Math 210, Fall 2008
Exam 2 "Solutions"

1. [25 points (10+10+5)]
Let $f(x,y,z) = \sin(xy-8) - \ln(z+1) + \frac{2x}{y-z}$.

(a) Compute the gradient $\vec{\nabla} f$ as a function of x, y, and z.

(b) Find the equation of the tangent plane to the surface $f(x,y,z) = 4$ at $(4,2,0)$.

(c) Compute the directional derivative $D_{\hat{\mathbf{u}}} f(4,2,0)$ where $\hat{\mathbf{u}}$ is a unit vector in the direction of $\langle -2,1,0\rangle$.

Solution:

(a) The partial derivatives of f are:

$$
\begin{aligned}
f_x &= y\cos(xy-8) + \frac{2}{y-z}\\
f_y &= x\cos(xy-8) - \frac{2x}{(y-z)^2}\\
f_z &= -\frac{1}{z+1} + \frac{2x}{(y-z)^2}
\end{aligned}
$$

The gradient is then $\vec{\nabla} f = \langle f_x, f_y, f_z\rangle$.

(b) At the point $(4,2,0)$, we have:

$$\vec{\nabla} f(4,2,0) = \langle 3,2,1\rangle$$

This vector is perpendicular to the tangent plane. Using this vector and the given point, the equation for the tangent plane is:

$$\boxed{3(x-4) + 2(y-2) + 1(z-0) = 0}$$

(c) To compute the directional derivative, we first compute $\hat{\mathbf{u}}$:

$$\hat{\mathbf{u}} = \frac{\langle -2,1,0\rangle}{||\langle -2,1,0\rangle||} = \frac{\langle -2,1,0\rangle}{\sqrt{5}}$$

The directional derivative at $(4,2,0)$ is then:

$$D_{\hat{\mathbf{u}}} f = \vec{\nabla} f(4,2,0)\cdot \hat{\mathbf{u}} = \langle 3,2,1\rangle \cdot \frac{\langle -2,1,0\rangle}{\sqrt{5}} = \boxed{-\frac{4}{\sqrt{5}}}$$

2. [30 points (10+5+15)]
Let $f(x,y) = x^2 + y^2 - y$, and let $\mathcal{D}$ be the bounded region defined by the inequalities $y \geq 0$ and $y \leq 1 - x^2$.

(a) Find and classify the critical points of $f(x,y)$.

(b) Sketch the region $\mathcal{D}$.

(c) Find the absolute maximum and minimum values of f on the region $\mathcal{D}$, and list the points where these values occur.

Solution:

(a) To find the critical points of f, set the partial derivatives to zero and solve:

$$f_x = 2x = 0 \quad \Rightarrow \quad x = 0$$
$$f_y = 2y - 1 = 0 \quad \Rightarrow \quad y = \frac{1}{2}$$

Therefore, the only critical point is $\boxed{(0, \frac{1}{2})}$. Now calculate the second derivatives:

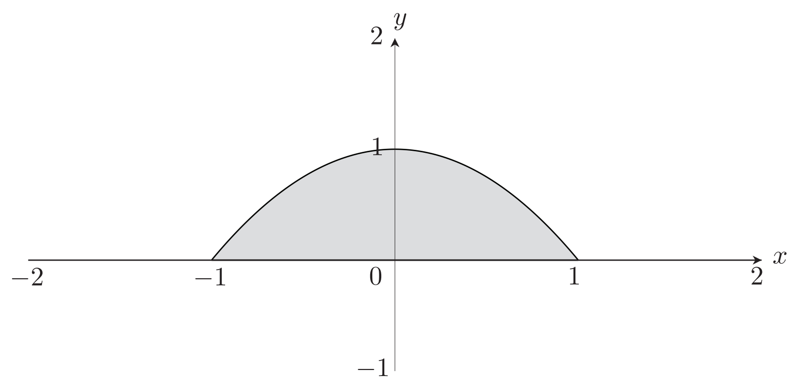
$$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0$$

The discriminant is:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0^2 = 4$$

Since $D(0, \frac{1}{2}) > 0$ and $f_{xx}(0, \frac{1}{2}) > 0$, the point $(0, \frac{1}{2})$ corresponds to a **local minimum**.

(b) A plot of the region is shown below:



(c) To find the absolute minimum and maximum values of f on the region, we must find the critical points inside $\mathcal{D}$ and on its boundary. We've already found the critical points in the interior. Let's focus on the boundary.

I. First, on the line $y = 0$, $-1 \leq x \leq 1$ we have:

$$f(x, 0) = x^2 + 0^2 - 0 = x^2, \quad -1 \leq x \leq 1$$

The function has a minimum value at $x = 0$ and maximum values at both $x = -1$ and $x = 1$. Therefore, the critical points on $y = 0$ are $(0, 0)$, $(-1, 0)$, and $(1, 0)$.

II. Second, on the parabola $y = 1 - x^2$, $-1 \leq x \leq 1$ we have:

$$f(x, 1 - x^2) = x^2 + (1 - x^2)^2 - (1 - x^2) = x^4, \quad -1 \leq x \leq 1$$

The function has a minimum value at $x = 0$ and maximum values at $x = -1$ and $x = 1$. Therefore, the critical points on $y = 1 - x^2$ are $(0, 1)$, $(-1, 0)$, and $(1, 0)$.

Evaluating f at each critical point we find:

$$f\left(0, \frac{1}{2}\right) = 0^2 + \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$$
$$f(0, 0) = 0^2 + 0^2 - 0 = 0$$
$$f(-1, 0) = (-1)^2 + 0^2 - 0 = 1$$
$$f(1, 0) = 1^2 + 0^2 - 0 = 1$$
$$f(0, 1) = 0^2 + 1^2 - 1 = 0$$

Therefore, $\boxed{\text{the absolute minimum of } f \text{ is } -\frac{1}{4}}$ and $\boxed{\text{the absolute maximum is } 1}$.

3. [15 points (5+10)]

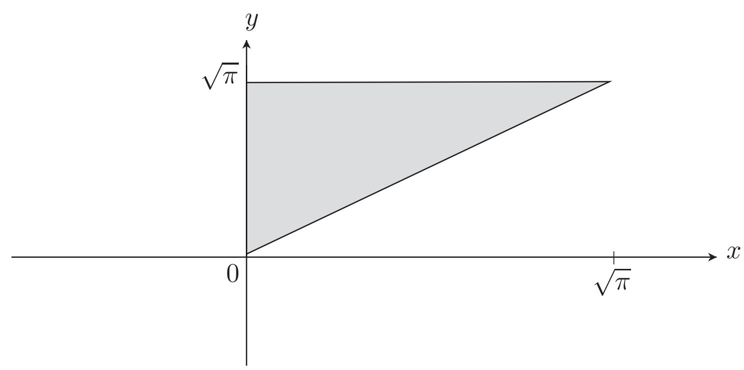
Consider the iterated integral $\displaystyle\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos(y^2)\, dy\, dx$.

(a) Sketch the region of integration.

(b) Compute the integral. (Hint: First reverse the order of integration.)

Solution:

(a) A plot of the region is shown below:



(b) Changing the order of integration and solving, we get:

$$\begin{aligned}\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos(y^2)\, dy\, dx &= \int_0^{\sqrt{\pi}} \int_0^{y} \cos(y^2)\, dx\, dy \\ &= \int_0^{\sqrt{\pi}} \left[x\cos(y^2)\right]_0^y\, dy \\ &= \int_0^{\sqrt{\pi}} y\cos(y^2)\, dy \\ &= \left[\frac{1}{2}\sin(y^2)\right]_0^{\sqrt{\pi}} \\ &= \frac{1}{2}\left[\sin(\pi) - \sin 0\right] \\ &= \boxed{0}\end{aligned}$$

4. [30 points (10+10+10)]
Let $\mathcal{Q}$ be the part of the unit disk that lies in the second quadrant, i.e.

$$\mathcal{Q} = \{(x,y) \mid x \le 0 \text{ and } y \ge 0 \text{ and } x^2 + y^2 \le 1\}.$$

(a) Write an iterated integral *in polar coordinates* that represents the area of $\mathcal{Q}$, and compute this area.

(b) Compute $\displaystyle\iint_{\mathcal{Q}} \left(3x^2 + 3y^2\right) dA$.

(c) Compute the average value of $f(x,y) = x^2 + y^2$ over $\mathcal{Q}$:

$$\operatorname{avg}(f) = \frac{\displaystyle\iint_{\mathcal{Q}} f(x,y)\, dA}{\displaystyle\iint_{\mathcal{Q}} 1\, dA}$$

Solution:

(a) The area is:

$$\boxed{\text{Area} = \int_{\pi/2}^{\pi} \int_0^1 r \, dr \, d\theta = \frac{\pi}{4}}$$

(b) The integral is:

$$\begin{aligned}
\iint_Q dA &= \int_{\pi/2}^{\pi} \int_0^1 3r^2 \cdot r \, dr \, d\theta \\
&= 3 \int_{\pi/2}^{\pi} \int_0^1 r^3 \, dr \, d\theta \\
&= 3 \int_{\pi/2}^{\pi} \left[\frac{1}{4} r^3 \right]_0^1 d\theta \\
&= \frac{3}{4} \int_{\pi/2}^{\pi} d\theta \\
&= \frac{3}{4} \left(\pi - \frac{\pi}{2} \right) \\
&= \boxed{\frac{3\pi}{8}}
\end{aligned}$$

(c) The average of f is:

$$\text{avg}(f) = \frac{\frac{\pi}{8}}{\frac{\pi}{4}} = \boxed{\frac{1}{2}}$$

where we used the fact that the numerator is one-third of the answer to part (b) and the denominator is the area which we found in part (a).