Exceptional Trajectories for the Teichmüller Geodesic Flow and Hausdorff Dimension

Osama Khalil

Ohio State University

JMM San Diego

Joint with: H. Al-Saqban, P. Apisa, A. Erchenko, S. Mirzadeh, C. Uyanik

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What we will talk about

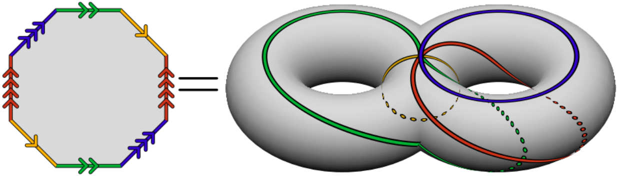
- **Space**: $X = \mathcal{H}_1(\alpha)$ a stratum of area 1 abelian differentials on a surface S.
- **Transformations**: $G = SL(2, \mathbb{R})$ acts on X,

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

- **Central question**: Given $\omega \in X$, study the set of directions θ for which the trajectory $g_t r_\theta \omega$ exhibits a deviation from the correct limit in Birkhoff's (and Oseledets') theorems.

What is an abelian differential?

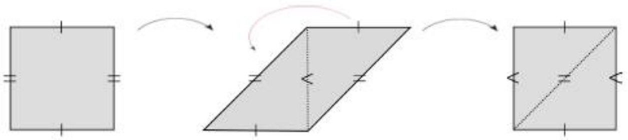
- S closed surface of genus $g \in \mathbb{N}$.
- An abelian differential: a collection of polygons in $\mathbb{C}$, identify parallel sides by **translations**



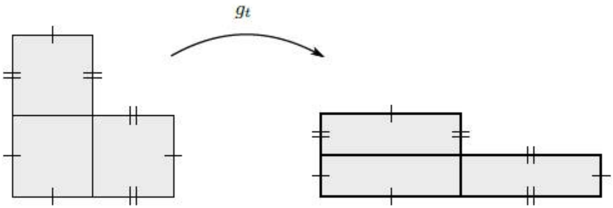
- $dz = d(z + c)$: pulling dz back gives an abelian differential with zeros at vertices
- Orders of zeros add up to $2g - 2$
- A partition α of $2g - 2$ into positive integers defines a **stratum** $\mathcal{TH}(\alpha)$ of abelian differentials

The $Mod(S)$ and $SL(2,\mathbb{R})$ actions

- $Mod(S) = Diff^{+}(S)/Diff_0(S)$: abelian differentials are identified by cutting and pasting. $\mathcal{H}(\alpha) = \mathcal{TH}(\alpha)/Mod(S)$.



- g_t action generates all geodesics in the Teichmüller metric.



Period coordinates, affine measures

- *Affine* local coordinates on the stratum $\mathcal{TH}(\alpha)$:

$$\omega \mapsto \left(\int_{\beta_1} \omega, \ldots, \int_{\beta_{2g}} \omega, \int_{\alpha_1} \omega, \ldots, \int_{\alpha_{n-1}} \omega \right) \in \mathbb{C}^{2g+n-1}$$

- Affine submanifolds of $\mathcal{H}(\alpha)$: lifts to $\mathcal{TH}(\alpha)$ are cut out by linear equations in these local coordinates.
- Affine measures are supported on and absolutely cts w.r.t. Lebesgue on some affine submanifold.

The work of Eskin, Mirzakhani and Mohammadi

- **Eskin-Mirzakhani**: the **only** $SL(2,\mathbb{R})$-invariant ergodic measures are affine.
- **Eskin-Mirzakhani-Mohammadi**: The closure $\mathcal{M}$ of the $SL(2,\mathbb{R})$ orbit of **every** $\omega \in \mathcal{H}_1(\alpha)$ is an affine submanifold supporting an affine invariant measure $\nu_{\mathcal{M}}$ and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{2\pi}\int_0^{2\pi} f(g_t r_\theta \omega)\, d\theta dt = \int f\, d\nu_{\mathcal{M}}$$

for all $f \in C_c(\mathcal{H}_1(\alpha))$.

Birkhoff holds in almost every direction

- any affine measure $\nu_{\mathcal{M}}$ is g_t-ergodic $\xRightarrow{\text{Birkhoff}}$ for $\nu_{\mathcal{M}}$ almost every ω:

$$\frac{1}{T}\int_0^T f(g_t\omega)\, dt \xrightarrow{T\to\infty} \int f \, d\nu_{\mathcal{M}}$$

- **Chaika-Eskin**: Fix **every** $\omega \in \mathcal{H}_1(\alpha)$ and for **almost every** θ,

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Hausdorff dimension of exceptional trajectories

- Fix any $\omega \in \mathcal{H}_1(\alpha)$, $\mathcal{M} = \overline{SL(2,\mathbb{R})\omega}$, $\nu_{\mathcal{M}}$ affine measure on $\mathcal{M}$.

Theorem (AAEKMU '17)

For every bounded Lipschitz function f and every $\varepsilon > 0$, the Hausdorff dimension of the set

$$\left\{ \theta \in [0, 2\pi] : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_t r_\theta \omega)\, dt \geqslant \int f \, d\nu_{\mathcal{M}} + \varepsilon \right\}$$

is strictly less than 1.

Strategy of the proof

- $B(f,T,\varepsilon) = \left\{\theta \in [0,2\pi] : \frac{1}{T}\int_0^T f(g_t r_\theta \omega)\, dt \geqslant \int f\, d\nu_{\mathcal{M}} + \varepsilon\right\}$
- **Main Estimate:** For every $\varepsilon > 0$, there is a $\delta > 0$:

$$|B(f,T,\varepsilon)| \ll e^{-\delta T}$$

- Deviation of a Lipschitz function is a **locally constant** property:

$$\theta \in B(f,T,\varepsilon) \Rightarrow \left[\theta - e^{-2T}, \theta + e^{-2T}\right] \subseteq B(f,T,C\varepsilon)$$

- Finishing the proof: measure estimate + locally constant $\Rightarrow$ $B(f,T,\varepsilon)$ can be covered by $\ll e^{(2-\delta)T}$ intervals of radius e^{-2T}

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Towards the main estimate

- For $M \in \mathbb{N}$, $N > 0$:

$$\frac{1}{MN}\int_0^{MN} f(g_t r_\theta \omega)\, dt = \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{\frac{1}{N}\int_{iN}^{(i+1)N} f(g_t r_\theta \omega)\, dt}_{f_i(\theta)}$$

- $\theta \in B(f, MN, \varepsilon) \Rightarrow f_i(\theta) \geqslant \nu_{\mathcal{M}}(f) + C\varepsilon$ for a positive proportion of $\{0, \dots, M-1\}$:

$$B(f, MN, \varepsilon) \subseteq \bigcup_{|I| \gg M} \bigcap_{i \in I} \underbrace{\{\theta : f_i(\theta) \geqslant \nu_{\mathcal{M}}(f) + C\varepsilon\}}_{Bad_i}$$

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A uniform pointwise theorem

- **Chaika-Eskin**: For every $x \in \mathcal{H}_1(\alpha)$ and almost every θ,

$$\frac{1}{T}\int_0^T f(g_t r_\theta x)\, dt \xrightarrow{T\to\infty} \int f\, d\nu_{\mathcal{N}}$$

where $\mathcal{N} = \overline{SL(2,\mathbb{R})x}$, $\nu_{\mathcal{N}}$ affine measure on $\mathcal{N}$.

- **Key Step 1:** Chaika-Eskin's result holds uniformly over compact sets:

Theorem (AAEKMU '17)

There exist finitely many affine invariant submanifolds $\mathcal{N}_i$ such that

$$\left|\left\{\theta : \left|\frac{1}{N}\int_0^N f(g_t r_\theta x)\, dt - \int f\, d\nu_{\mathcal{M}}\right| > \varepsilon\right\}\right| \xrightarrow{N\to\infty} 0$$

uniformly *as x varies over any fixed compact subset in the complement of $\cup_i \mathcal{N}_i$*

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A uniform pointwise theorem $\Rightarrow$ measure bound for Bad_i

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$$Bad_i = \{\theta : f_i(\theta) \geqslant \nu_{\mathcal{M}}(f) + C\varepsilon\}$$

- J an interval of size e^{-2iN}:

$$|J|^{-1}\int_J f_i(\theta)\, d\theta \asymp \int_0^1 \frac{1}{N}\int_0^N f(g_t r_\theta \underbrace{(g_{iN} r_{\theta_0}\omega)}_{x})\, dt\, d\theta$$

- If $g_{iN} r_{\theta_0}\omega$ lands in a good compact set for $N \Rightarrow$ get a bound on measure of Bad_i by uniform pointwise theorem.

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Applying the uniform pointwise theorem

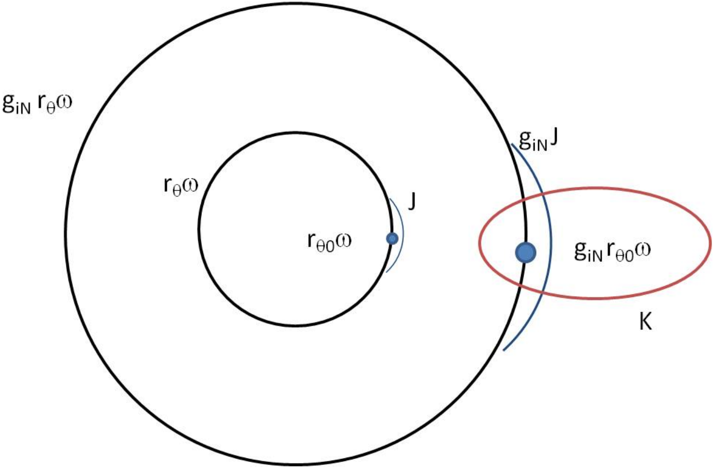

Figure: K is a good compact set

What if we don't land in the good compact set?

- **Key Step 2:** control the dimension of the directions in which geodesics frequently miss a good compact set
- For $K \subset \mathcal{H}_1(\alpha)$, $\rho, N > 0$:

$$Z_\omega(K, N, \rho) = \left\{ \theta : \limsup_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} \chi_K(g_{iN} r_\theta omega) \leqslant 1 - \rho \right\}$$

Theorem (AAEKMU '17)

Given a finite collection $\mathcal{C}$ of affine invariant submanifolds and $\rho > 0$, there exists a compact set K in the complement of $\cup\mathcal{C}$ such that the Hausdorff dimension of the set $Z_omega(K, N, \rho)$ is strictly less than 1.

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Application: deviation of exponents of the KZ cocycle

- The main result can help control the dimension of the set of directions which frequently miss an **open** set where desirable properties hold.

Theorem (AAEKMU '17)

Let A be the Kontsevich-Zorich cocycle over $\mathcal{M}$. Denote by λ_i the Lyapunov exponents of A (with multiplicities) with respect to $\nu_{\mathcal{M}}$. For any $\theta \in [0, 2\pi]$, suppose $\psi_1(t,\theta) \leq \cdots \leq \psi_{2g}(t,\theta)$ are the eigenvalues of the matrix $A^*(g_t, r_\theta\omega)A(g_t, r_\theta\omega)$. Then, the Hausdorff dimension of the set

$$\left\{ \theta \in [0, 2\pi] : \limsup_{t\to\infty} \frac{\log \|\psi_i(t,\theta)\|}{t} \geq 2\lambda_i + \varepsilon \right\}$$

is strictly less than 1.

Thanks!