Exponential Mixing Via Additive Combinatorics

Question: $g_t : X \circlearrowleft$ topologically mixing Anosov flow. μ = equillibrium state. Is μ exponentially mixing, i.e. $\exists \delta > 0 : \forall t \geq 0, \varphi, \psi ; \int \psi = 0$

$$\left| \int \varphi \cdot \psi \circ g_t \, d\mu \right| \lesssim_{\varphi,\psi} e^{-\delta t}.$$

Dolgopyat '98:

- g_t = geodesic flows on -ve curved surfaces $\Rightarrow$ Yes
- C^1-foliations + μ = SRB $\Rightarrow$ Yes
- Thermodynamic formalism

Liverani '04:

- g_t = geodesic flow on -ve curvature + μ = SRB $\Rightarrow$ Yes.
- intrinsic/geometric method → more precise
- suited for perturbations.

Giullietti – Liverani – Pollicott '12: Small perturbation of const. curv. μ = MMÉ $\Rightarrow$ Yes

Scheme: Non-joint integrability → Cancellations → mixing

Goal: Introduce new cancellation mechanism

Question: What about MME in higher dimensions?

Setting of Main Thm:

g_t = geodesic flow on geometrically finite quotients of -ve curved symmetric spaces

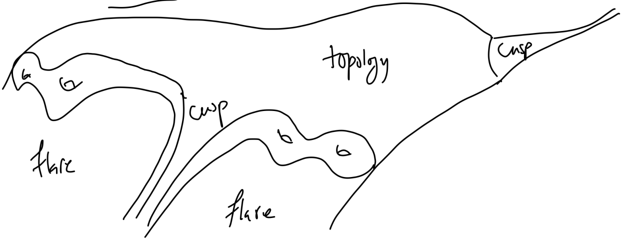
μ = MME

Thm (K. '23): g_t is exp. mixing w.r.t. μ.

- Naud, Stoyanov, Chow, Sarkar, Winter: no cusps, symbolic
- Mohammadi–Oh, Edwards–Oh: rep theory, large entropy
- Li–Pan '21: Cusped real hyperbolic mflds

Geom. fin. Mflds for Dynamicists:

- Supp(μ) not cpt in general
- Curvature non-const. $\Longrightarrow$ $Dg_t|_{\text{unstable}}$ non-conformal
- $\mu|_{\text{unstable}}$ not abs. cts.
- Picture:



Proof Ideas:

Dream: $\mathcal{L}_t \psi := \psi \circ g_t$, $\int \psi \, d\mu = 0$

Find a norm $\|\cdot\|$ on C^r-fns:

$$\left| \underbrace{\int \varphi \cdot \mathcal{L}_t \psi \, d\mu}_{e(t)} \right| \lesssim_\varphi \|\mathcal{L}_t \psi\| \qquad (1)$$

and $\|\mathcal{L}_t\| \lesssim e^{-\delta t}$ $\quad \forall t \geq 0 \qquad (2).$

Candidate (Blank-Keller-Liverani, Gouzel-Liverani)

$$\|\psi\| = \sup_{\substack{x,\varphi,\\ D^{cs}}} \int_{W^u_{loc}(x)} \varphi \, D^{cs}\psi \, d\mu^u_x$$

$\Rightarrow$ satisfies (1)

* $\mathcal{L}_t$ contracts D^s & $\|\psi\|$
* D^c .... $\Rightarrow$ Can't expect (2)

New Dream:

Paley–Wiener Thm (Pollicott)

$$\hat{\rho}(z) := \int_0^\infty e^{-zt} \rho(t)\, dt$$

$z \in \mathbb{C}$, $Re(z) > 0$

$\hat{\rho}(z)$ extends analytically to $Re(z)$ "$\Longleftrightarrow$" $|\rho(t)| \lesssim e^{-\delta t}$

Def: $R(z) := \int_0^\infty e^{-zt} \mathcal{L}_t \, dt$

resolvent

$$\hat{\rho}(z) = \int \varphi \cdot R(z)(\psi)\, d\mu$$

$$\{\text{poles of } \hat{\rho}(z)\} = \{\text{poles of } z \mapsto R(z) \in Bdd(\|\cdot\|)\}$$

Lem: $\mathcal{L}_t = e^{tX}$, X = flow-derivative

$$= \sum_{n \geq 0} \frac{(tX)^n}{n!}$$

obs. $R(z) = \int_0^\infty e^{-zt + tX}\,dt$

$= (X - zId)^{-1}$

$\{\text{Poles of } R(z)\} = \mathrm{Spec}(X)$

$= \left\{ z + \frac{1}{\mathrm{Spec}(R(z))} \right\}$ $(\star)$

obs.: $\lambda \in \mathrm{Spec}(R(z))$ eigenvalu

$R(z) f = \lambda f = (X - zId)^{-1} f$

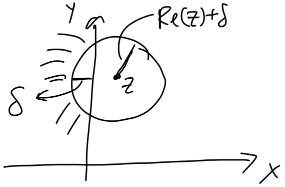
$(X - zId) f = \frac{1}{\lambda} f$

$X f = \left(z + \frac{1}{\lambda}\right) f$

Upshot of $(\star)$:

If $\|R(z)\| \leq \frac{1}{Re(z) + \delta}$ $(\star\star)$

$\Rightarrow \hat{\rho}(z)$ has no poles in the disk:

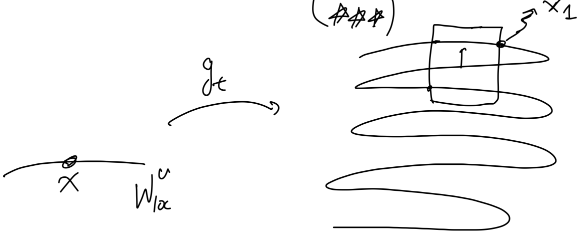


After lots of hard but manageable work:

$(\star\star)$ reduces to showing: $z = a + ib$

$$\left| \int_{\underline{T}}^{T+1} \underline{\underline{e}}^{-zt} \int_{\underline{W^u_{loc}(x)}} \varphi \, \mathcal{L}_t \psi \, d\mu^u_x \right| \lesssim_\varphi e^{-aT} \|\nabla\psi\| \, \underline{\underline{|b|}}^{-\varepsilon}$$

$(\ast\ast\ast)$



localize & stable holonomy:

$$\sum_k \int_0^1 \int_{W^u_{loc}(x')} e^{-z(t+\tau_k(y))} \psi \, d\mu^u_{x'}(y)$$

τ_k = temporal dist. function

Non-joint integrability $\Longrightarrow$ τ_k non-const.

Main estimate:

$$\sum_k \int_{W^u_{loc}(x')} e^{-ib\tau_k(y)} \, d\mu^u_{x'}(y)$$

C^1-foliations / Contact flow:

τ_k = linear term + controlled error

$$\Rightarrow \sum_{k} \int_{W^u_{loc}} e^{-ib\langle v_k, y\rangle} \, d\mu^u_{x'}$$

= Average over Fourier coefficients of $\mu^u_{x'}$

$\mu = \text{SRB} \Rightarrow \mu^u_{x'} = \text{Lebesgue} \to$ by-parts

Thm^2 (K. '23)

ν = Borel prob. meas on $\mathbb{R}^n$, cpt supp.

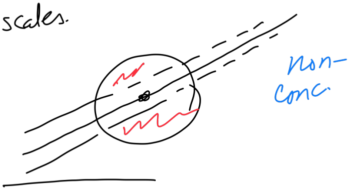
Then, either

(1) $\forall \varepsilon > 0, \exists \delta > 0 : \forall T :$

$$\left| \left\{ \|\xi\| \lesssim T : |\hat{\nu}(\xi)| \geq T^{-\delta} \right\} \right| \lesssim T^{\varepsilon}$$

OR

(2) ν concentrates near proper subspaces at many scales.



Remarks:

Affine non-concentration holds for many dynamically defined measures:

- self-similar
- self-conformal
- Prop. (K.): Patterson-Sullivan meas. in setting of main thm

Proof: Enough to show: $\exists N$

$$\int_{|\xi|\le T} \left|\widehat{\nu^{*N}}(\xi)\right|^2 d\xi \lesssim T^{\varepsilon}$$

$$\|$$

$$\left|\widehat{\nu^{*N}}\right| = |\hat{\nu}|^N$$

$\operatorname{supp}(\nu * \nu) = \operatorname{supp}\nu + \operatorname{supp}\nu$

ν^{*N} = spreads out unless ν is supp. near a subgp. (Inverse Thm).

Remarks: 1) Inverse thm was proved indep. by Shmerkin

2) 1-dim'l of inverse thm $\rightsquigarrow$ Shmerkin's pf of Furstenberg's intersection conjecture.