Paschke Categories, K-homology and Riemann-Roch Theorem

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C^*-Categories

A C^*-*category* is a category enriched in the category of Banach spaces with a $*$-operation on the morphisms. For example:

- Category $\mathfrak{B}$ of Hilbert spaces H where morphisms are bounded operators $\mathfrak{B}(H^1, H^2)$ between them.
- Category $\mathfrak{B}/\mathfrak{K}$ of Hilbert spaces H where morphisms are bounded operators modulo compact operators $\mathfrak{B}(H^1, H^2)/\mathfrak{K}(H^1, H^2)$ between them.
- For a C^*-algebra A, and a C^*-category $\mathfrak{A}$, the category $\mathfrak{Rep}_{\mathfrak{A}}(A)$ of representations $\rho : A \to \mathfrak{A}(H)$, where H is an object in $\mathfrak{A}$, and morphisms from $\rho^1 : A \to \mathfrak{A}(H^1)$ to $\rho^2 : A \to \mathfrak{A}(H^2)$ are $\mathfrak{A}(H^1, H^2)$.

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Definition of Paschke Categories

Let A be a C^*-algebra, and let $\rho^1 : A \to \mathfrak{B}(H^1)$ and $\rho^2 : A \to \mathfrak{B}(H^2)$ be two representations. Then a bounded operator $T : H^1 \to H^2$ is called **pseudo-local** if for each $a \in A$, $\rho^2(a)T - T\rho^1(a)$ is compact.
It is called **locally compact** if for each $a \in A$, both $\rho^2(a)T$ and $T\rho^1(a)$ are compact.

Definition

Let $\mathfrak{D}$ be the category of representations of A with pseudo-local operators between them and $\mathfrak{C}$ be the ideal subcategory of representations of A with locally compact operators between them. Then their quotient category $\mathfrak{D}/\mathfrak{C}$ is called the Paschke category.

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Exact Structure on Paschke Categories

Define an exact structure on the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$ by saying that a complex

$$\dots \xrightarrow{T^{i-1}} (\rho^i, H^i) \xrightarrow{T^i} (\rho^{i+1}, H^{i+1}) \xrightarrow{T^{i+1}} (\rho^{i+2}, H^{i+2}) \xrightarrow{T^{i+2}} \dots$$

is **exact** if there is a contracting homotopy.

Let $f : A \to B$ be a map of C^*-algebras. Then by precomposing a representation $\rho : B \to \mathfrak{B}(H)$ by f, we obtain an exact pull-back functor $f^* : (\mathfrak{D}/\mathfrak{C})_B \to (\mathfrak{D}/\mathfrak{C})_A$.

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K-Theory of Paschke Categories

Since $(\mathfrak{D}/\mathfrak{C})_A$ is an exact C^*-category, we can perform the Waldhausen $S.$ construction on it, and consider its *fat* geometric realization to get a bisimplicial space $\Omega\|wS.(\mathfrak{D}/\mathfrak{C})_A\|$.

Theorem

Let A be a C^*-algebra. Then for $i \geq 1$, the Waldhausen K-theory groups $K_i((\mathfrak{D}/\mathfrak{C})_A)$ are isomorphic to topological K-homology groups $K^{1-i}(A)$.

Corollary

Let X be a "nice" topological space Then

$$K_i((\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \cong K_{i-1}(X) \text{ for } i \geq 1,$$

where the latter is the Steenrod topological K-homology of X.

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Proof Sketch

Let $\rho : A \to \mathfrak{B}(H)$ be an *ample* representation. Then define **Paschke Dual** $\mathfrak{Q}_\rho(A)$ to be the quotient of pseudo-local operators modulo locally compact operators with respect to ρ.

Voiculescu's theorem says that isomorphism class of the C^*-algebra $\mathfrak{Q}_\rho(A)$ doesn't depend on the choice of the ample representation ρ.

Theorem (Paschke)

The K-theory groups $K_i(\mathfrak{Q}(A))$ of the Paschke dual are isomorphic to the K-homology groups $K^{1-i}(A)$ of the C^*-algbera A.

The subcategory of ample representations in $(\mathfrak{D}/\mathfrak{C})_A$ is strictly cofinal. The rest follows from Waldhausen's cofinality theorem.$\square$

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The Dolbeault Complex

Goal: Construct the Riemann-Roch map $K_0^{hol}(X) \to K_0^{top}(X)$ for a complex space X. A complicated construction for this was given by Roni Levy in Acta Math, vol 158(1987).

Let X be a complex manifold of dimension n (not necessarily compact), and let E be a holomorphic vector bundle on X. Then we have the **Dolbeault resolution** of $\mathcal{O}(E)$:

$$0 \to \mathcal{O}(E) \hookrightarrow \mathcal{A}^{0,0}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(E) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}^{0,n}(E) \to 0.$$

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Completed Dolbeault Complex on Compact Manifolds

If X is compact, fix a hermitian metric on X and on the vector bundle E. Define:

- The Hilbert spaces $H^i = L^2$-completion of $\Gamma(X, \mathcal{A}^{0,i}(E))$.
- The representations $\rho^i : C_0(X) \to \mathfrak{B}(H^i)$ by point-wise multiplication.
- The bounded operator $\chi(D)$ by applying *functional calculus* on $D = \bar{\partial} + \bar{\partial}^*$, where $\chi(t) = \frac{t}{\sqrt{1+t^2}}$. Let $\chi_i(D)$ denote the restriction to $\mathfrak{B}(H^i, H^{i+1})$.

Then the sequence $\tau'_X(E)$

$$0 \to \rho^0 \xrightarrow{\chi_0(D)} \rho^1 \xrightarrow{\chi_1(D)} \dots \xrightarrow{\chi_{n-1}(D)} \rho^n \to 0$$

is an exact sequence in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$.

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The Functor τ'

The sequence $\tau'_X(E)$ in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$ doesn't depend on the choice of hermitian metric.
If we choose a *metric of bounded geometry* then by a result of Engel, the argument of why $\tau'_X(E)$ is an exact sequence when X is compact, goes through in the non-compact case as well.

τ' extends to a functor $\tau'_X : \mathbf{M}(X) \to Ch^b(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$ from the exact category of vector bundles on X to the category of acyclic bounded chain-complexes in $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$, which is exact.

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Double Exact Sequences

Let $\mathcal{A}$ be an exact category, and let $Bi(\mathcal{A})$ be the exact category of bounded binary (double) acyclic chain complexes in $\mathcal{A}$. Grayson shows that there is a map $\mathbb{K}(Bi(\mathcal{A})) \to \Omega\mathbb{K}(\mathcal{A})$.

So far, we have a map $\tau'_X : \mathbb{K}^{hol}(X) \to \mathbb{K}^{top}(Ch^b(\mathfrak{D}/\mathfrak{C})_{C_0(X)})$.

For a bounded operator $T : H^1 \to H^2$, we can assign an automorphism to it:

$$\begin{array}{ccccccccccccc} \dots & \oplus & H^1 & \oplus & H^1 & \oplus & H^1 & \oplus & H^2 & \oplus & H^2 & \oplus & \dots \\ & & & \searrow & & \searrow & & \overset{T}{\searrow} & & \searrow & & & \\ \dots & \oplus & H^1 & \oplus & H^1 & \oplus & H^1 & \oplus & H^2 & \oplus & H^2 & \oplus & \dots \end{array}$$

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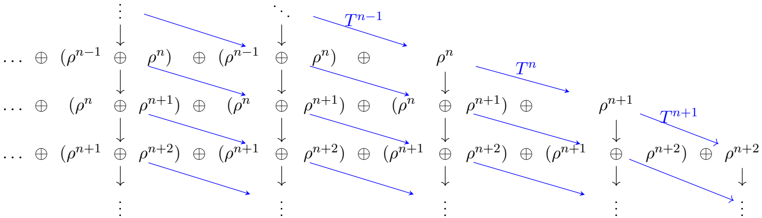
The Large Diagram

We can generalize this process to assign a binary (acyclic) chain complex to a (an acyclic) chain complex $(\rho^{\cdot}, T^{\cdot})$ in the Paschke category as follows:



The Riemann-Roch Theorem

Hence we get the functor

$$\tau_X : \mathbb{K}^{hol}(X) \xrightarrow{\tau'_X} \mathbb{K}^{top}(Ch^b(\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \to \mathbb{K}^{top}(Bi(\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \to \Omega\mathbb{K}^{top}((\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \cong \mathbb{K}^{top}(X).$$

Theorem (Riemann-Roch)

Let X, Y be complex manifolds, $\iota : U \to X$ be inclusion of an open subset, and let $f : X \to Y$ be a proper, smooth and surjective morphism. Then we have the commutative diagrams below.

$$\begin{array}{ccc} K^{hol}(X) & \xrightarrow{\iota^!} & K^{hol}(U) \\ \downarrow \tau_X & & \downarrow \tau_U \\ K^{top}(X) & \xrightarrow[\iota^*]{} & K^{top}(U) \end{array} \qquad \begin{array}{ccc} K^{hol}(X) & \xrightarrow{Rf_*} & K^{hol}(Y) \\ \downarrow \tau_X & & \downarrow \tau_Y \\ K^{top}(X) & \xrightarrow[f_*]{} & K^{top}(Y) \end{array}$$

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Thank You for Listening!