On Zero-sum Flows in Hypergraphs

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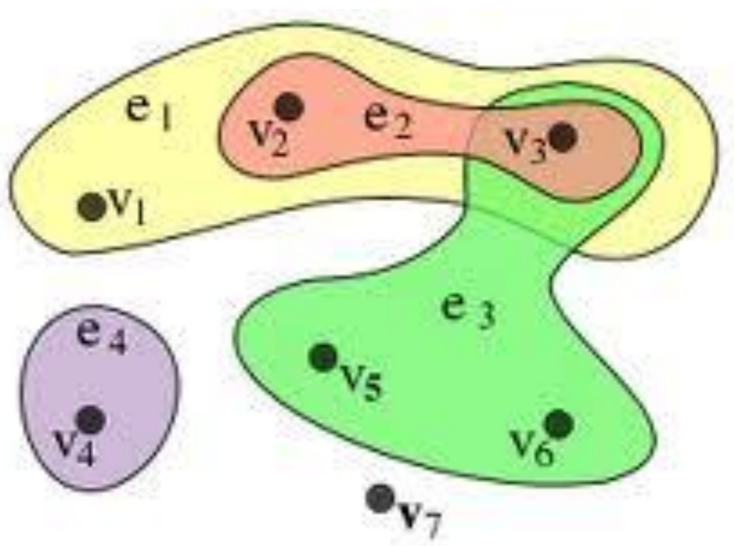
ICTP-IPM Workshop and Conference in Combinatorics and Graph Theory, Trieste, Italy

3 September-14 September 2012

- **Definition.** A hypergraph H is an ordered pair (V, E) where V is a set of vertices and E is a set of hyperedges such that $E \subseteq P(V)$.

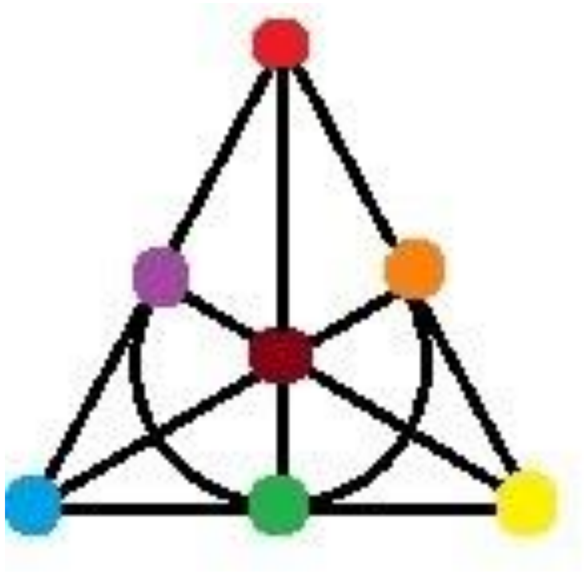
- **Definition.** The incidence matrix of a hypergraph is a $(0,1)$ –matrix which has a row for each vertex and a column for each hyperedge, and $(v, E)=1$ if and only if vertex v is incident upon hyperedge E.



	e_1	e_2	e_3	e_4
v_1	1	0	0	0
v_2	1	1	0	0
v_3	1	1	0	1
v_4	0	0	1	0
v_5	0	0	0	1
v_6	0	0	0	1
v_7	0	0	0	0

- **Definition.** The hypergraph H is called k-uniform if the size of every hyperedge is k.

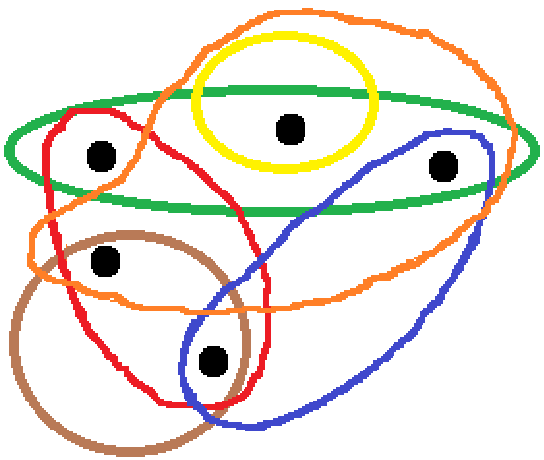
- **Definition.** The hypergraph H is called m-regular if the degree of every vertex is m.



A 3-uniform 3-regular hypergraph

- **Definition.** Let H be a hypergraph with the incidence matrix N. H admits zero-sum flow if there exists a nowhere-zero vector u in the null space of N.

- **Definition.** The hypergraph H admits a zero-sum k-flow if the enteries of u are in the set $\{\pm 1, \ldots, \pm(k-1)\}$



Hypergraph H

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A zero-sum 4-flow for H.

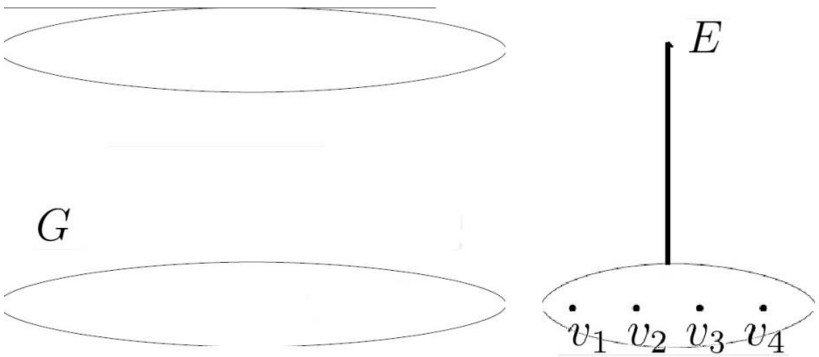
- **Theorem.** For every hypergraph H, there exists a 3-regular 3-uniform hypergraph H' such that H' has a zero-sum flow if and only if H has a zero-sum flow.

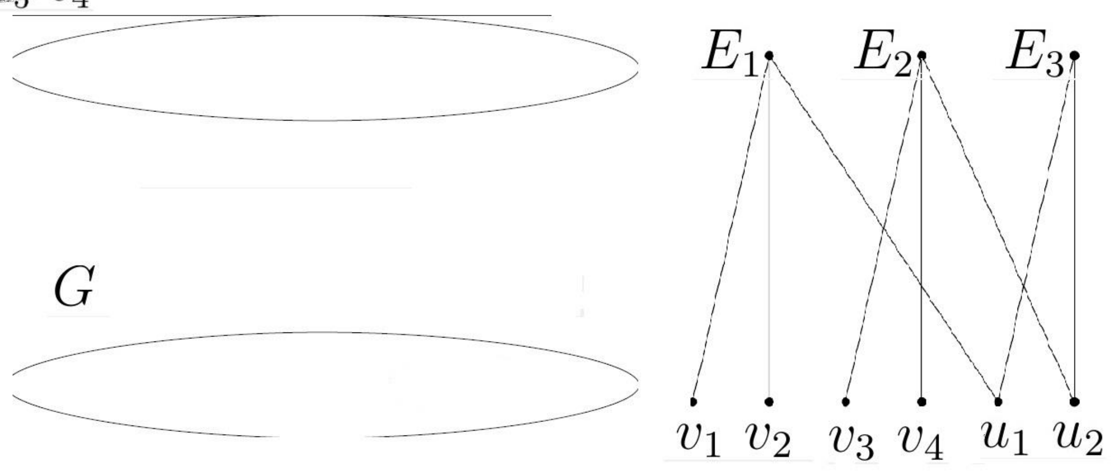
- **Lemma 1.** For every hypergraph H_1 there exists a hypergraph H_2 such that:

i. The size of hyperedges of H_2 do not exceed 3.

ii. The hypergraph H_1 has a zero-sum flow if and only if H_2 has a zero-sum flow.

- Example for the case |E|=4

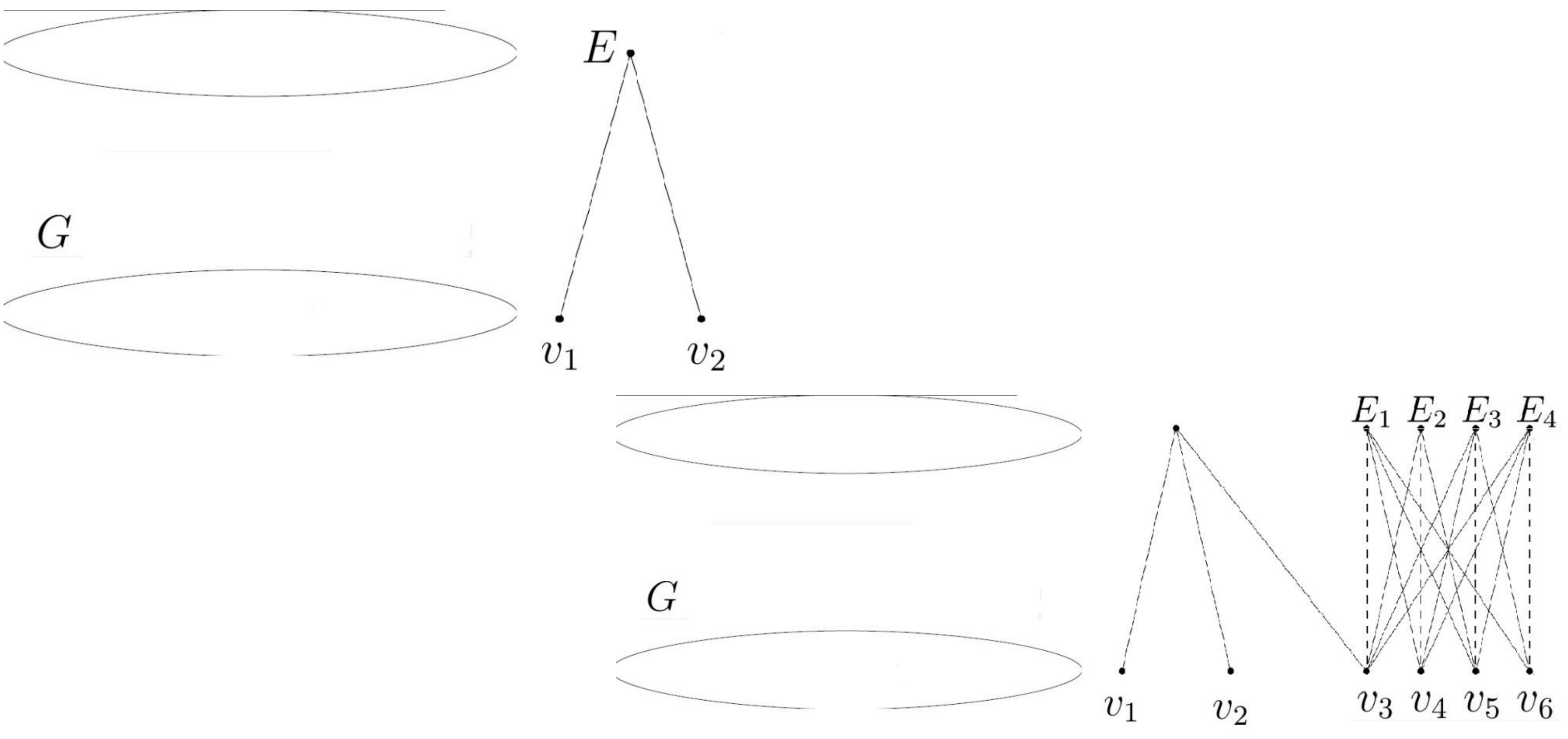




❑**Lemma 2.** For hypergraph H_2 (defined in the previous lemma) there exists a hypergraph H_3 such that:

i. The hypergraph H_3 is 3-uniform.
ii. The hypergraph H_3 has a zero-sum flow if and only if H_2 has a zero-sum flow.

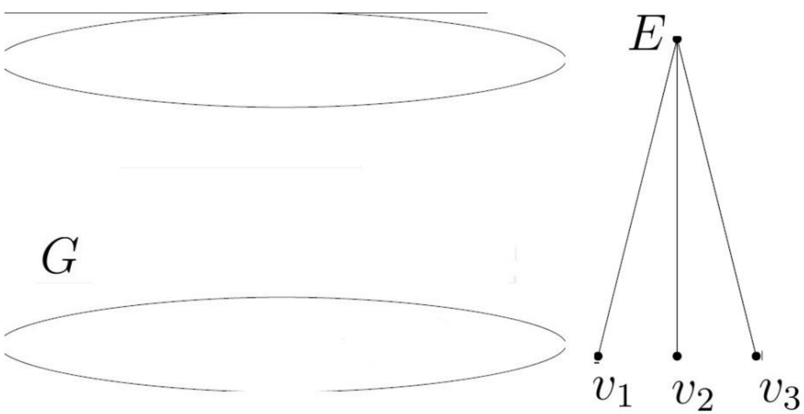
- Example for the case |E|=2

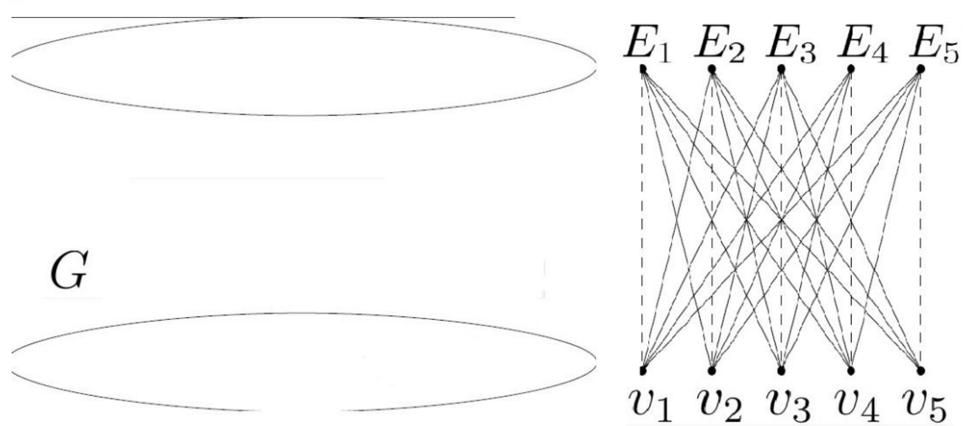


- **Lemma 3.** For the hypergraph H_3 (defined in the previous lemma) and $k \geq 3$, there exists a hypergraph H_4 such that:

i. The hypergraph H_4 is k-uniform.
ii. The hypergraph H_4 has a zero-sum flow if and only if H_3 has a zero-sum flow.

- Example for the case k=4

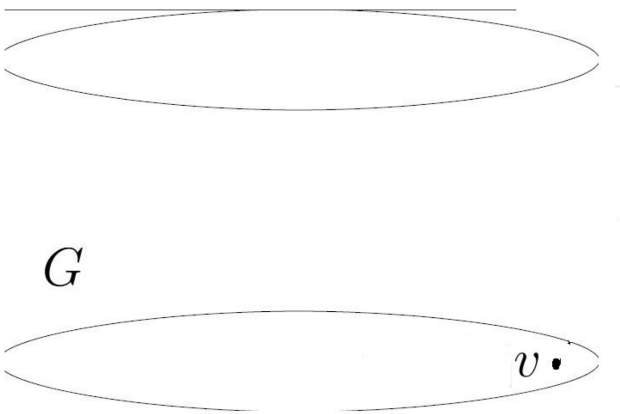


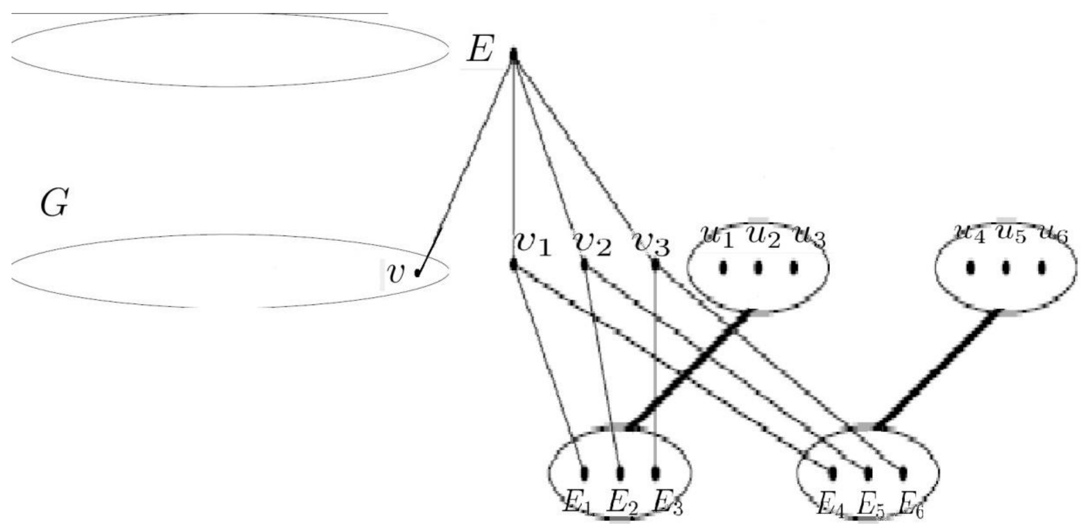


- **Lemma 4.** For the hypergraph H_4 (defined in the previous lemma) and integers $k \geq m \geq 3$, there exists a hypergraph H_5 such that:

i. The hypergraph H_5 is k-uniform.
ii. The degree of each vertex of H_5 is divisable by m.
iii. The hypergraph H_5 has a zero-sum flow if and only if H_4 has a zero-sum flow.

- Example for the case k=4, m=3, d(v)=5

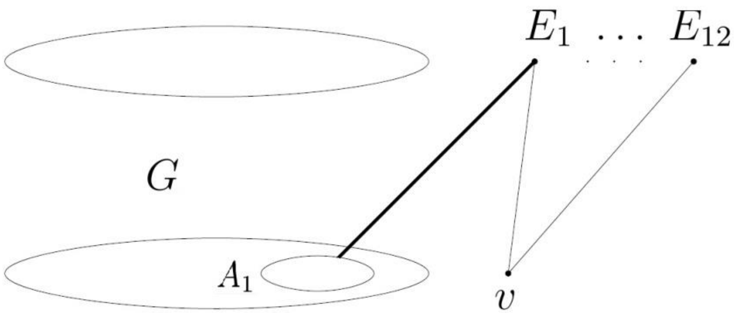


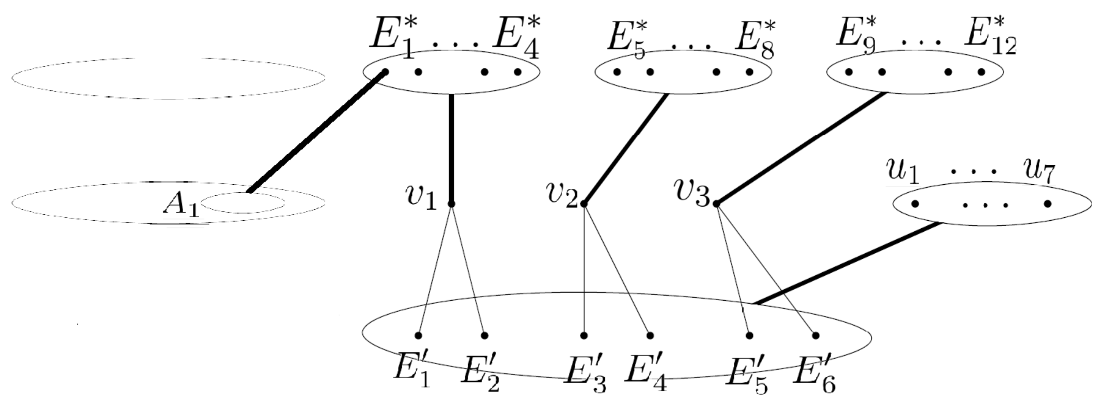


❑ **Lemma 5.** For the hypergraph H_5 (defined in the previous lemma) and integers $k \geq m \geq 3$, there exists a hypergraph H_6 such that:

i. The hypergraph H_6 is k-uniform.
ii. The hypergraph H_6 is m-regular.
iii. The hypergraph H_6 has a zero-sum flow if and only if H_5 has a zero-sum flow.

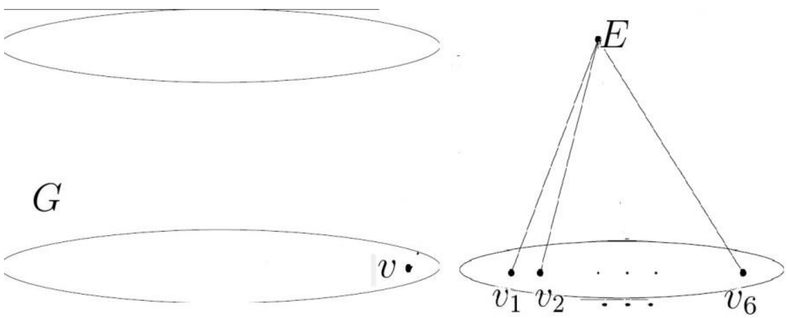
- Example for the case m=6, k=8 and d(v)=12

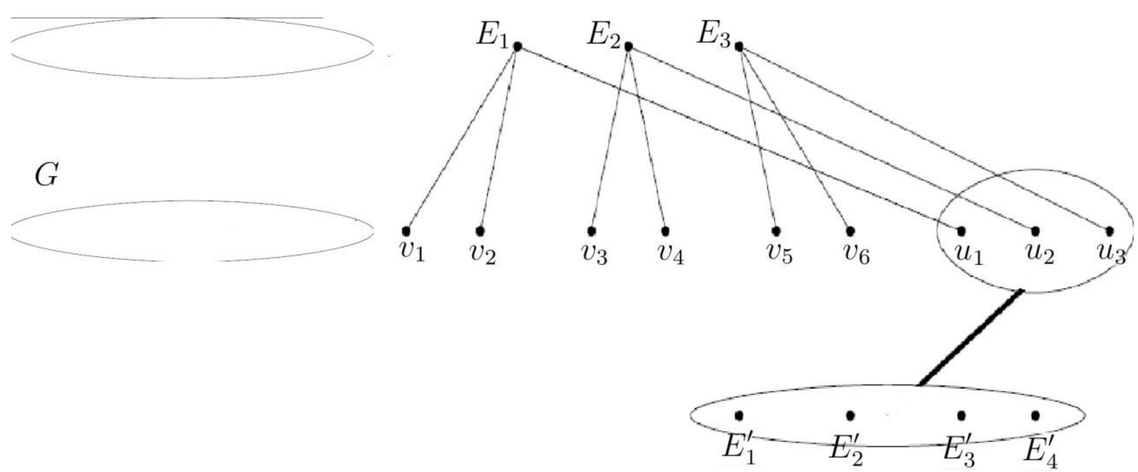




❑ **Lemma 6.** The previous lemma holds in the case $m > k \geq 3$.

- Example for the case k=3, m=5





- **Theorem.** For every hypergraph H, and two integers k , $m \geq 3$, there exists a k-regular m-uniform hypergraph H' such that:

  The hypergraph H has a zero-sum flow, if and only if H' has a zero-sum flow.

- Furthermore, if H' has a zero-sum n-flow, then H has a zero-sum n-flow as well.

- For every positive integer n, there exists a 3-uniform 3-regular hypergraph which has a zero-sum flow, but has no zero-sum n-flow.

THANK YOU FOR YOUR ATTENTION