# Finite Volume Method for Hyperbolic PDEs

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# Outline

# 1D Advective transport

1D fluid flow

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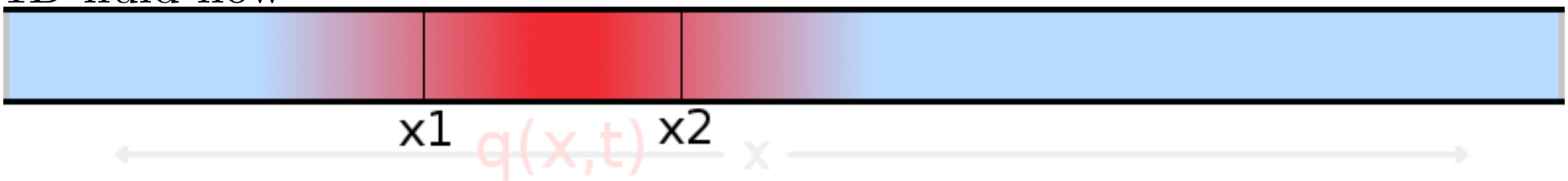
Introduce a tracer that moves in the fluid. Let $\rho(x)$ denote the density of the tracer.

# 1D Advective transport

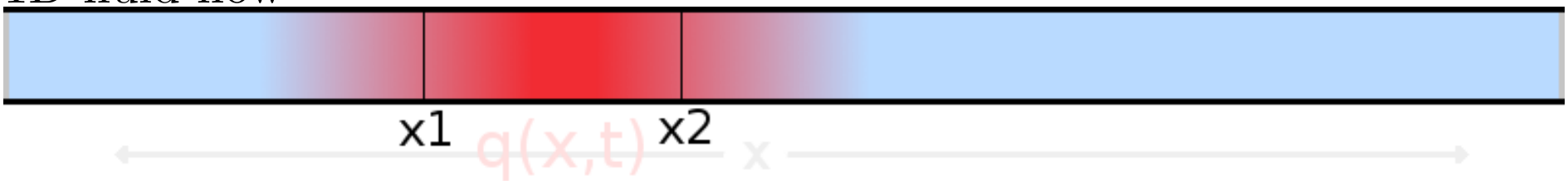
1D fluid flow




$$\int_{x_1}^{x_2} q(x,t)dx = \text{mass of tracer between } x_1 \text{ and } x_2.$$

# 1D Advective transport

1D fluid flow




$$\int_{x_1}^{x_2} q(x,t)dx = \text{mass of tracer between } x_1 \text{ and } x_2.$$

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = F_1(t) - F_2(t),$$

where $F_i$ is the flux of mass from right to left at $x_i$.

# Conservation law

For general autonomous flux $F = f(q)$, we have

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,\mathrm{d}x = f(q(x_1,t)) - f(q(x_2,t)).$$

For $f$ sufficiently smooth, we have:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,\mathrm{d}x = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(q(x,t))\,\mathrm{d}x,$$

which we can write as

$$\int_{x_1}^{x_2} \left[\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} f(q(x,t))\right] \mathrm{d}x = 0.$$

# Conservation law

Differential form of the 1D conservation law:

$$q_t + f(q)_x = 0.$$

# Hyperbolic systems

A 1D *quasilinear* system

$$q_t + A(q, x, t)q_x = 0$$

is *hyperbolic* at $(q, x, t)$ if $A(q, x, t)$ is diagonalizable with real eigenvalues.

The 1D nonlinear conservation law

$$q_t + f(q)_x = 0$$

is hyperbolic if the Jacobian matrix $\frac{\partial f}{\partial q}$ is diagonalizable with real eigenvalues for each physically relevant $q$.

# Linear hyperbolic systems

Consider the linear hyperbolic IVP

$$\begin{cases} q_t + Aq_x = 0, \\ \quad q(x,0) = q_0(x) \end{cases}$$

Then we can write $A = R\Lambda R^{-1}$, where $R \in \mathbb{R}^{m\times m}$ is the matrix of eigenvectors and $\Lambda \in \mathbb{R}^{m\times m}$ is the matrix of eigenvalues. Making the substitution $q = Rw$, we get the decoupled system

$$w_t^p + \lambda^p w_x^p = 0, \quad p = 1 \dots m.$$

The 1D advection equation

$$\begin{cases} w_t + \lambda w_x = 0, \\ w(x,0) = w_0(x) \end{cases}$$

is easily solved using the method of characteristics:

$$w(x,t) = w_0(x - \lambda t).$$

We can now write

$$\begin{aligned} q(x,t) &= \sum_{p=1}^{m} w^p(x,t) r^p \\ &= \sum_{p=1}^{m} w_0^p(x-\lambda^p t) r^p \\ &= \sum_{p=1}^{m} [\ell^p q_0(x-\lambda^p t)] r^p \end{aligned}$$

So the solution $q(x,t)$ is a superposition of waves with speeds $\lambda^p$.

# Domain of dependence and Range of Influence

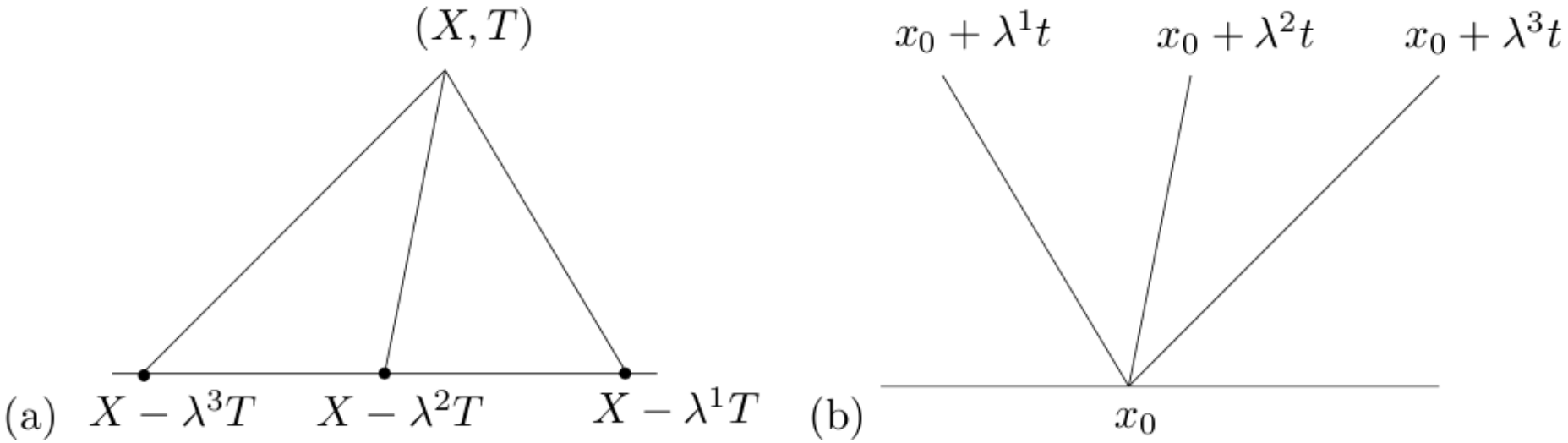



Fig. 3.2. For a typical hyperbolic system of three equations with $\lambda^1 < 0 < \lambda^2 < \lambda^3$, (a) shows the domain of dependence of the point $(X, T)$, and (b) shows the range of influence of the point $x_0$.

(R. Leveque, 2002)

# Riemann Problem

The hyperbolic equation with initial data

$$q_0(x) = \begin{cases} q_\ell & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

is known as the *Riemann problem*.

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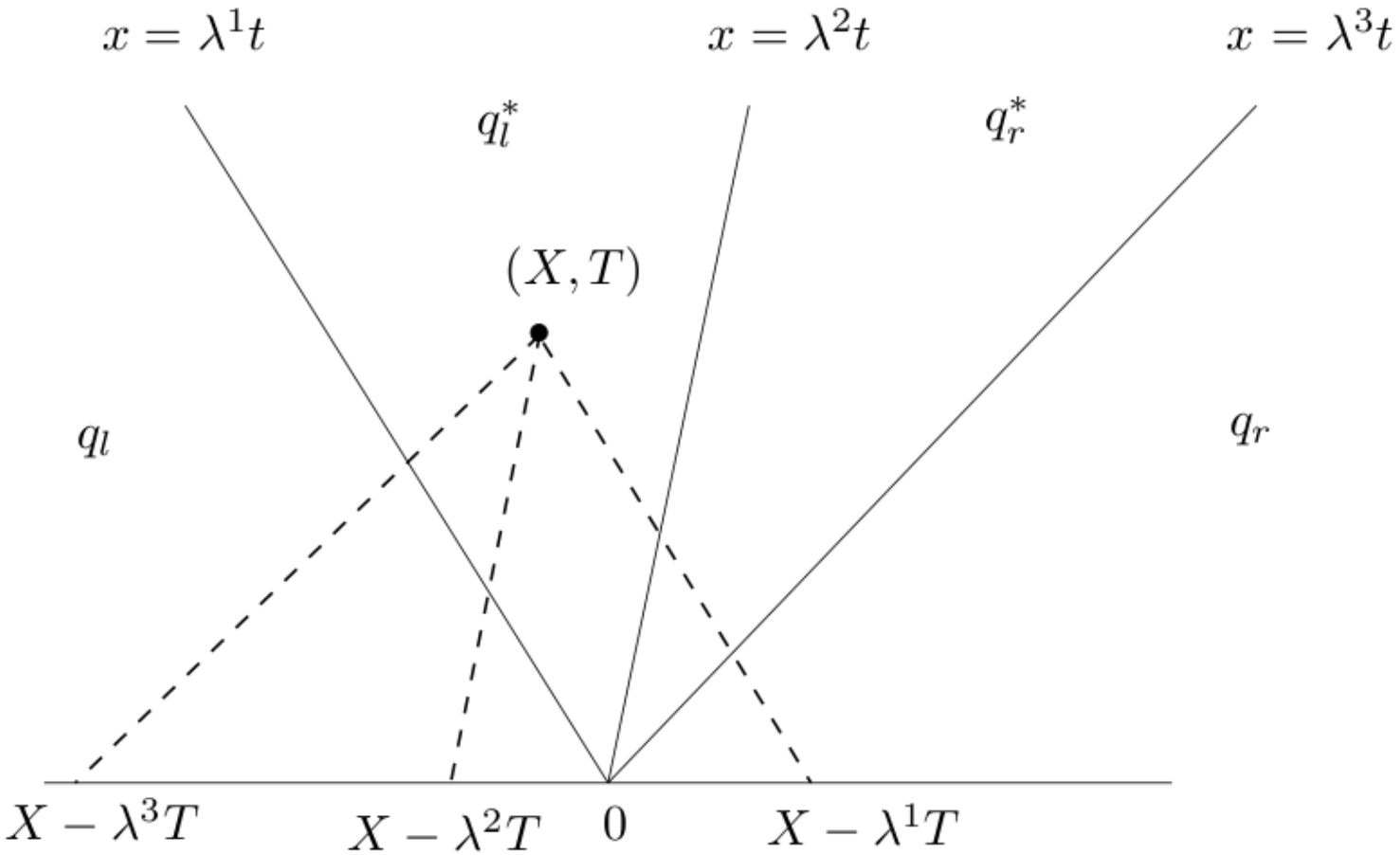
For the linear constant-coefficient system, the solution is given by:

$$\begin{aligned} q(x,t) &= q_\ell + \sum_{p:\lambda^p < x/t} \left[\ell^p (q_r - q_\ell)\right] r^p \\ &= q_r - \sum_{p:\lambda^p \geq x/t} \left[\ell^p (q_r - q_\ell)\right] r^p. \end{aligned}$$

# Riemann Problem

Solution to Riemann problem using characteristics:
$(\lambda^1 < 0 < \lambda^2 < \lambda^3)$



# Initial-Boundary Value problems

For problems with bounded domains $a \leq x \geq b$, we also need boundary conditions. The signs of the wave speeds dictate how many conditions are required at each boundary.
For example, the advection equation

$$q_t + q_x = 0$$

requires only boundary conditons at $x = a$.

# Outline

# Finite Volume Method

We subdivide the spatial domain into grid cells $C_i$, and in each cell we approximate the average of $q$ at time $t_n$:

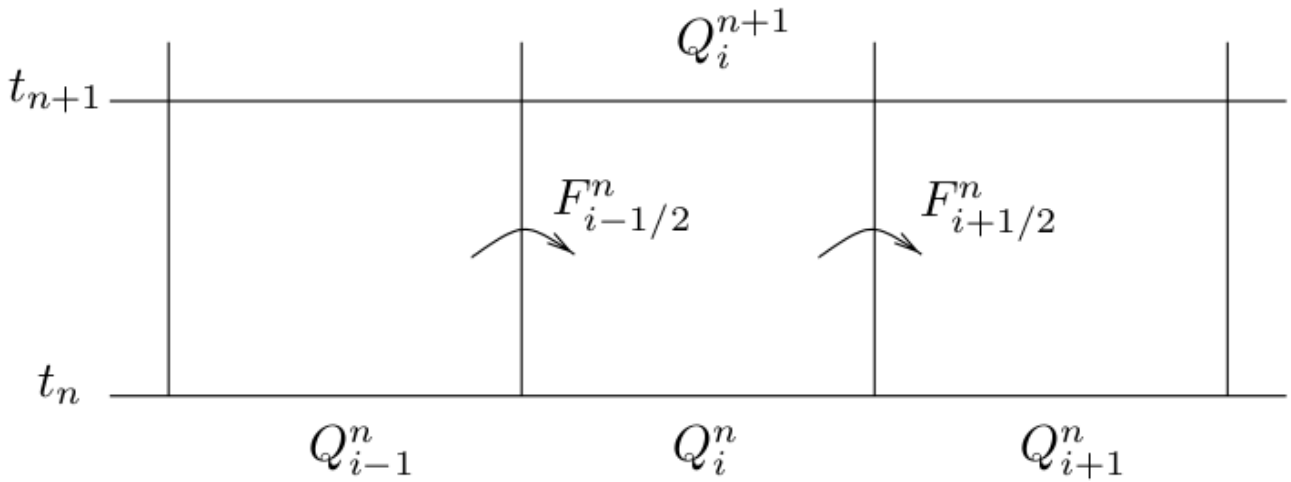
$$Q_i^n \approx \frac{1}{m(C_i)} \int_{C_i} q(x, t_n)\, \mathrm{d}x.$$

At each time step we update these values based on fluxes between cells.

# 1D Finite Volume Method

In 1D, $C_i = (x_{i-1/2}, x_{i+1/2})$.

Assume uniform grid spacing $\Delta x = x_{i+1/2} - x_{i-1/2}$.



# 1D conservation law

Recall the autonomous conservation law:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,\mathrm{d}x = f(q(x_1,t)) - f(q(x_2,t)).$$

Integrating in time from $t_n$ to $t_{n+1}$ and dividing by $\Delta x$, we can derive

$$\frac{1}{\Delta x}\int_{C_i} q(x,t_{n+1})\,\mathrm{d}x = \frac{1}{\Delta x}\int_{C_i} q(x,t_n)\,\mathrm{d}x - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,\mathrm{d}t - \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,\mathrm{d}t\right].$$

# Approximation to flux term

Note that we cannot in general evaluate the time integrals exactly. However, it does suggest numerical methods of the form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right),$$

where

$$F_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t))\,\mathrm{d}t.$$

# Numerical flux

For a hyperbolic problem, information propagates at a finite speed. So it is reasonable to assume that we can obtain $F^n_{i-1/2}$ using only the values $Q^n_{i-1}$ and $Q^n_i$:

$$F^n_{i-1/2} = \mathcal{F}(Q^n_{i-1}, Q^n_i)$$

where $\mathcal{F}$ is some *numerical flux function*. Then our numerical method becomes

$$Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(Q^n_i, Q^n_{i+1}) - \mathcal{F}(Q^n_{i-1}, Q^n_i)\right].$$

# Necessary conditions for convergence

If we want our numerical solution to converge to the true solution as $\Delta x, \Delta t \to 0$, then

- The method must be *consistent*, i.e. the local truncation error goes to 0 as $\Delta t \to 0$
- The method must be *stable*, i.e. small errors in each time step do not grow too quickly.

# Consistency

Suppose we have a numerical method $Q^{n+1} = \mathcal{N}(Q^n)$ and "true" values $q^n$ and $q^{n+1}$. Then the local truncation error is defined to be:

$$\tau = \frac{\mathcal{N}(q^n) - q^{n+1}}{\Delta t}.$$

The method is consistent if $\tau$ vanishes as $\Delta t \to 0$ for all smooth $q(x,t)$ that satisfy the differential equation. This is usually easy to check using Taylor expansions.

# CFL condition

A necessary condition for stability is the *Courant-Friedrichs-Levy condition*: A numerical method can be stable only if its numerical domain of dependence contains the true domain of dependendence of the PDE, at least in the limit as $\Delta t, \Delta x \to 0$.

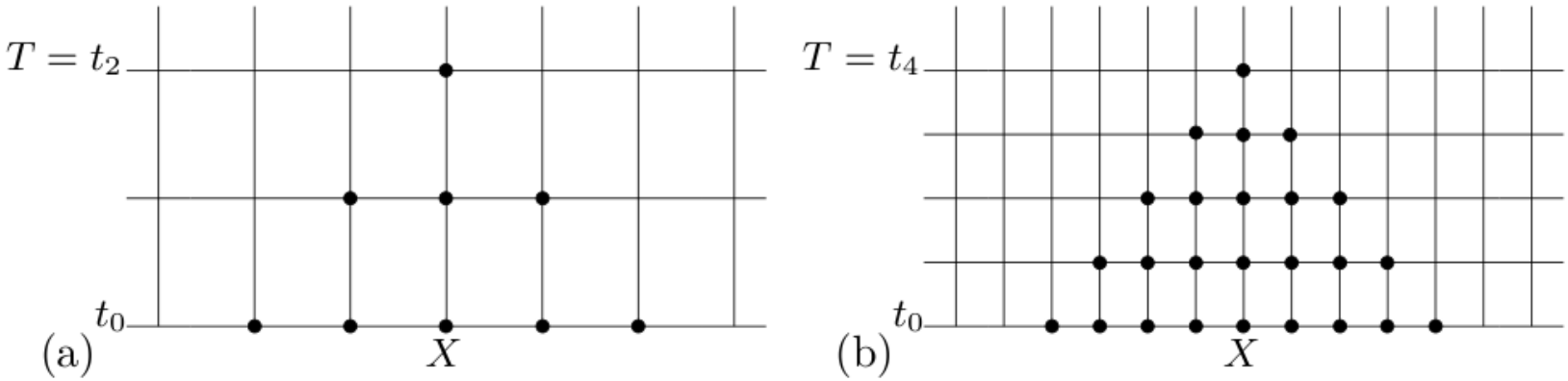



Fig. 4.3. (a) Numerical domain of dependence of a grid point when using a three-point explicit finite difference method, with mesh spacing $\Delta x^a$. (b) On a finer grid with mesh spacing $\Delta x^b = \frac{1}{2}\Delta x^a$.

# CFL condition

For a hyperbolic system of equations, we have wave speeds $\lambda^1, \ldots, \lambda^m$ with domain of dependence

$$\mathcal{D}(x,t) = \{x - \lambda^p t \mid p = 1, \ldots, m\}.$$

Then the CFL condition is

$$\frac{\Delta x}{\Delta t} \geq \max_p |\lambda^p|.$$

Note: This is necessary but not sufficient for stability.

# Choosing the numerical flux

A naive choice for the numerical flux function might be:

$$\mathcal{F}(Q_{i-1}^n, Q_i^n) = \frac{1}{2}\left[f(Q_{i-1}^n) + f(Q_i^n)\right].$$

# Choosing the numerical flux

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$$\mathcal{F}(Q_{i-1}^n, Q_i^n) = \frac{1}{2}\left[f(Q_{i-1}^n) + f(Q_i^n)\right].$$

. . . but this leads to an unstable scheme.

(shown via Von Neumann analysis)

# Upwind methods

Since information is propagated along characteristics, symmetric numerical flux functions won't be effective. We seek to use *upwind* methods where information for each characteristic variable is obtained by looking in the direction from which it should be coming.
The *first-order upwind method* for the constant-coefficient advection equation $q_t + \lambda q_x = 0$ with $\lambda > 0$ is given by

$$Q_i^{n+1} = Q_i^n - \lambda \frac{\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n)$$

# Godunov's Method for Linear Systems

The following *REA algorithm* was proposed by Godunov (1959):

1. **Reconstruct** a piecewise polynomial function $\tilde{q}^n(x, t_n)$ from the cell averages $Q_i^n$. In the simplest case, $\tilde{q}^n(x, t_n)$ is piecewise constant on each grid cell:

$$\tilde{q}^n(x, t_n) = Q_i^n, \qquad \text{for all } x \in C_i.$$

2. **Evolve** the hyperbolic equation with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$.
3. **Average** this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{C_i} \tilde{q}^n(x, t_{n+1})\, \mathrm{d}x.$$

# Godunov's Method for Linear Systems

Note: the Evolve step (2) requires solving the Riemann problem, provided $\Delta t$ is small enough so waves from adjacent cells don't interact. Recall that the solution to the Riemann problem for a linear system can be written as a set of waves:

$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \left[\ell^p (Q_{i+1} - Q_i)\right] r^p \equiv \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^p$$

Note that after time $\Delta t$ the $p$th wave has moved a distance $\lambda^p \Delta t$. Then the effect of this wave on the cell average $Q$ is a change by the amount

$$-\lambda^p \frac{\Delta t}{\Delta x} \mathcal{W}^p_{i-1/2}.$$

So, the Average step (3) can be easily computed due to the simple geometries of the problem.

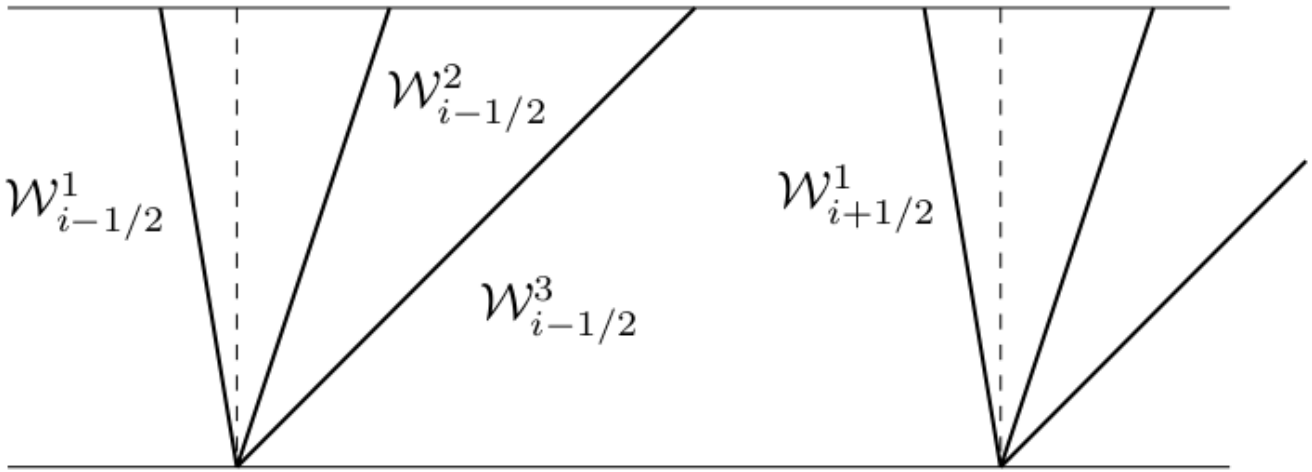


Fig. 4.7. An illustration of the process of Algorithm 4.1 for the case of a linear system of three equations. The Riemann problem is solved at each cell interface, and the wave structure is used to determine the exact solution time $\Delta t$ later. The wave $\mathcal{W}^2_{i-1/2}$, for example, has moved a distance $\lambda^2 \Delta t$ into the cell.

# Wave-propagation Form of Godunov's Method

Define $\lambda^+ = \max(\lambda, 0)$ and $\lambda^- = \min(\lambda, 0)$. Combining (2) and (3), we have the update algorithm:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+ \mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(\lambda^p)^- \mathcal{W}_{i+1/2}^p\right].$$

# Godunov's Method for General Conservation Laws

For the general conservation law

$$q_t + f(q)_x = 0,$$

we use the solution to the Riemann problem to define

$$F^n_{i-1/2} = f(Q_{i-1}) + \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}^p_{i-1/2}$$

or

$$F^n_{i-1/2} = f(Q_i) + \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}^p_{i-1/2}$$

# Roe's Method

Define $|A| = R|\Lambda|R^{-1}$, where $|\Lambda| = \text{diag}(|\lambda^p|)$.
Then we can derive the formula

$$F^n_{i-1/2} = \frac{1}{2}\left[f(Q_{i-1}) + f(Q_i)\right] - \frac{1}{2}|A|(Q_i - Q_{i-1}).$$

Used in nonlinear problems, this is known as *Roe's method*.

# References

# Thanks for listening!

Content was taken liberally from:

*Finite-Volume Methods For Hyperbolic Problems*
Randall LeVeque, 2002