Calculus I | # Midterm II Review | **Spring 2015**

1. Let $y = \tan^{-1}(x)$. Use implicit differentiation to find dy/dx.

Solution: This one wasn't worded very carefully. Sorry. I wanted you to *prove* that the derivative of $\tan^{-1}$ is $1/(1+x^2)$. Here's how:

$$\begin{aligned} y &= \tan^{-1} x \\ \tan y &= x \\ \sec^2 y \cdot y' &= 1 \\ y' &= \frac{1}{\sec^2 y}. \end{aligned}$$

At this point you need to show that $\sec^2 y = 1 + x^2$. There are a couple ways to do this. You can draw a triangle as we did in discussion. The other way is to use some trig identities. For example, $\sec^2 y = 1 + \tan^2 y$ and in this problem $\tan y = x$.

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2. Find and classify all critical points of $f(x) = x/\ln(x)$.

Solution: How do we find critical points? Critical points occur when the derivative doesn't exist or when it's zero. Let's calculate:

$$f'(x) = \frac{\ln(x) - x\frac{1}{x}}{\ln(x)^2} = \frac{\ln(x) - 1}{\ln(x)^2}$$

This is 0 precisely when $\ln(x) = 1$ (set the numerator to 0). When is $\ln(x) = 0$? When $x = e$. Thus $x = 1e$ is a critical point of $f(x)$. The other critical points are when the derivative doesn't exist. This happens at $x = 1$ (then the denominator is 0). To classify $x = e$, we could do a sign diagram or compute the second derivative. I'll do the latter:

$$f''(x) = \frac{2 - \ln(x)}{x \log^3(x)}.$$

When $x = e$ the numerator is 1 and the denominator is positive. Thus $f''(e) > 0$, so e is a *local minimum.*

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3. Find the derivative:

(a) $\sin(x)^x$ (b) $\log_7(x^2+1)$ (c) $(3x^2+4x)^{18}$ (d) $\cos^{-1}(\sqrt{x})$

Solution:

(a) The trick to remember is $\sin(x)^x = e^{x\ln(\sin(x))}$. The derivative of this (using the chain and product rules) is

$$e^{x\ln(\sin(x))} \cdot \left(\ln(\sin(x)) + \frac{x\cos(x)}{\sin(x)}\right)$$

(a) This is the chain rule and the log derivative rule. Here's the answer:

$$\frac{1}{\ln(7)(x^2+1)} \cdot 2x$$

(a) Chain rule, baby:

$$18(3x^2+4x)^{17} \cdot (6x+4)$$

(a) Ditto:

$$-\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-3/2}.$$

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4. Let $f(x) = e^{2x-1}$. Find $(f^{-1})'(1)$.

Solution: We'll use the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

We need two pieces of information: $f'(x)$ and $f^{-1}(1)$. By inspection or by taking logs, we see that $f(1/2) = 1$, so $f^{-1}(1) = 1/2$. Further,

$$f'(x) = e^{2x-1} \cdot 2.$$

Notice $f'(1/2) = 1$. Now we can use the formula:

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(1/2)} = \frac{1}{2}$$

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5. Let

$$f(x) = \frac{x^3}{3} + x^2 - 3x + 1.$$

(a) Find and classify all critical points. (b) On what intervals is $f(x)$ increasing? (c) On what interval is $f(x)$ decreasing?

Solution:

(a) Critical points occur when the derivative is zero:

$$f'(x) = x^2 + 2x - 3.$$

We can find the roots of this easily: $f'(x) = (x+3)(x-1)$. So f has critical points at -3 and 1. Let's classify with the second derivative test:

$$f''(x) = 2x + 2$$
$$f''(-3) = -4$$
$$f''(1) = 4$$

Since $f''(-3) < 0$ it's a local maximum. Since $f''(1) > 0$ it's a local minimum.

(a) Remember f increases when its derivative is positive. We can check between the zeroes of the derivative to figure out what f is doing. For example, when $x = -4$ we have $f'(-4) = 5 > 0$ so $f(x)$ is increasing for ALL x in $(\infty, -3)$. Similarly, f is decreasing on $(-3, 1)$ and increasing again on $(1, \infty)$.

(a) As I said above, f decreases on $(-3, 1)$.

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6. A joyous calculus student throws her calculus textbook into the air in a fit of exuberance. She is standing on the roof of BSB, which is 50 feet in the air. The height of the book at time t is given by

$$h(t) = -15t^2 + 25t + 50.$$

(a) Find the velocity of the book at t seconds. (b) When is the book at its highest point? (c) When does the book hit the ground?

Solution:

(a) Velocity is defined as the derivative of distance, so we take derivatives:

$$v(t) = h'(t) = -30t + 25.$$

(a) The book hits its highest point at an absolute maximum! We know how to find these: find all *local* maxima, and compare those with the limits as $x \to \pm\infty$. Well,

$$0 = -30t + 25 \Rightarrow t = 25/30 = 5/6.$$

You can check (use whatever method you like) that this is a local maximum. Further, $\lim_{t\to\pm\infty} h(t) = -\infty$ so $t = 5/6$ is an *absolute* maximum.

(a) This is just algebra. The book hits the ground when $h(t) = 0$. This has two roots, $t = \frac{1}{6} \cdot (5 \pm \sqrt{145})$. Only one of these makes sense physically. You pick.

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7. Consider the function $f(x) = -x^4 + 2x^2 - 3$. Does it have an absolute maximum? Absolute minimum?

Solution: Like in 6 (b) we have to find local maxima. Take derivative:

$$f'(x) = -4x^3 + 4x.$$

This has roots $x = -1, 0, 1$. You can check on your own that -1 and 1 are local maxima, while $x = 0$ is a local minimum. Notice that $f(1) = f(-1) = -2$. If f is to have an absolute maximum, the limits as $x \to \infty$ have to be less than -2. Indeed, $\lim_{x\to\pm\infty} f(x) = -\infty$. Thus -2 is the absolute maximum of $f(x)$ and is attained when $x = \pm 1$.

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8. Draw the graph of a function satisfying ALL of these properties: (i) $f(0) = 0$. (ii) $f'(0) = 1$. (iii) $f'(2) = 0$. (iv) $x = 4$ is an absolute maximum.

Solution: I don't know how to TeX graphs! You're on your own.

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9. A differentiable function $f(x)$ satisfies $f(0) = 0$, $f(\pi) = 3$, $f'(\pi) = 2$, $f'(0) = 4$. Find the equation of the line tangent to $f(\sin(x))$ at the point $x = \pi$.

Solution: We need to know a few things: the coordinates of the point $(\pi, f(\sin(\pi))$ and the derivative of $f(\sin(x))$ at that point. Behold:

$$f(\sin(\pi)) = f(0) = 0 \qquad\qquad (f(\sin(x))' = f'(\sin(x)) \cdot \cos(x).$$

Thus at $x = \pi$, the derivative is $f'(0) \cdot 1 = f'(0) = 4$. We have the equation:

$$y = 4(x - \pi).$$