1. (20 pts) Complete each of the following:

(a) Compute the directional derivative \(D_u f\) of the function \(f(x, y) = xy^2 + \ln(xy)\) at the point \((1, 1)\) in the direction of \(\nabla = (1, 2)\).

(b) Use the Chain Rule to compute \(\frac{\partial w}{\partial s}\) when \(s = 1, t = 2\) if

\[w(x, y) = x + x^2y^3, \ x(s, t) = st, \ y(s, t) = s^2\]

Solution:

(a) The first partial derivatives of \(f\) are:

\[f_x = y^2 + \frac{1}{x}\]
\[f_y = 2xy + \frac{1}{y}\]

The gradient of \(f\) at \((1, 1)\) is:

\[\nabla f(1, 1) = \langle f_x(1, 1), f_y(1, 1) \rangle = \langle 1^2 + \frac{1}{1}, 2(1)(1) + \frac{1}{1} \rangle = \langle 2, 3 \rangle\]

Next, make \(\nabla\) into a unit vector:

\[\hat{u} = \frac{\nabla}{||\nabla||} = \frac{(1, 2)}{\sqrt{5}}\]

The directional derivative is:

\[D_u f = \nabla f \cdot \hat{u} = \langle 2, 3 \rangle \cdot \frac{(1, 2)}{\sqrt{5}} = \frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}} = \frac{8}{\sqrt{5}}\]

(b) First, we compute the necessary partial derivatives:

\[\frac{\partial w}{\partial x} = 1 + 2xy^3, \ \frac{\partial w}{\partial y} = 3x^2y^2, \ \frac{\partial x}{\partial s} = t, \ \frac{\partial y}{\partial s} = 2s\]

When \(s = 1\) and \(t = 2\), we have:

\[x(1, 2) = (1)(2) = 2, \ y(1, 2) = 1^2 = 1\]

Therefore, when \(s = 1\) and \(t = 2\) we have:

\[\frac{\partial w}{\partial x} = 1 + 2(2)(1)^3 = 5\]
\[\frac{\partial w}{\partial y} = 3(2)^2(1)^2 = 12\]
\[\frac{\partial x}{\partial s} = t = 2\]
\[\frac{\partial y}{\partial s} = 2s = 2(1) = 2\]

Using the Chain Rule, we have:

\[\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}\]
\[= (5)(2) + (12)(2)\]
\[= 34\]
2. (20 pts) Consider the function:

\[ f(x, y) = x^2 - xy - y^2 \]

(a) Find the equation of the plane tangent to the surface \( z = f(x, y) \) at the point \((1, 1, -1)\).

(b) Use the linearization of \( f(x, y) \) about \((1, 1)\) to estimate \( f(1.1, 0.95) \).

**Solution:**

(a) The equation of the tangent plane has the form:

\[ z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \]

where \( a = 1 \) and \( b = 1 \). The first derivatives of \( f \) are:

\[ f_x = 2x - y, \quad f_y = -x - 2y \]

Then we have:

\[ f(1, 1) = -1 \]
\[ f_x(1, 1) = 2(1) - 1 = 1 \]
\[ f_y(1, 1) = -(1) - 2(1) = -3 \]

The equation of the tangent plane is:

\[ z = -1 + 1(x - 1) - 3(y - 1) \]

(b) The linearization is:

\[ L(x, y) = -1 + 1(x - 1) - 3(y - 1) \]

which is the same answer we obtained in part (a). Plugging in \( x = 1.1 \) and \( y = 0.95 \), we get:

\[ L(1.1, 0.95) = -1 + 1(1.1 - 1) - 3(0.95 - 1) = -1 + 0.1 - 3(-0.05) = 0.75 \]

3. (20 pts) Find the critical points of \( f(x, y) \) and specify for each whether it corresponds to a local maximum, local minimum or saddle point, given that the partial derivatives of \( f \) are:

\[ f_x = 2x - 4y, \quad f_y = -4x + 5y + 3y^2 \]

**Solution:** To find the critical points, we solve the system:

\[ f_x = 2x - 4y = 0 \]
\[ f_y = -4x + 5y + 3y^2 = 0 \]

Solving the first equation for \( x \), we get \( x = 2y \). Plugging this into the second equation, we get:

\[ -4(2y) + 5y + 3y^2 = 0 \]
\[-8y + 5y + 3y^2 = 0 \]
\[ 3y^2 - 3y = 0 \]
\[ y^2 - y = 0 \]
\[ y(y - 1) = 0 \]
\[ y = 0, \quad y = 1 \]
When \( y = 0 \), we get \( x = 0 \) from the first equation. When \( y = 1 \), we get \( x = 2 \). Therefore, the critical points are:

\[
(0, 0), \ (2, 1)
\]

To classify the points, we use the second derivative test. The discriminant \( D(x, y) \) is:

\[
D(x, y) = f_{xx}f_{yy} - f_{xy}^2
\]

\[
D(x, y) = (2)(5 + 6y) - (-4)^2
\]

\[
D(x, y) = 10 + 12y - 16
\]

\[
D(x, y) = 12y - 6
\]

Computing \( D(x, y) \) at each critical point, we have:

\[
D(0, 0) = 12(0) - 6 = -6 < 0 \quad \Rightarrow \quad (0, 0) \text{ is a saddle point}
\]

\[
D(2, 1) = 12(1) - 6 = 6 > 0, \ f_{xx}(2, 1) = 2 > 0 \quad \Rightarrow \quad (2, 1) \text{ corresponds to a local minimum}
\]

4. (20 pts) For the following integral:

\[
\int_0^2 \int_y^4 y^3 x e^{x^2} \, dx \, dy
\]

sketch the region of integration, reverse the order of integration, and evaluate the resulting integral.

The region of integration is shown below.

![Region of Integration](image)

Reversing the order of integration and evaluating the integral we have:

\[
\int_0^2 \int_y^4 y^3 x e^{x^2} \, dx \, dy = \int_0^4 \int_0^{\sqrt{x}} y^3 x e^{x^2} \, dy \, dx
\]

\[
= \int_0^4 \left[ \frac{1}{4} y^4 e^{x^2} \right]_0^{\sqrt{x}} \, dx
\]

\[
= \frac{1}{4} \int_0^4 x e^{x^2} \, dx
\]

\[
= \frac{1}{4} \left[ \frac{1}{2} e^{x^2} \right]_0^4
\]

\[
= \frac{1}{8} (e^{16} - 1)
\]
5. (20 pts) Find the volume of the region bounded above by the sphere \( x^2 + y^2 + z^2 = 4 \) and below by the plane \( z = 1 \). (Hint: use cylindrical coordinates)

**Solution:** The intersection of the sphere and the plane is:

\[
\begin{align*}
    x^2 + y^2 + 1^2 &= 4 \\
    x^2 + y^2 &= 3
\end{align*}
\]

Therefore, the projection of the region onto the \( xy \)-plane is the disk \( x^2 + y^2 \leq 3 \). Also, the equation for the sphere in cylindrical coordinates is:

\[
r^2 + z^2 = 4 \quad \Rightarrow \quad z = \sqrt{4 - r^2}
\]

The volume is then:

\[
V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{3 - r^2}} r \, dz \, dr \, d\theta
\]

\[
= 2\pi \int_0^{\sqrt{3}} \left[ r \sqrt{4 - r^2} - r \right] dr
\]

\[
= 2\pi \left[ -\frac{1}{3} (4 - r^2)^{3/2} - \frac{1}{2} r^2 \right]_0^{\sqrt{3}}
\]

\[
= 2\pi \left[ -\frac{1}{3} - \frac{1}{2} (3) + \frac{1}{3} (8) \right]
\]

\[
= \frac{5\pi}{3}
\]