# MATH 210
## Sample exam problems for the 2nd hour exam
## Fall 2009
## "Solutions"

1. The partial derivatives are $f_x = 6x + y$ and $f_y = x + 4y$. At the point $(1, 1)$ we have $f_x(1, 1) = 7$, $f_y(1, 1) = 5$. The linearization is $L(x, y) = 6 + 7(x - 1) + 5(y - 1)$. Then, $f(1.1, 1.2) \approx L(1.1, 1.2) = 7.7$.

2. We must solve the equations $f_x = 3x^2 - 3y = 0$ and $f_y = -3x + 3y^2 = 0$. The solutions are $(0, 0)$ and $(1, 1)$. The discriminant at $(0, 0)$ is $D(0, 0) = -9$ and the discriminant at $(1, 1)$ is $D(1, 1) = 27$. Furthermore, $f_{xx}(1, 1) = 6$. Therefore, $f$ has a saddle at $(0, 0)$ and a local minimum at $(1, 1)$.

3. To solve the double integral, it is necessary to change the order of integration:

$$\int_0^2 \int_0^{x^2} \sin\left(x^3\right)\, dy\, dx = \int_0^2 x^2 \sin\left(x^3\right)\, dx = -\frac{1}{3}\cos\left(x^3\right)\Big|_0^2 = \frac{1}{3} - \frac{1}{3}\cos 8.$$

4. Using the method of Lagrange multipliers, we must solve the equations:

$$1 = \lambda\left(\frac{1}{2}x\right), \qquad 1 = \lambda\left(\frac{2}{9}y\right), \qquad -1 = \lambda(2z), \qquad \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

Solving the first three equations for $x$, $y$, and $z$, respectively, and then plugging into the last equation we get $\lambda = \pm\sqrt{\frac{7}{2}}$. Substituting each value of $\lambda$ back into the first three equations, we get the points

$$P_1 = \left(2\sqrt{\frac{2}{7}}, \frac{9}{2}\sqrt{\frac{2}{7}}, -\frac{1}{2}\sqrt{\frac{2}{7}}\right), \quad P_2 = \left(-2\sqrt{\frac{2}{7}}, -\frac{9}{2}\sqrt{\frac{2}{7}}, \frac{1}{2}\sqrt{\frac{2}{7}}\right)$$

The minimum value of $f$ is $-\sqrt{14}$ at $P_2$ and the maximum value of $f$ is $\sqrt{14}$ at $P_1$.

5. Let $F(x, y, z) = x^2 + y^3 - 2z$. The gradient is $\overrightarrow{\nabla} F = \langle 2x, 3y^2, -2\rangle$. At the point $(1, 2, 4)$ we have $\overrightarrow{\nabla} F(1, 2, 4) = \langle 2, 12, -2\rangle$. The equation for the tangent plane is $2(x - 1) + 12(y - 2) - 2(z - 4) = 0$.

6. The gradient is $\overrightarrow{\nabla} F = \langle 6x, 2y, -8z \rangle$. At the point $(1, -4, 3)$ we have $\overrightarrow{\nabla} F(1, -4, 3) = \langle 6, -8, -24 \rangle$. The equation for the tangent plane is $6(x-1) - 8(y+4) - 24(z-3) = 0$.

7. We must solve the equations $f_x = x^2 - y = 0$ and $f_y = 2y - x = 0$. The solutions are $(0,0)$ and $(\frac{1}{2}, \frac{1}{4})$. The discriminant at $(0,0)$ is $D(0,0) = -1$ and the discriminant at $(\frac{1}{2}, \frac{1}{4})$ is $D(\frac{1}{2}, \frac{1}{4}) = 1$. Furthermore, $f_{xx}(\frac{1}{2}, \frac{1}{4}) = 1$. Therefore, $f$ has a saddle at $(0,0)$ and a local minimum at $(\frac{1}{2}, \frac{1}{4})$.

8. Using the method of Lagrange multipliers, we must solve the equations:

$$2x = \lambda(2x), \qquad -1 = \lambda(2y), \qquad x^2 + y^2 = 4$$

The first equation tells us that $x = 0$ or $\lambda = 1$. If $x = 0$, then the third equation tells us that $y = \pm 2$. If $\lambda = 1$, then the second equation tells us that $y = -\frac{1}{2}$ and the third equation tells us that $x = \pm\frac{\sqrt{15}}{2}$. Therefore, we must evaluate $f$ at the four points

$$P_1 = (0,2), \quad P_2 = (0,-2), \quad P_3 = \left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right), \quad P_4 = \left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$$

The minimum value of $f$ is $-1$ at $P_1$ and the maximum value of $f$ is $\frac{17}{4}$ at $P_3$ and $P_4$.

9. The projection of the region onto the $xy$-plane is the portion of the circle of radius 1 centered at $(0,0)$ in the first quadrant. Therefore, the volume is:

$$V = \iint_{\mathcal{D}} (1 - x^2 - y^2)\, dA = \int_0^{\pi/2} \int_0^1 (1 - r^2) r \, dr\, d\theta = \frac{\pi}{8}$$

10. Using the method of Lagrange multipliers, we must solve the equations:

$$2x = \lambda(2x), \qquad -2y = \lambda(2y), \qquad 4z = \lambda(2z), \quad x^2 + y^2 + z^2 = 1$$

There are 6 sets of solutions to these equations:

$$\begin{aligned} &P_1 = (0,0,1), \quad P_2 = (0,0,-1), \quad P_3 = (0,1,0) \\ &P_4 = (0,-1,0), \quad P_5 = (1,0,0), \quad P_6 = (-1,0,0) \end{aligned}$$

The minimum value of $f$ is $-1$ at $P_3$ and $P_4$ and the maximum value of $f$ is 2 at $P_1$ and $P_2$.

11. The region is bounded below by $z = 0$ and above by $z = y = r\sin\theta$. The projection of $\mathcal{W}$ onto the $xy$-plane is the upper half of the circle of radius 2 centered at $(0, 0)$. Therefore, the integral is:

$$\iiint_{\mathcal{W}} (x^2 + y^2)^{1/2}\, dV = \int_0^{\pi} \int_0^2 \int_0^{r\sin\theta} r \cdot r\, dz\, dr\, d\theta = 8$$

12. The region is bounded below by $z = -\sqrt{1 - r^2}$ and above by $z = \sqrt{1 - r^2}$. The projection of $\mathcal{B}$ onto the $xy$-plane is the circle of radius 1 centered at $(0, 0)$. Therefore, the integral is:

$$\iiint_{\mathcal{B}} x^2\, dV = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} (r\cos\theta)^2 \cdot r\, dz\, dr\, d\theta = \frac{4\pi}{15}$$

13. The projection of the region onto the $xy$-plane is the circle of radius 1 centered at $(0, 0)$. Therefore, the volume is:

$$V = \iint_{\mathcal{D}} \left[\left(2 - r^2\right) - r^2\right]\, dA = \int_0^{2\pi} \int_0^1 \left(2 - 2r^2\right) \cdot r\, dr\, d\theta = \pi$$

14. The first partial derivatives of $f$ are $\dfrac{\partial f}{\partial x} = ye^{xy}$ and $\dfrac{\partial f}{\partial y} = xe^{xy}$. The equations for polar coordinates are $x = r\cos\theta$ and $y = r\sin\theta$. Therefore, $\dfrac{\partial x}{\partial r} = \cos\theta$ and $\dfrac{\partial y}{\partial r} = \sin\theta$. Therefore, the partial derivative $\dfrac{\partial f}{\partial r}$ is:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = (2r\sin\theta\cos\theta)e^{r^2\sin\theta\cos\theta} = \frac{2xy}{\sqrt{x^2 + y^2}}e^{xy}$$

15. The projection of $\mathcal{A}$ onto the $xy$-plane is the portion of the circle of radius 1 centered at $(0, 0)$ in the first quadrant. Using polar coordinates, the average value of $f$ on $\mathcal{A}$ is:

$$\bar{f} = \frac{\iint_{\mathcal{A}} f(x, y)\, dA}{\iint_{\mathcal{A}} dA} = \frac{\int_0^{\pi/2} \int_0^1 (2 + r\cos\theta - r\sin\theta) \cdot r\, dr\, d\theta}{\int_0^{\pi/2} \int_0^1 r\, dr\, d\theta} = 2$$

16. The region of integration is a triangle bounded by the lines $y = 0$, $y = x$, and $x + y = 2$.

The integral is then:

$$\iint\limits_{\mathcal{D}} \frac{x}{y+1}\,dA = \int_0^1 \int_y^{2-y} \frac{x}{y+1}\,dx\,dy = 4\ln 2 - 2$$

17. (a) $f_x = 2x - 1 = 0,\ f_y = 2y = 0 \implies x = \frac{1}{2},\ y = 0$

    (c) On the boundary $x = 0$, the function becomes $f(0, y) = y^2$ and attains a maximum value of 1 at $y = \pm 1$ and a minimum value of 0 at $y = 0$. On the boundary $x = 1 - y^2$, the function becomes $f(1 - y^2, y) = (1 - y^2)^2 - (1 - y^2) + y^2 = y^4$ and attains a maximum value of 1 at $y = \pm 1$ and a minimum value of 0 at $y = 0$. Therefore, we have:

$$f\left(\frac{1}{2}, 0\right) = -\frac{1}{4}, \quad f(0,0) = 0, \quad f(0, \pm 1) = 1, \quad f(1, 0) = 0$$

    The minimum value of $f$ is $-\frac{1}{4}$ at $(\frac{1}{2}, 0)$ and the maximum value of $f$ is 1 at $(0, \pm 1)$.

18. The gradient of $F$ is $\overrightarrow{\nabla} F = \left\langle 2x + 4x^2 e^{4x-y^2}, -2x^2 y e^{4x-y^2} \right\rangle$. At the point $(1, 2)$ we have $\overrightarrow{\nabla} F(1, 2) = \langle 6, -4 \rangle$. The direction of fastest growth is:

$$\hat{\mathbf{u}} = \frac{\overrightarrow{\nabla} F(1,2)}{\left\| \overrightarrow{\nabla} F(1,2) \right\|} = \left\langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right\rangle$$

19. The gradient of $f$ is $\overrightarrow{\nabla} f = \langle 2xy - e^{x+y}, x^2 - e^{x+y} \rangle$. The derivative of $\overrightarrow{\mathbf{r}}(t)$ is $\overrightarrow{\mathbf{r}}'(t) = \langle -e^{-t}, -\sin t \rangle$. At $t = 0$ we have $\overrightarrow{\mathbf{r}}(0) = \langle 1, 1 \rangle \implies x = 1,\ y = 1$ and $\overrightarrow{\mathbf{r}}'(0) = \langle -1, 0 \rangle$. Using the Chain Rule, we have:

$$\left. \frac{d}{dt} f\left(\overrightarrow{\mathbf{r}}(t)\right) \right|_{t=0} = \overrightarrow{\nabla} f(1,1) \cdot \overrightarrow{\mathbf{r}}'(0) = \left\langle 2 - e^2, 1 - e^2 \right\rangle \cdot \langle -1, 0 \rangle = e^2 - 2$$

20. In spherical coordinates, the equation for sphere is $\rho = 1$ and the equation for the cone is $\phi = \frac{\pi}{4}$. Therefore, the mass is:

$$\text{mass} = \iiint\limits_{\mathcal{A}} f(x, y, z)\,dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho \cdot \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$