Sparse Equidistribution, Birkhoff's Theorem and Unipotent Dynamics

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What we will talk about

- **Homogeneous Spaces**: $X = G/\Gamma$, G a Lie group, Γ a lattice. Ex: $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.
- **Transformations**: elements g_n of G acting by left multiplication.
- **Measures**: μ the Haar measure on X.
- **Central question**: when equidistribution of sequences of the form $g_n x$ for some $x \in X$ holds i.e.

$$\frac{1}{N}\sum_{n=1}^{N} \delta_{g_n x} \xrightarrow{weak-*} \mu$$

A motivating question

$G = SL(2,\mathbb{R})$, $\Gamma < G$ a lattice, $p : \mathbb{N} \to \mathbb{N}$ an increasing sequence,

$$u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Conjecture (Shah '94)

If $p(n)$ is a polynomial, then for **every** $x \in G/\Gamma$ which is not $u(t)$-periodic,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{u(p(n))x} = \mu_{G/\Gamma}$$

Equidistribution of exponential sequences

$G = SL(2,\mathbb{R})$, $\Gamma \subset G$ a lattice, $H = SO(2)$. Let $\lambda, \varepsilon > 0$ and let

$$g_n = \begin{pmatrix} 1 & e^{\lambda n^{\varepsilon}} \\ 0 & 1 \end{pmatrix}, k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Theorem (K.'17)

For every $x \in G/\Gamma$ and for almost every $\theta \in [0, 2\pi]$, there exists a sequence $A(\theta) \subseteq \mathbb{N}$ of full upper density such that

$$\lim_{\substack{N\to\infty \\ N\in A(\theta)}} \frac{1}{N} \sum_{n=1}^{N} \delta_{g_n k_\theta x} = \mu_{G/\Gamma}$$

Moreover, if G/Γ is compact then $A(\theta) = \mathbb{N}$.

What we know

- **Dani-Smilie '84, Ratner '91**: The only $u(t)$ invariant measures are $\mu_{G/\Gamma}$ and measures supported on periodic orbits.
- **Bourgain 1986**: The conjecture holds for $\mu_{G/\Gamma}$-**almost every** x.
- **Venkatesh 2010**: $p(n) = n^{1+\varepsilon}$, compact quotients of $SL(2, \mathbb{R})$. Improvements: Flaminio, Forni, Tanis, Vishe, Zheng.
- **Sarnak-Ubis 2011**: $p(n) = n^{th}$ prime, $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$, absolutely continuous weak limits with mass at least $1/10$.

A middle ground

Question

Does the conjecture hold ν-**almost everywhere** *where* ν *is singular w.r.t.* $\mu_{G/\Gamma}$ *but satisfies*

$$\frac{1}{N}\sum_{n=1}^{N}(u(p(n)))_*\nu \to \mu_{G/\Gamma}$$

Question (More generally)

If ν *is a probability measure on* G/Γ *satisfying* $\frac{1}{N}\sum_{n=1}^{N}(g_n)_*\nu \to \mu_{G/\Gamma}$ *for a sequence* g_n *of elements* G*. When does equidistribution of* $\{g_n x : n \in \mathbb{N}\}$ *hold for* ν*-almost every* x*?*

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Examples of ν

Author(s)	ν	**Pushforward by**
Eskin-McMullen '93	Closed orbits of a symmetric subgroup H of G	$g_n \to \infty$ in G/H
Shah '09, '10	Curves on horospherical subgroups of G	diagonal elements
Eskin-Mirzakhani-Mohammadi '13	$SO(2)$ orbits on a stratum of abelian differentials	geodesic flow

- Many others: Eskin-Mozes-Shah, Kleinbock-Weiss, ...

Abstract Setup

- $(X, \mathfrak{B}, \mu)$ a standard Borel space.
- $T_n : X \to X$ a sequence of continuous transformations.
- $\nu \in \mathcal{P}(X)$ satisfying

$$\frac{1}{N}\sum_{n=1}^{N}(T_n)_*\nu \to \mu$$

What can we prove with abstract ergodic theoretic tools?

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The Birkhoff Case: $T_n = T^n$

$(X, \mathfrak{B}, \mu)$ standard Borel space, $T : X \to X$ μ-ergodic, $\nu \in \mathcal{P}(X)$,

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Theorem (K.'17)

For ν-almost every $x \in X$, there exists a sequence $A(x) \subseteq \mathbb{N}$, of full upper density, such that

$$\lim_{\substack{N\to\infty \\ N\in A(x)}} \frac{1}{N}\sum_{n=1}^{N}\delta_{T^n x} = \mu$$

- Does the limit exist?

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Example: Expanding Curves

$G = SL(d+1,\mathbb{R})$, $\Gamma = SL(d+1,\mathbb{Z})$, $\varphi : [0,1] \to \mathbb{R}^d$ an analytic curve. $\nu = \varphi_*(Leb)$. For $v \in \mathbb{R}^d$ and $t \in \mathbb{R}$, define

$$u(v) = \begin{pmatrix} 1 & v^t \\ 0 & I_d \end{pmatrix}, \; a(t) = \begin{pmatrix} e^{dt} & 0 \\ 0 & e^{-t} I_d \end{pmatrix}$$

By Shah'09, the pointwise theorem applies in this case and gives:

Corollary (K.'17)

If the image of φ is not contained in finitely many proper affine subspaces of $\mathbb{R}^d$, then for almost every $s \in [0,1]$, there exists $A(s) \subseteq \mathbb{N}$ of full upper density such that

$$\lim_{\substack{N\to\infty \\ N \in A(s)}} \frac{1}{N} \sum_{n=1}^{N} \delta_{a(n)u(\varphi(s))x} = \mu_{G/\Gamma}$$

What about sequences of transformations?

Same setup but need two more conditions:

1. There exists $S : X \to X$ such that for ν-almost every x, all limit points of the sequence $\frac{1}{N}\sum_{n=1}^{N} \delta_{T_n x}$ is S invariant.
2. μ is S ergodic and there exists a σ-compact μ-null set $Z \subset X$ on which all other S-ergodic measures live.

Theorem (K.'17)

For ν-almost every x, there exists a sequence $A(x) \subseteq \mathbb{N}$, of full upper density, such that

$$\lim_{N \in A(x)} \frac{1}{N} \sum_{n=1}^{N} \delta_{T_n x} = \mu$$

Unipotent invariance and Ratner's Theorem

H a symmetric subgroup, $\nu = \mu_H$ is H-invariant probability measure on a closed H-orbit.

Definition (Ratner Sequences)

We say a sequence g_n of elements of G is a **Ratner Sequence** for H if there exists a one parameter unipotent subgroup U such that for μ_H almost every $x \in G/\Gamma$, any limit point of the measures $\frac{1}{N}\sum_{n=1}^{N}\delta_{g_n x}$ is invariant by U.

Existence of Ratner Sequences

Generalizing a technique due to J. Chaika and A. Eskin, we prove

Theorem (K.'17)

If the sequence g_n grows exponentially with bounded H component, then g_n contains a Ratner sequence as a subsequence.

- In most examples, the sequence $g_n = g^{p(n)}$ for some $g \in G$ and $p : \mathbb{N} \to \mathbb{N}$ increasing. No need to pass to a subsequence.

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Proof of main theorem

- **Step 1**: the elements

$$g_n = \begin{pmatrix} 1 & e^{\lambda n^{\varepsilon}} \\ 0 & 1 \end{pmatrix}$$

  form a Ratner sequence for $H = SO(2)$.

- **Step 2**: Apply the ergodic theorem to conclude that for *every* $x \in G/\Gamma$ and for almost every $\theta \in [0, 2\pi]$, there exists a sequence $A(\theta) \subseteq \mathbb{N}$ of full upper density such that

$$\lim_{\substack{N \to \infty \\ N \in A(\theta)}} \frac{1}{N} \sum_{n=1}^{N} \delta_{g_n k_\theta x} = \mu_{G/\Gamma}$$

Thanks!