u-Substitution

1. Evaluate the following integrals:

(a) $\displaystyle\int_1^{\ln 10} \frac{e^{\ln x}}{x}\,dx$ (b) $\displaystyle\int \sin(\cos(x))\sin(x)\,dx$

(c) $\displaystyle\int_1^3 x\sqrt{10-x^2}\,dx$ (d) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

Solutions:

(a) Let $u = \ln x$ so that $du = 1/x\,dx$. Then:

$$\int e^{\ln x}/x\,dx = \int e^u\,du = e^u = e^{\ln x}\Big]_1^{\ln 10} = \ln 10 - 1.$$

(You can also observe that $e^{\ln x} = x$ and go from there)

(b) Let $u = \cos x$, $du = -\sin x\,dx$. Then:

$$\int -\sin(u)\,du = \cos(u) + C = \cos(\cos(x)) + C.$$

(c) Let $u = x^2$, $du = 2x\,dx$. We get:

$$\int_1^9 \frac{1}{2}\sqrt{10-u}\,du$$

This is hard to integrate, so we do *another* substitution, $v = 10-u$, $dv = -du$. Now we integrate:

$$\frac{1}{2}\int_1^9 -\sqrt{v}, dv = -\frac{v^{3/2}}{3}\Bigg]_{v=1}^9 = \frac{26}{3}.$$

(d) Let $u = e^x$, $du = e^x\,dx$.

$$\int \frac{du}{1+u^2} = \arctan u + C = \arctan e^x + C.$$

Optimization

1. Suppose you want to construct a window in the shape of a rectangle topped with a semi-circle. Find the window with maximal area subject to the constraint that the window's perimeter must be 6 feet.

Solution: It's a good idea to draw a picture. I won't. Let r be the radius of the semicircle and ℓ the length of the side of the rectangle, so that the perimeter is given by $2\ell + 2r + \pi r$. The function we want to maximize is

$$A = \frac{1}{2}\pi r^2 + 2r\ell,$$

subject to the constraint

$$6 = 2\ell + 2r + \pi r.$$

Solve the latter equation for ℓ. We get $\ell = 3 - r - \pi/2r$. Substitute this into the area equation:

$$A = \frac{1}{2}\pi r^2 + 2r(3 - r - (1/2)\pi r) = \frac{1}{2}\pi r^2 + 6r - 2r^2 - \frac{pi}{2}r^2 = 6r - (2+\pi/2)r^2.$$

Then

$$\frac{dA}{dr} = 6 - (4+\pi)r.$$

Set this equation to 0 and solve:

$$0 = 6 - (4+\pi)r \qquad\qquad r = \frac{6}{4+\pi}.$$

Now we plug this back into our equation for area:

$$A = 6\left(\frac{6}{4+\pi}\right) - \left(2+\frac{\pi}{2}\right)\left(\frac{6}{4+\pi}\right)^2 \approx 2.52\ldots$$

You should verify that this is indeed a maximum (and not a minimum!)

---

2. You have 40 feet of fencing to create a rectangular pig pen. You use the wall of your house for one side of the fence. What is the maximal area pig pen you can create (you may assume your house is very long).

Solution: Again it's a good idea to draw a picture. The area of a rectangle is given by $A = \ell \cdot w$ and the perimeter is $2\ell + 2w$. In our case, only three sides of pen are made of fence, so we have the constraint $2\ell + w = 40$. That is, $w = 40 - 2\ell$. Now we have

$$A = \ell(40 - 2\ell) = 40\ell - 2\ell^2.$$

Take derivatives and solve:

$$\begin{aligned} \frac{dA}{d\ell} &= 40 - 4\ell \\ 0 &= 40 - 4\ell \\ 10 &= \ell. \end{aligned}$$

Thus the maximal possible area is $A = 40(10) - 2(10)^2 = 200$ square feet. Again you should check that this is a maximum (second derivative test).

Limits

1. Find all (vertical and horizontal) asymptotes of the following functions:

(a) $f(t) = t^{-2}$ (b) $q(x) = \dfrac{x^2 - 4x + 3}{x^2 - 5x + 6}$ (c) $p(s) = \dfrac{1}{\ln(s)}$

Solution: Remember, vertical asymptotes occcur where the function blows up; horizontal asymptotes are the limits as $x \to \pm\infty$.

(a) It's pretty clear that $f(t)$ blows up at $t = 0$, so that's a vertical asymptote. As $t \to \pm\infty$ we have $f(t) \to 0$, so $y = 0$ is the only horizontal asymptote.

(b) First we factor and simplify:

$$q(x) = \frac{x^2 - 4x + 3}{x^2 - 5x + 6} = \frac{(x-3)(x-1)}{(x-3)(x-2)} = \frac{x-1}{x-2}.$$

Certainly $x = 2$ is a vertical asymptote (if you don't believe me, compute the left and right limits as $x \to 2$). You can also verify that

$$\lim_{x\to\infty} = \lim_{x\to-\infty} = 1.$$

(c) A vertical asymptote will occur when the denominator is 0. We know that $ln(s) = 0$ when $s = 1$ (why?) so that's our vertical asymptote. Now we compute

$$\lim_{s\to\infty} \frac{1}{\ln(s)} = 0.$$

Note that there's no need to calculate the limit as $s \to -\infty$ because $\ln(s)$ isn't defined for negative numbers.

2. Evaluate the following limits:

$$\text{(a)} \lim_{x\to\infty} \frac{3x^9 - 12x^3 + 1}{7x^4 - 6x^9} \qquad \text{(b)} \lim_{x\to 0} \frac{\sin(x)}{e^x - 1} \qquad \text{(c)} \lim_{x\to 0} \frac{|x|}{x}$$

Solutions:

(a) It's fine to use L'ôpital's rule here or remember that we can compare the highest power terms in the numerator and denominator. In either case, we see the limit is ∞.

(b) Here we need L'ôpital's rule because both numerator and denominator go to 0. Taking derivatives we get

$$\lim_{x\to 0} \frac{\cos(x)}{e^x} = \frac{1}{1} = 1.$$

(c) The limit does not exist. Look at the right and left limits and observe that they are different.

Fundamental Theorem of Calculus

1. State the Fundamental Theorem of Calculus (both flavors). What properties must a function $f(x)$ have in order for the fundamental theorem of calculus to apply?

Solution: (Flavor 1) Let $f(x)$ be a function continuous on $[a, b]$. Define

$$F(x) = \int_a^x f(t)\, dt.$$

Then for all x in (a, b), $F'(x) = f(x)$. (Flavor 2) Suppose $f(x)$ and $F(x)$ be functions such that $F'(x) = f(x)$. If f is integrable then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

---

2. Use the fundamental theorem of calculus to evaluate the following expressions:

$$\text{(a)} \int_0^4 9x^2 - 6x + 1\, dx \qquad \text{(b)} \int_{-3}^3 x^3 + 4x\, dx$$

$$\text{(c)} \int_0^\pi \cos(x)\, dx \qquad \text{(d)} \frac{d}{dt} \int_{17}^t e^{(x^2)}\, dx$$

$$\text{(e)} \frac{d}{dq} \int_{2q}^5 e^{8x} \tan(x)\, dx \qquad \text{(f)} \frac{d}{dy} \int_{2y}^{\cos(3y^2)} e^{(-1/x^2)}\, dx$$

Solutions: I won't write out the whole solutions. If you don't understand a step, try to work it out yourself!

(a) $3x^3 - 3x^2 + x]_0^4 = 148$.

(b) 0 (it's an odd function!)

(c) $\sin(x)]_0^\pi = 0$.

(d) $e^{(t^2)}$.

(e) $-2e^{16q} \tan(2q)$.

(f) $-e^{-1/(\cos(3y^2))} \cdot \sin(3y^2) \cdot 6y - e^{-1/(2y)^2} \cdot 2$.

Related Rates

1. Gregor Clegane fills a cylindrical tank with some lava. Suppose the tank has radius 2 meters. If Gregor is pouring lava at a rate of 3 cubic meters per second, at what rate is the lava level rising? (You may assume the tank is made of mithril.)

Solution: The volume of a cylinder is $V = 2\pi r^2 h$. We are given $dV/dt = 3$ and $r = 2$. Taking derivatives,

$$\begin{aligned} \frac{dV}{dt} &= 2\pi r^2 \frac{dh}{dt} \\ 3 &= 2\pi \cdot 4 \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{3}{8\pi} \quad \text{(cubic meters per second).} \end{aligned}$$

Derivatives

1. Compute the derivative of the following functions:

(a) $\cos(\tan(x))$ (b) $\sqrt{\dfrac{e^x - 4x}{7x^3 + 4x}}$ (c) $\tan^{-1}(\ln(x))$

Solutions: Chain rule baby!

(a) $-\sin(\tan(x)) \cdot \sec^2(x)$.

(b)

$$\frac{1}{2}\left(\frac{e^x - 4x}{7x^3 + 4x}\right) \cdot \frac{(7x^3 + 4x)(e^x - 4) - (e^x - 4x)(21x^2 + 4)}{(7x^3 + 4x)^2}$$

(c)

$$\frac{1}{1 + (\ln x)^2} \cdot \frac{1}{x}$$

---

2. Use implicit differentiation to find dy/dx, where $xy^2 + \cos(y) - \sin(x) = \ln(2y)$.

Solution

$$\begin{aligned} xy^2 + \cos(y) - \sin(x) &= \ln(2y) \\ 2xyy' + y^2 - \sin(y)y' - \cos(x) &= \frac{1}{2y} 2y' \\ y'(2xy - \sin(y) - (1/y)) + y^2 - \cos(x) &= 0 \\ y' &= \frac{\cos(x) - y^2}{2xy - \sin(y) - (1/y)}. \end{aligned}$$

The Shape of Functions

1. Let $f(x) = -x^3 - 4x^2 + 3x + 18$.

(a) Find and classify all critical points of $f(x)$.

(b) On what intervals is $f(x)$ increasing? Decreasing?

(c) Where is $f(x)$ concave up? Concave down?

(d) Sketch a graph of $f(x)$.

Solutions:

(a) $f'(x) = -3x^2 - 8x + 3 = -(3x-1)(x+3)$. Critical points occur at $x = -3$ and $x = 1/3$. We can tell $x = -3$ is a minimum, either by the second derivative test or by plugging in points to either side and observing that the first derivative changes sign from - to +. The point $x = 1/3$ is a maximum.

(b) We consider the intervals $(-\infty, -3)$, $(-3, 1/3)$, $(1/3, \infty)$. By checking the sign of the first derivative, we see that $f(x)$ is decreasing on $(-\infty, -3) \cup (1/3, \infty)$ and increasing on $(-3, 1/3)$.

(c) Now we compute $f''(x) = -6x - 8$. Points of inflection are at $x = 8/6 = 4/3$. Here the second derivative changes from positive to negative, so that f is concave up on $(-\infty, 4/3)$ and concave down on $(4/3, \infty)$.

---

2. Draw a function $g(x)$ satisfying ALL of the following: $g(-2) = 0$; $g'(x) > 0$ for all x; $g''(-1) = 0$; $g''(x) > 0$ for $x < -1$; $g''(x) < 0$ for $x > -1$.

Solution: I can't draw, but this should look something like a cube root function.

Integration

1. Suppose f and g are continuous functions and

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

True or false: $f(x) = g(x)$.

Solution: *False.* For example, let $f(x) = \sin(x)$, $g(x) = x$, $a = 1$ $b = -1$. Then both integrals are 0, but these functions are clearly different.

---

2. Let

$$f(x) = \begin{cases} 1 & x > 3 \\ 2 & x \leq 3 \end{cases}$$

Compute $\displaystyle\int_0^5 f(x)\,dx$.

Solution: We have to break this up into two integrals:

$$\begin{aligned} \int_0^5 f(x)\,dx &= \int_0^3 2\,dx + \int_3^5 dx \\ &= 2x]_0^3 + x]_3^5 \\ &= 6 + 2 \\ &= 8. \end{aligned}$$

---

3. Compute the following integrals:

(a) $\displaystyle\int xy^2\ dx$

(b) $\displaystyle\int xy^2\,dy$

Solution: The whole point of this problem is that these integrals are *different*. It matters which variable you integrate.

(a)

$$\int xy^2 \; dx = y^2 \int x \, dx = \frac{1}{2} y^2 x^2$$

(b)

$$\int xy^2 \, dy = x \int y^2 \, dy = \frac{1}{3} x y^3$$