On the Mozes-Shah phenomenon for horocycle flows on moduli spaces

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The Mozes-Shah phenomenon

Given a Lie group G, a discrete subgroup Γ, and a 1-parameter unipotent flow $u_t \subset G$:

- E.g: image of $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ under embedding $\mathrm{SL}_2(\mathbb{R}) \hookrightarrow G$.

Theorem (Mozes-Shah '95)

Let μ_n be a sequence of u_t-ergodic probability measures converging to a probability measure μ on G/Γ. Then,

1. *μ is ergodic for a subgroup $L \subseteq G$ generated by unipotents.*
2. *The supports of μ_n converge to the support of μ.*

This phenomenon is very rare

- Contrast this with say the geodesic flow on a compact hyperbolic surface: ergodic measures are dense in space of invariant measures.
- Davis-Lelievre, McMullen: this phenomenon fails even for billiard flows on regular polygons (apart from few exceptions like tori).
- It has deep applications beyond dynamics: e.g. counting integral points on varieties, much harder than counting lattice points.

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What happens on moduli spaces?

- $S-$closed genus g surface.
- Holomorphic differential: a collection of polygons in $\mathbb{C}$, identifying parallel sides by **translations** gives S.

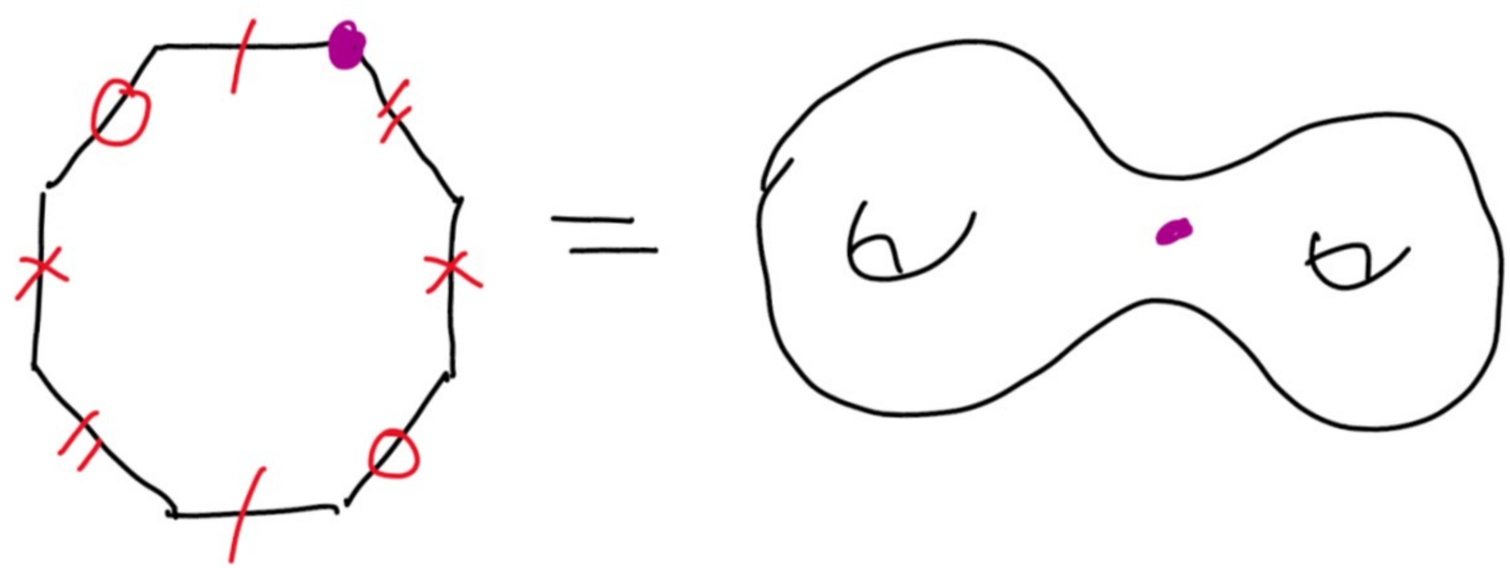

Figure: Regular octagon

- $\Omega\mathcal{M}_g-$space of all holomorphic differentials (on S).

The $\mathrm{SL}_2(\mathbb{R})$ action

- $dz = d(z+c)$: dz gives a 1-form, may vanish at vertices.
- Orders of zeros add up to $2g-2$.
- An integral partition α of $2g-2$ defines a **stratum** $\mathcal{H}(\alpha) \subset \Omega\mathcal{M}_g$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on polygons via action on $\mathbb{R}^2$, preserving the above data: $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathcal{H}(\alpha)$.

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \qquad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

What happens on moduli spaces?

Eskin-Mirzakhani-Mohammadi: the space of ergodic measures for the full upper triangular subgroup of $\mathrm{SL}_2(\mathbb{R})$ is closed and

$$\frac{1}{T}\int_0^T\int_0^1 f(g_t u_s x)\, dsdt \xrightarrow{T\to\infty} \int_{\overline{\mathrm{SL}_2(\mathbb{R})\cdot x}} f, \qquad \forall x \in \mathcal{H}(\alpha).$$

Question

What about the horocycle flow? Can we drop extra averaging in t?

Why limits of horocycle invariant measures?

Eskin-Masur showed that if

- the g_t pushforward of the horocycle arc through a flat surface (X, ω) equidistributes to some $\mathrm{SL}_2(\mathbb{R})$ invariant measure

then, the limit

$$\lim_{T \to \infty} \frac{\# \{\text{closed geodesics on } (X, \omega) \text{ of length } \leq T\}}{T^2}$$

exists.

What happens on moduli spaces?

Theorem (Chaika-Khalil-Smillie)

There exists a sequence of periodic horocycle measures on $\mathcal{H}(2)$ whose weak-$*$ limit is a non-trivial convex combination of

1. *The $\mathrm{SL}_2(\mathbb{R})$-ergodic measure on a closed $\mathrm{SL}_2(\mathbb{R})$-orbit, and*
2. *The Masur-Veech measure on $\mathcal{H}(2)$.*

- This limit measure cannot be ergodic for any group of bilipschitz homeomorphisms.

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What happens on moduli spaces?

By a very general argument:

Corollary (Chaika-Khalil-Smillie)

There is a dense G_δ set of points in $\mathcal{H}(2)$ whose horocycle orbits are ***not*** *generic to any measure, i.e. ergodic averages do not converge.*

- Sharp contrast with Ratner's theorem on homogeneous spaces.
- New, more flexible, mechanism for failure of genericity compared with work of Chaika-Smillie-Weiss.

Proof Ideas

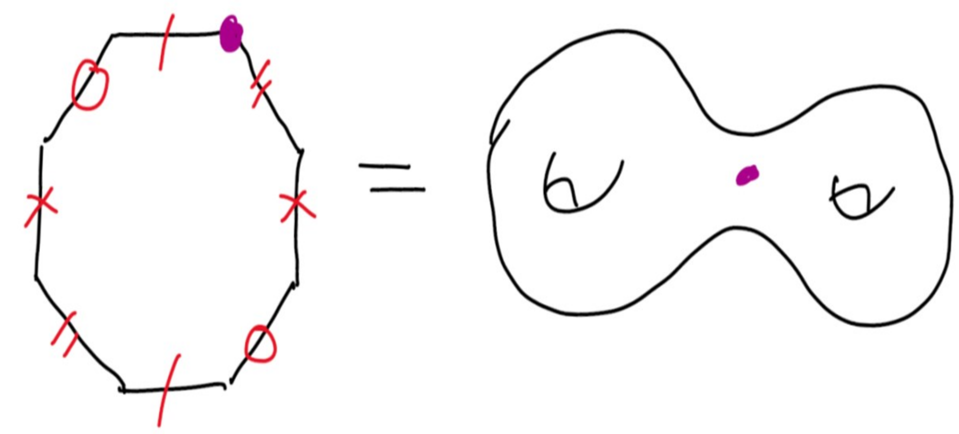
The Kontsevich-Zorich cocycle

- The tangent space to $\mathcal{H}(\alpha)$ carries an invariant **symplectic** form.
- The Kontsevich-Zorich cocycle is essentially the derivative of the $\mathrm{SL}_2(\mathbb{R})$ action:

$$\mathrm{KZ} : \mathrm{SL}_2(\mathbb{R}) \times \mathcal{H}(\alpha) \to \mathbf{Sp}(\mathrm{T}\mathcal{H}(\alpha))$$
$$\mathrm{KZ}(gh, x) = \mathrm{KZ}(g, hx) \cdot \mathrm{KZ}(h, x).$$

The construction: horocycles near the Octagon locus

(X_0, ω_0)—the regular octagon in $\mathcal{H}(2)$



- **Veech**: $\mathcal{O} := \mathrm{SL}_2(\mathbb{R}) \cdot (X_0, \omega_0)$ is closed in $\mathcal{H}(2)$ and

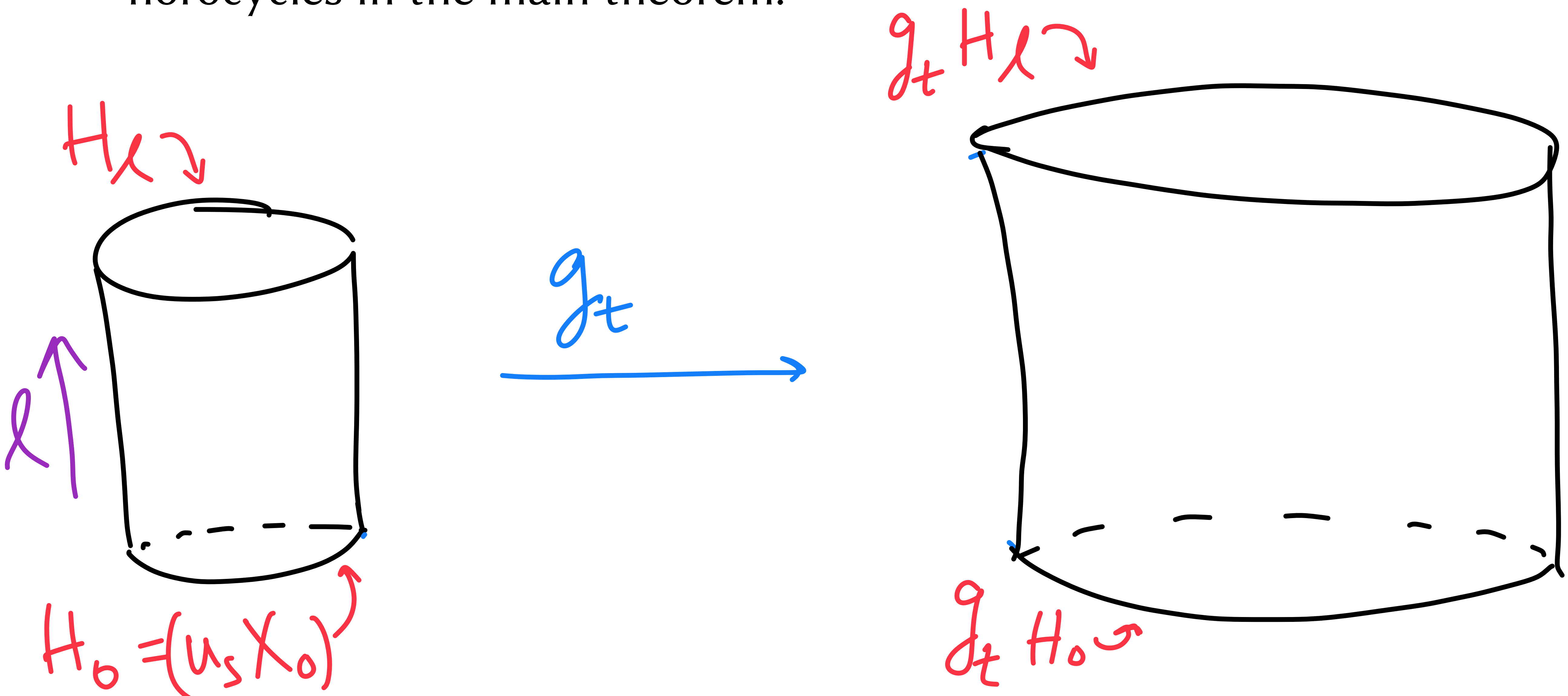
$$\mathcal{O} \cong \mathrm{SL}_2(\mathbb{R})/\Gamma,$$

  for a lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$ with 2 cusps.

- The u_s orbit of (X_0, ω_0) itself is periodic.

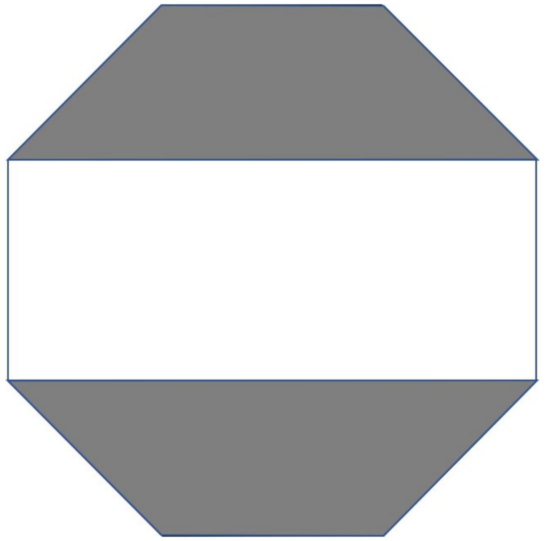
Construction in a nutshell

- Produce a 1-parameter family of periodic horocycles $\ell \mapsto H_\ell$ which are **outside** $\mathcal{O}$ via **tremors** and $H_0 = (u_s X_0) \subset \mathcal{O}$.
- Choose $t_n \to \infty$ and $\ell_n \to 0$ carefully so that $g_{t_n} H_{\ell_n}$ give us the periodic horocycles in the main theorem.



What are tremors?

The octagon has 2 horizontal cylinders.



- We'll shear one of them to the **right** by an amount ℓ (and shear the other to the **left** by the same amount).
- We get a path $\ell \mapsto (X_\ell, \omega_\ell)$, with tangent vector field β:

$$(X_\ell, \omega_\ell) =: \mathrm{Trem}_{X_0}(\beta; \ell).$$

What are tremors?

- Tremoring **commutes** with the horocycle flow:
  - (X_ℓ, ω_ℓ) has a periodic u_t orbit H_ℓ.
- $\beta \perp \mathrm{T}\mathcal{O}$ infinitesimally $\Longrightarrow$ for small ℓ, $(X_\ell, \omega_\ell) \notin \mathcal{O}$.

Local normalizer

Tremors are also **"normalized"** by the geodesic flow, i.e.

Lemma

The pushforward of a tremor path $\ell \mapsto (X_\ell, \omega_\ell)$ starting at (X_0, ω_0) by g_t is also tremor path starting at $g_t \cdot (X_0, \omega_0)$. Moreover,

$$g_t u_s X_\ell = \mathrm{Trem}_{g_t u_s X_0}(\mathbf{KZ}(\mathbf{g_t}, \mathbf{u_s X_0}) \cdot \beta; \ell).$$

- Magic happens because $s \mapsto \mathrm{KZ}(g_t, u_s X_0)$ is far from constant.
- For comparison, if β were tangent to a **horocycle** path, then $\mathrm{KZ}(g_t, u_s X_0) \cdot \beta = e^t \cdot \beta$.

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The cocycle is bad news for the horocycle flow

The distance of $g_t u_s X_\ell$ to the Octagon locus is essentially*

$$\|\mathrm{KZ}(g_t, u_s X_0) \cdot \beta\| .$$

Oseledets + Chaika-Eskin:

$$\frac{\log \|\mathrm{KZ}(g_t, u_s X_0) \cdot \beta\|}{t} \xrightarrow{t\to\infty} \lambda > 0, \qquad \text{for a.e. } s.$$

But...these norms exhibit **significant fluctuations** around the mean $e^{\lambda t}$ as s varies!

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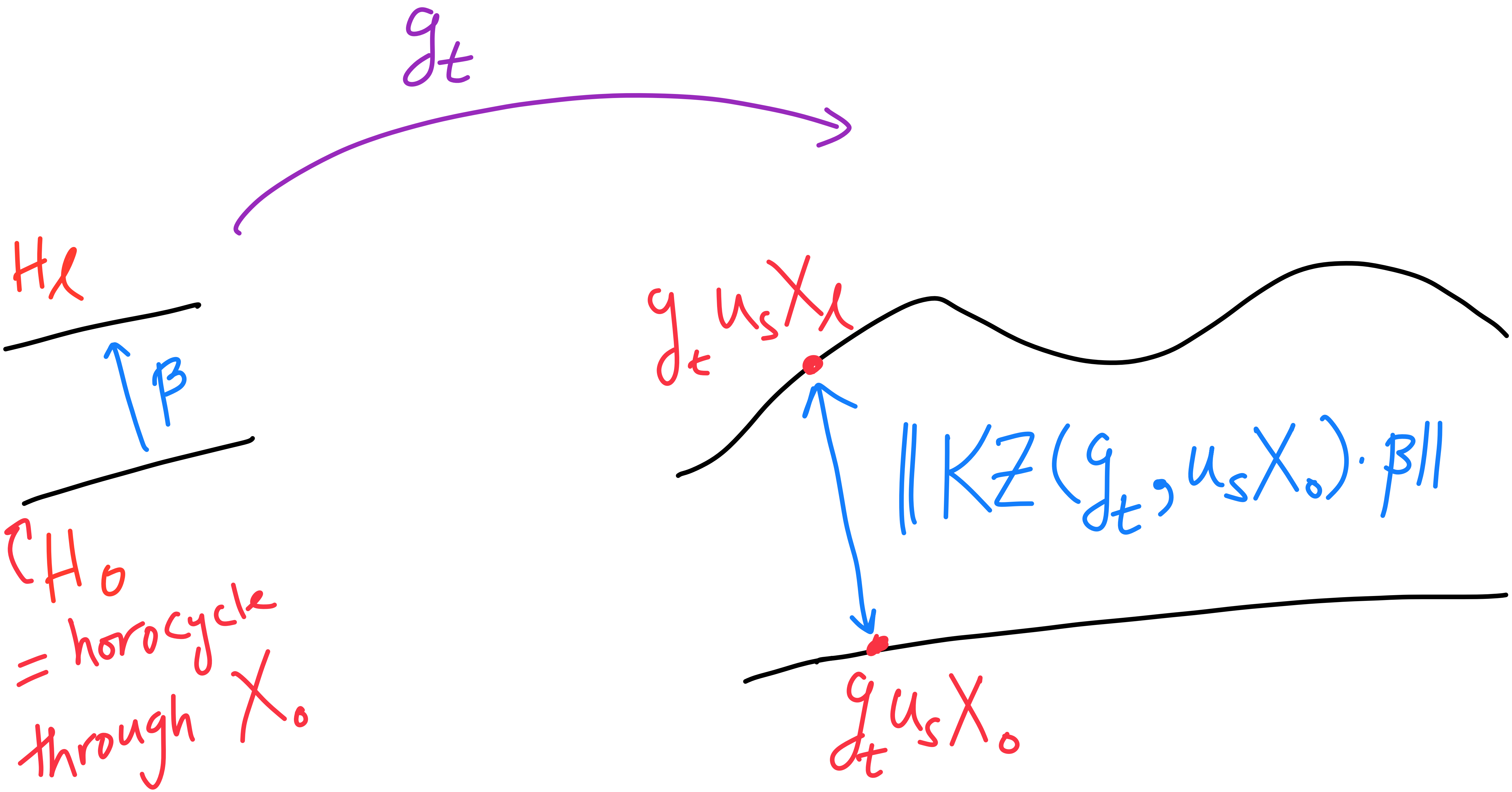
Fluctuations $\Longrightarrow$ Chaotic divergence of u_s-orbits



Non-concentration of the cocycle

The heart of the proof is:

Theorem (Chaika-Khalil-Smillie)

For every $K \geq 1$*, for all large enough* t*,*

$$\sup_{r \geq 0} \left|\{s \in [0,1] : r \leq \|\mathrm{KZ}(g_t, u_s X_0) \cdot v\| \leq Kr\}\right| < 0.99,$$

for any $0 \neq v \perp \mathrm{T}\mathcal{O}$*.*

Remark

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Moral of non-concentration:

- Both the sets where the cocycle is large/small have definite mass.
- When the norm is small $\Longrightarrow$ definite mass of the horocycle $g_t H_\ell$ is in the ϵ neighborhood of the Octagon locus.
- When the norm is large (+more work) $\Longrightarrow$ definite mass is far from $\mathcal{O}$.

Central Limit Theorem - In progress

- $\mu =$ prob. measure on a closed $SL_2(\mathbb{R})$-orbit in a stratum $\mathcal{H}(\alpha)$.
- $V \subset \mathrm{T}\mathcal{H}(\alpha)-$ strongly irreducible subbundle.

Theorem (Khalil, in progress)

Assume top Lyapunov exponent λ_1 is simple. Then, there exists $\sigma \geq 0$, for all $0 \neq v \in V$,

$$\frac{\log \|\mathrm{KZ}(g_t, \bullet)v\| - \lambda_1 t}{\sqrt{t}} \xrightarrow[t\to\infty]{\textit{in law}} \mathrm{Gaussian}(0, \sigma^2).$$

- Approach is via spectral theory of transfer operators: flows pose serious difficulties.

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Positive Variance

Theorem (Khalil 2022+)

Assume further that

1. *there are two periodic g_t-orbits with distinct Lyapunov exponents,*
2. $\dim(V) = 2$.

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Proof of non-concentration - rough idea

- Observation: if s_1, s_2 have norms in $[r, Kr]$, then

$$\frac{1}{K} \leqslant \frac{\|\mathrm{KZ}(g_t, u_{s_2} X_0) \cdot v\|}{\|\mathrm{KZ}(g_t, u_{s_1} X_0) \cdot v\|} \leqslant K.$$

- Strategy: build disjoint, definite measure sets $A, B \subset [0,1]$ and a **measure preserving** $\Phi : A \longrightarrow B$:

$$\frac{\left\|\mathrm{KZ}(g_t, u_{\Phi(s)} X_0) \cdot v\right\|}{\|\mathrm{KZ}(g_t, u_s X_0) \cdot v\|} > K.$$

- Upshot: $\Phi\Big(\text{ Concentrated set } \cap A\Big)$ misses the concentrated set.

Non-concentration: 1) Lyapunov spectrum

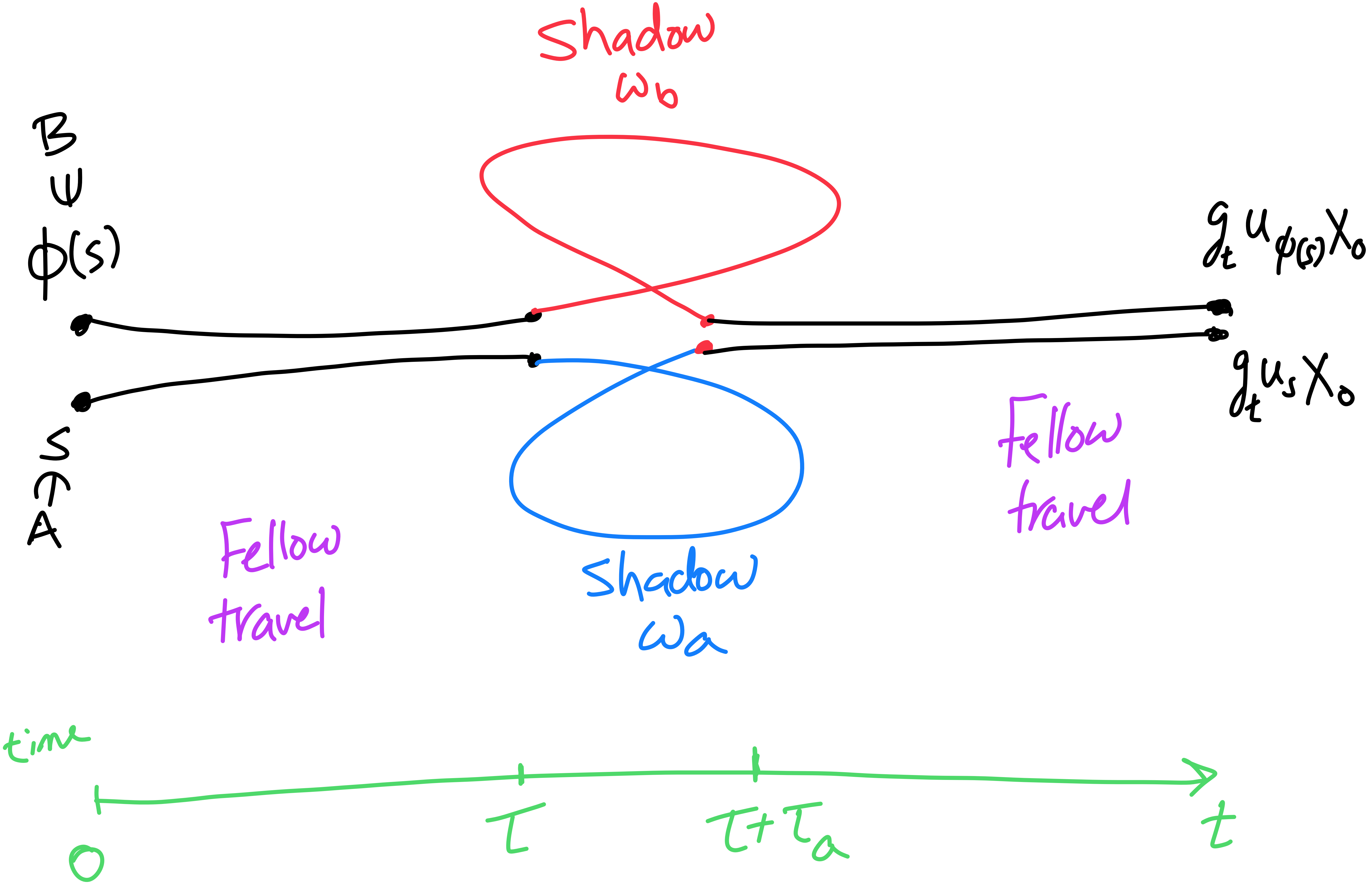
Proposition (Chaika-Khalil-Smillie)

For every $\varepsilon > 0$, there are two nearby points $\omega_a, \omega_b \in \mathcal{O}$ with periodic g_t orbits and (possibly non-primitive) periods $\tau_a \approx \tau_b$, but

$$\|\mathrm{KZ}(g_{\tau_a}, \omega_a)\| < \varepsilon \, \|\mathrm{KZ}(g_{\tau_b}, \omega_b)\| \, .$$

- These periodic orbits have distinct Lyapunov exponents on $\mathrm{T}\mathcal{O}^{\perp}$.
- Impossible in homogeneous dynamics.

Non-concentration: 2) Matching



Non-concentration: 3) Strong Irreducibility

Strong irreducibility $\Longrightarrow$ norm of products $\approx$ product of norms:

For $s \in A$

$$\|\mathrm{KZ}(g_T, u_s X_0)\| \approx \|\text{after shadowing}\| \, \|\mathrm{KZ}(g_{\tau_a}, \omega_a)\| \, \|\text{before shadowing}\| .$$

Similarly for $s \in B$.

So, if $s \in A$,

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Questions

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Is this the worst that can happen for limits of horocycle ergodic measures (on $\mathcal{H}(2)$)?

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Are there exotic horocycle flow orbit closures in $\mathcal{H}(2)$?

Thanks!