_2} \cdot \exp(w_{\rho,\ell}) y_\rho)}.$$

By (3.12), the points $\exp(w_{\rho,\ell}) \cdot y_\rho$ all belong to Ω. In particular, by Proposition 3.3, we have

$$\mu^s_{y_\rho}\left(N^-_{e^{-t}} \cdot \exp(w_{\rho,\ell}) y_\rho\right) \asymp e^{-\delta t}.$$

On the other hand, by [11, Lemma 3.8], we have that $N^-_{\epsilon_1} \cdot S$ has measure $O(\epsilon_1^\alpha)$[7]. The lemma now follows. □

Reduction to L^2-flattening. To simplify our error terms, we make the following choices:

$$\eta = 1/12, \qquad e^t = \|\xi\|^{1/24}. \tag{3.38}$$

In view of Lemmas 3.11 and 3.12, we introduce the following notation:

$$\mathrm{Close}_\eta := \left\{(k,\ell) \in I^2_{\rho,t} : \left\| m^{-1}_{\rho,\ell} w_{\rho,\ell} - m^{-1}_{\rho,k} w_{\rho,k} \right\| < \|\xi\|^{-\eta} \right\}. \tag{3.39}$$

We also define the following set of indices parametrizing measures $\mu^u_{y^j_\rho}$ for which many of the frequencies $\beta^j_{k,\ell}$ are close together. Let $\mathcal{J}$ denote the index set for the indices j of the measures $\mu^u_{y^j_\rho}$ and set

$$\mathrm{Bad}_\eta := \left\{ j \in \mathcal{J} : \# \left\{(k,\ell) \in I^2_{\rho,t} : (k,\ell,j) \in \mathrm{Small}\right\} > \|\xi\|^{-\alpha/48} \times (e^{\delta t} + \# I_{\rho,t}) \# I_{\rho,t} \right\}. \tag{3.40}$$

The following corollary allows us to estimate estimate the part of the sum corresponding to Bad_η.

Corollary 3.13. *We have the following counting estimate on* Bad_η*:*

$$\sum_{j \in \mathrm{Bad}_\eta} r^\delta \ll \|\xi\|^{-\alpha/48} \mu^u_{y_\rho}(W_\rho).$$

Proof. The corollary will follow from an application of Markov's inequality to the estimates in Lemmas 3.11 and 3.12 as follows. First, we note that

$$\begin{aligned} &\sum_{j \in \mathcal{J}} \sum_{(k,\ell) \in I^2_{\rho,t}} r^\delta \mathbb{1}_{\mathrm{Small}}(k,\ell,j) \\ &= \underbrace{\sum_{(k,\ell) \in \mathrm{Close}_\eta} \sum_{j \in \mathcal{J}} r^\delta \mathbb{1}_{\mathrm{Small}}(k,\ell,j)}_{\text{(I)}} + \underbrace{\sum_{(k,\ell) \notin \mathrm{Close}_\eta} \sum_{j \in \mathcal{J}} r^\delta \mathbb{1}_{\mathrm{Small}}(k,\ell,j)}_{\text{(II)}}. \end{aligned}$$

Then, by Lemma 3.12 and our choices in (3.38), the first sum is estimated as follows:

$$\text{(I)} \ll \left(e^{-t} + \|\xi\|^{-\eta}\right)^\alpha \times e^{\delta t} \# I_{\rho,t} \mu^u_{y_\rho}(W_\rho) \ll \|\xi\|^{-\alpha/24} \times e^{\delta t} \# I_{\rho,t} \mu^u_{y_\rho}(W_\rho).$$

For the second sum, we use Lemma 3.11 and the definition of Close_η to get

$$\text{(II)} \ll \|\xi\|^{(\eta - 1/6)\alpha} e^{\alpha t} \times \# I^2_{\rho,t} \mu^u_{y_\rho}(W_\rho) \ll \|\xi\|^{-\alpha/24} \times \# I^2_{\rho,t} \mu^u_{y_\rho}(W_\rho).$$

[7]Note that, similarly to the case of affine subspaces in Theorem 3.4, this estimate can be deduced from the fact that PS measures give 0 mass to proper subvarieties of the boundary using the argument in [18, Section 8].

Hence, the corollary follows by Markov's inequality. □

To simplify notation, we set

$$E_1 := \max \left\{ \|\xi\|^{-1/24}, \|\xi\|^{-\alpha/48} \right\}. \tag{3.41}$$

For $w \in \mathbb{R}^d$, we let

$$\nu_j := \mu^u_{y^j_\rho} \Big|_{N_1^+}, \qquad \hat{\nu}_j(w) := \int_{N^+} e^{i\langle w,n\rangle} \, d\nu_j(n). \tag{3.42}$$

Then, by (3.33) and Corollary 3.13, we obtain

$$\int_{W_\rho} |\Psi_\rho(t,n)|^2 \, d\mu^u_{y_\rho} = r^\delta \sum_{j\notin \mathrm{Bad}_\eta} \sum_{k,\ell\in I_{\rho,t}} |\hat{\nu}_j(\beta^j_{k,\ell})| + O\left(((\|\psi\|_{C^1}+1) \times E_1 \times \# I^2_{\rho,t} \mu^u_{y_\rho}(W_\rho) \right). \tag{3.43}$$

For each j, the sum on the right side of the above estimate can be viewed as an average, when properly normalized, over Fourier coefficients of the measure ν_j. Moreover, Corollary 3.13 guarantees that the frequencies $\beta^j_{k,\ell}$ are sampled from a well-separated set. Hence, this average can be estimated using the L^2-Flattening Theorem, Theorem 2.3.

The role of L^2-flattening. Let $\eta_2 > 0$ be a small parameter to be chosen using Lemma 3.14 below. Note that the total mass of ν_j, denoted $|\nu_j|$, is $\mu^u_{y^j_\rho}(N_1^+)$. For each $k \in I_{\rho,t}$, define the following set, which roughly speaking, consists of frequencies where $\hat{\nu}_j$ is large:

$$B(j,k,\eta_2) := \left\{ \ell \in I_{\rho,t} : |\hat{\nu}_j(\beta^j_{k,\ell})| > \|\xi\|^{-\eta_2} |\nu_j| \right\}. \tag{3.44}$$

Then, splitting the sum over frequencies according to the size of the Fourier transform $\hat{\nu}_j$ and reversing our change variables to go back to integrating over A_j, we obtain

$$\begin{aligned} r^\delta \sum_{j\notin \mathrm{Bad}_\eta} \sum_{k,\ell\in I_{\rho,t}} |\hat{\nu}_j(\beta^j_{k,\ell})| & \\ \ll \left(\max_{j\notin \mathrm{Bad}_\eta, k\in I_{\rho,t}} \# B(j,k,\eta_2) + \|\xi\|^{-\eta_2} \# I_{\rho,t} \right) \# I_{\rho,t} \mu^u_{y_\rho}(W_\rho), & \end{aligned} \tag{3.45}$$

The following key counting estimate for $B(j,k,\eta_2)$ is a consequence of the L^2-flattening theorem, Theorem 2.3.

Lemma 3.14. *For every $\varepsilon > 0$, there is $\eta_2 > 0$ such that for all $j \notin \mathrm{Bad}_\eta$ and $k \in I_{\rho,t}$, we have*

$$\# B(j,k,\eta_2) \ll_\varepsilon \|\xi\|^{\varepsilon-\alpha/96} \times \sqrt{(e^{\delta t} + \# I_{\rho,t}) \# I_{\rho,t}},$$

where $\alpha > 0$ is the exponent provided by Theorem 3.4. Here, η_2 is the constant provided by Theorem 2.3 (denoted by δ in the notation of the theorem).

Proof. Recall the definition of the frequencies $\beta^j_{k,\ell}$ in (3.34). The rough idea behind the proof is that. since $j \notin \mathrm{Bad}_\eta$, the frequencies $\beta^j_{k,\ell}$ are well-separated. This allows us to apply Theorem 2.3 on the Lebesgue measure of the set of frequencies where the Fourier transform is large to conclude that the sets $B(j,k,\eta_2)$ are relatively small in size.

More precisely, Proposition 2.4 and Theorem 2.3 imply that there exists $\eta_2 > 0$, depending on ε (but not on the index j), such that the set

$$Q := \left\{ \beta^j_{k,\ell} : \ell \in B(j,k,\eta_2) \right\}$$

can be covered by $O_\varepsilon(\|\xi\|^\varepsilon)$ balls B_i of radius $1/2$.

Let $\tilde{B}_i$ denote the set of indices $\ell \in B(j,k,\eta_2)$ such that $\beta^j_{k,\ell} \in B_i$. In particular, we have

$$\#B(j,k,\eta_2) \leq \sum_i \#\tilde{B}_i. \tag{3.46}$$

Moreover, we note that for $\ell_1, \ell_2 \in \tilde{B}_i$, we have that $\beta^j_{k,\ell_1} - \beta^j_{k,\ell_2} = \beta^j_{\ell_2,\ell_1}$. Since B_i has radius $1/2$, we get that $\left\|\beta^j_{\ell_2,\ell_1}\right\| < 1$. Hence, recalling the definition of the sets Small in (3.36), and letting Small_j denote the set of pairs $(p,q) \in I^2_{\rho,t}$ with $(p,q,j) \in \mathrm{Small}$, we obtain

$$\#\tilde{B}_i^2 \leq \#\mathrm{Small}_j.$$

On the other hand, since $j \notin \mathrm{Bad}_\eta$, then by definition, we have that

$$\#\mathrm{Small}_j \leq \|\xi\|^{-\alpha/48} \times (e^{\delta t} + \#I_{\rho,t})\#I_{\rho,t}.$$

Since the sum in (3.46) has at most $O_\varepsilon(\|\xi\|^\varepsilon)$ terms, this estimate completes the proof. $\square$

Combining estimates and concluding the proof. Recall that α is the exponent provided by Theorem 3.4. Let $\eta_2 > 0$ be the exponent provided by Lemma 3.14 when applied with $\varepsilon = \alpha/200$ and let κ be defined as follows:

$$\kappa = \min\left\{1/24, \alpha/200, \eta_2\right\}. \tag{3.47}$$

Then, by combining (3.25), (3.27), (3.43), (3.45), and Lemma 3.14, we obtain the following bound:

$$\begin{aligned}\int_{N_1^+} e^{i\langle\xi,n\rangle}\psi(n)\,d\mu^u_x(n) &\ll_\Gamma (\|\psi\|_{C^1}+1)\times\|\xi\|^{-\kappa} \\ &\quad\times\left(e^{-\delta t}\sum_\rho \mu^u_{y_\rho}(W_\rho)\#I_{\rho,t} + e^{-3\delta t/4}\sum_\rho \mu^u_{y_\rho}(W_\rho)(\#I_{\rho,t})^{3/4}\right).\end{aligned}$$

The first sum on the right side is $O_\Gamma(1)$ in light of (3.15) and the fact that $\mu^u_{y_\rho}(W_\rho) \asymp mu^u_{x_{\rho,\ell}}(W_\ell)$ for all $\ell \in I_{\rho,t}$. That the second sum is also $O_\Gamma(1)$ is proved in the following lemma. This concludes the proof of Theorem 1.1 apart from Lemma 3.7 which is proved in the next section.

Lemma 3.15. *For every $p \in (1,\infty)$, we have that*

$$e^{-\delta t/p}\sum_{\rho\in\mathcal{P}_\xi}\mu^u_{y_\rho}(W_\rho)\,(\#I_{\rho,t})^{1/p} \ll_\Gamma 1.$$

Proof. Indeed, letting q be such that $1/p + 1/q = 1$, we obtain by Hölder's inequality that

$$e^{-\delta t/p}\sum_{\rho\in\mathcal{P}_\xi}\mu^u_{y_\rho}(W_\rho)\,(\#I_{\rho,t})^{1/p} \leq e^{-\delta t/p}\left(\sum_{\rho\in\mathcal{P}_\xi}\mu^u_{y_\rho}(W_\rho)\right)^{1/q}\times\left(\sum_{\rho\in\mathcal{P}_\xi}\mu^u_{y_\rho}(W_\rho)\#I_{\rho,t}\right)^{1/p}.$$

Since $\mu^u_{y_\rho}(W_\rho) \asymp mu^u_{x_{\rho,\ell}}(W_\ell)$ for all $\ell \in I_{\rho,t}$, it follows by (3.15) that

$$e^{-\delta t}\sum_{\rho\in\mathcal{P}_\xi}\mu^u_{y_\rho}(W_\rho)\#I_{\rho,t} \ll_\Gamma 1.$$

Moreover, Proposition 3.3 implies that $\mu^u_{y_\rho}(W_\rho) \asymp \|\xi\|^{-\delta/3}$. Hence, the lemma follows in light of (3.5). $\square$

4. Explicit formula for stable holonomy maps and Proof of Lemma 3.7

In this section, we give explicit formulas for the commutation relations between stable and unstable subgroups which we need for the proof of Lemma 3.7.

Consider the following quadratic form on $\mathbb{R}^{d+2}$: for $x = (x_i) \in \mathbb{R}^{d+2}$,

$$Q(x) = 2x_0 x_{d+1} - |x_1|^2 - \cdots - |x_d|^2.$$

Let $\mathrm{SO}_{\mathbb{R}}(Q) \cong \mathrm{SO}(d+1,1)$ be the orthogonal group of Q; i.e. the subgroup of $\mathrm{SL}_{d+2}(\mathbb{R})$ preserving Q. Then, we have a surjective homomorphism $\mathrm{SO}_{\mathbb{R}}(Q) \to G = \mathrm{Isom}^+(\mathbb{H}^{d+1})$ with finite kernel. The geodesic flow is induced by the diagonal group

$$A = \left\{ g_t = \mathrm{diag}(e^t, \mathrm{I}_d, e^{-t}) : t \in \mathbb{R} \right\},$$

where I_d denotes the identity matrix in dimension d.

For $x \in \mathbb{R}^d$, viewed as a row vector, we write x^t for its transpose. We let $\|x\|^2 := x \cdot x$, and $x \cdot x$ denotes the sum of coordinate-wise products in the standard basis on $\mathbb{R}^d$. Hence, N^+ can be parametrized as follows:

$$N^+ = \left\{ n^+(x) := \begin{pmatrix} 1 & x & \frac{\|x\|^2}{2} \\ \mathbf{0} & \mathrm{I}_d & x^t \\ 0 & \mathbf{0} & 1 \end{pmatrix} : x \in \mathbb{R}^d \right\}. \tag{4.1}$$

The group N^- is parametrized by the transpose of the elements of N^+. Recall that $M = \mathrm{SO}_d(\mathbb{R})$ denotes the centralizer of A inside the standard maximal compact subgroup $K \cong \mathrm{SO}_{d+1}(\mathbb{R})$ of G. In particular, M is given by

$$M = \left\{ m(O) := \mathrm{diag}(1, O, 1) : O \in \mathrm{SO}_d(\mathbb{R}) \right\}.$$

Finally, we recall that the product map $N^- \times A \times M \times N^+ \to G$ is a diffeomorphism near identity.

We are now ready for the proof. Recall from (3.17) that $\phi_\ell(n)$ is defined to be the element of N^+ satisfying $np^-_{\rho,\ell} \in N^- AM\phi_\ell(n)$. In particular, $\phi_\ell^{-1}(n)$ is the unique element of N^+ satisfying

$$n(p^-_{\rho,\ell})^{-1} \in N^- AM \phi_\ell^{-1}(n).$$

Hence, in view of the explicit parametrization above, in order to compute $\phi_\ell^{-1}(n)$, it suffices to compute the top row of the matrix $n(p^-_{\rho,\ell})^{-1}$ and to note that

$$g_s n^+(x) = \begin{pmatrix} e^s & e^s x & \frac{e^s\|x\|^2}{2} \\ \mathbf{0} & \mathrm{I}_d & x^t \\ 0 & \mathbf{0} & e^{-s} \end{pmatrix},$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$. This allows us to extract the N^+ component from the top row of $n(p^-_{\rho,\ell})^{-1}$ by scaling it suitably so that the top left entry is 1. In particular, the claimed formula follows by a direct calculation upon recalling from (3.16) that $p^-_{\rho,\ell} = \exp(w_{\rho,\ell}) m_{\rho,\ell} g_{t_{\rho,\ell}}$.

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