3. Let $g(t) = \dfrac{|t^2 - 4|}{t - 5}$.

(a) What is the domain of g?

(b) Where is g differentiable? Where is g continuous?

(c) Find $g'(t)$ wherever it exists.

(d) Notice from your answers to (b) (if you got it right) that there is a point t_0 where g is continuous but *not* differentiable. Compute:

$$\lim_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h} \qquad \lim_{h \to 0^-} \frac{f(t_0 + h) - f(t_0)}{h}$$

(There was a typo in the original version of this problem that I gave you in part (d) – I had you compute a limit that wasn't very interesting)

Solution:

(a) The domain of g is the set of values t is allowed to take. The only thing t *cannot* be is 5. Thus the domain is $(-\infty, 5) \cup (5, \infty)$ (you may also write $\{x \in \mathbb{R} : x \neq 5\}$ or even $\mathbb{R} \setminus \{5\}$).

(b) The function g is continous everywhere on its domain; it is not continuous at 5. Thus g is continuous on $(-\infty, 5) \cup (5, \infty)$. Now g is differentiable everywhere except $t = \pm 2$. To see this, it might help to graph g. Alternatively, write:

$$g(t) = \begin{cases} \dfrac{t^2 - 4}{t - 5} & |t| \geq 2 \\ \\ \dfrac{4 - t^2}{t - 5} & |t| < 2 \end{cases}$$

We will compute in part (d) what the derivative of g is near ± 2 and see that it is not differentiable there.

(c) Using our formula from above, so long as $t \neq \pm 2$ we can use our derivative rules to compute $g'(t)$. We get:

$$g'(t) = \begin{cases} \dfrac{(t-5)2t - (t^2 - 4)}{(t-5)^2} & |t| > 2 \\ \\ \dfrac{-(t-5)2t - (t^2 - 4)}{(t-5)^2} & |t| < 2 \end{cases}$$

(d) Now we can compute the "left-" and "right-derivative" at ± 2. I'll do the case $t_0 = 2$, and you can try $t_0 = -2$ for yourself.

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} &= \frac{(2-5)(2\cdot 2)-(2^2-4)}{(2-5)^2} \\
&= \frac{-12}{9} \\
\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} &= \frac{-(2-5)(2\cdot 2)-(2^2-4)}{(2-5)^2} \\
&= \frac{12}{9}.
\end{aligned}$$

Notably, these limits are different!