1. Find and classify the critical points of the function [17 pts]

$$f(x, y) = x^3 + 3xy - y^3.$$

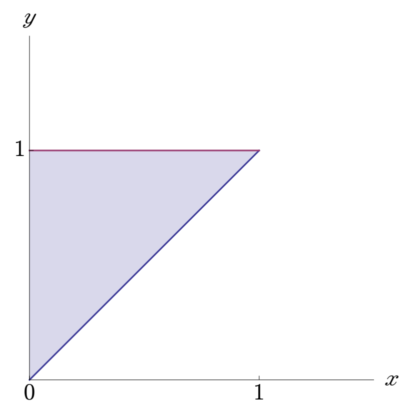
Solution: To find the critical points we solve the system of equations: $f_x = 3x^2 + 3y = 0$, $f_y = 3x - 3y^2 = 0$. Solving the first equation for y we get $y = -x^2$. After plugging into the second equation and simplifying we get the equation $x - x^4 = 0$ whose real solutions are $x = 0$, $x = 1$. When $x = 0$ the first equation gives us $y = 0$ and when $x = 1$ the first equation gives us $y = -1$. Therefore, the critical points are $(0, 0)$ and $(1, -1)$.

Now use the Second Derivative Test to classify each point. The second partial derivatives of $f(x, y)$ are $f_{xx} = 6x, \quad f_{yy} = -6y, \quad f_{xy} = 3$. The discriminant $D(x, y)$ is then $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -36xy - 9$. The values of $D(x, y)$ at the critical points are $D(0, 0) = -9$ and $D(1, -1) = 27$. Since $D(0, 0) < 0$, we know that $(0, 0)$ is a saddle point. Since $D(1, -1) > 0$ and $f_{xx}(1, -1) = 6 > 0$, we know that $(1, -1)$ corresponds to a local minimum.

2. Sketch the region of integration and compute $\displaystyle\int_0^1 \int_0^y e^{-y^2}\, dx\, dy$. [17 pts]

Solution: The region of integration is shown on the right. The value of the integral is

$$\int_0^1 \int_0^y e^{-y^2}\, dx\, dy = \int_0^1 y e^{-y^2}\, dy$$
$$= -\frac{1}{2} e^{-y^2}\Big|_0^1$$
$$= \frac{1}{2}\left(1 - e^{-1}\right)$$



3. Compute

$$\iiint_A z\, dV$$

where A is the region inside the sphere $x^2 + y^2 + z^2 = 2$, inside the cylinder $x^2 + y^2 = 1$, and above the xy-plane. [17 pts]

Solution: It's easiest to solve this problem using cylindrical coordinates. The region is bounded below by the plane $z = 0$ and above by the sphere $z = \sqrt{2 - r^2}$, written in cylindrical coordinates. The projection of A onto the xy-plane is a disk of radius 1 centered at the origin. Thus, the integral and its value is $\displaystyle\iiint_A z\, dV = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{2-r^2}} z\,(r\, dz\, dr\, d\theta) = \frac{3\pi}{4}$.

4. Compute the integral of the field $\vec{F}(x,y) = (x+y)\vec{\imath} + 0\vec{\jmath}$ along the curve $\vec{c}(t) = \cos t\vec{\imath} + \sin t\vec{\jmath}$. [17pts]

Solution: The derivative of $\vec{c}(t)$ is $\frac{d}{dt}\vec{c}(t) = -\sin t\vec{\imath} + \cos t\vec{\jmath}$ and the vector field written in terms of t is $\vec{F}(\vec{c}(t)) = (\cos t + \sin t)\vec{\imath} + 0\vec{\jmath}$. The integral of $\vec{F}$ over the curve is $\int_C \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{c}(t))\cdot \frac{d}{dt}\vec{c}(t)\,dt = \int_0^{2\pi} (-\sin t\cos t - \sin^2 t)\,dt = -\pi.$

5. Find the maximum and minimum of the function $f(x,y,z) = 2x - 3y + z$ subject to the condition $x^2 + 3y^2 + z^2 = 1$. [16 pts]

Solution: We'll use the Method of Lagrange Multipliers to solve the problem. We must solve the system:

$$2 = \lambda(2x), \quad -3 = \lambda(6y), \quad 1 = \lambda(2z), \quad x^2 + 3y^2 + z^2 = 1.$$

Solving the first three equations for x, y, and z in terms of λ we get $x = \frac{1}{\lambda}$, $y = -\frac{1}{2\lambda}$, $z = \frac{1}{2\lambda}$. After plugging these into the fourth equation we get:

$$\left(\frac{1}{\lambda}\right)^2 + 3\left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \iff \lambda^2 = 2 \iff \lambda = \pm\sqrt{2}$$

The corresponding values of x, y, and z are $x = \pm\frac{1}{\sqrt{2}}$, $\quad y = \mp\frac{1}{2\sqrt{2}}$, $\quad z = \pm\frac{1}{2\sqrt{2}}$. The values of $f(x,y,z)$ at each point are

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = 2\left(\frac{1}{\sqrt{2}}\right) - 3\left(-\frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2\sqrt{2}}\right) = \sqrt{8}$$
$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = 2\left(-\frac{1}{\sqrt{2}}\right) - 3\left(\frac{1}{2\sqrt{2}}\right) + \left(-\frac{1}{2\sqrt{2}}\right) = -\sqrt{8}$$

Therefore, the minimum value of f is $-\sqrt{8}$ and the maximum is $\sqrt{8}$.

6. For the vector field $\vec{F}(x,y) = (x+y)\vec{\imath} + (x-y)\vec{\jmath}$, find a function $\varphi(x,y)$ with $\operatorname{grad}\varphi = \vec{F}$ or use the partial derivative test to show that such a function does not exist. [16 pts]

Solution: By inspection, the function $\varphi(x,y) = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 + C$ is a potential function for $\vec{F}(x,y)$. To check this, we take the gradient of φ and find that

$$\operatorname{grad}\varphi = (x+y)\vec{\imath} + (x-y)\vec{\jmath} = \vec{F}.$$