Math 210, Spring 2008
Exam 2 "Solutions"

1. (20 pts) Complete each of the following:

   (a) Compute the directional derivative $D_{\mathbf{u}}f$ of the function $f(x,y) = xy^2 + \ln(xy)$ at the point $(1,1)$ in the direction of $\overrightarrow{\mathbf{v}} = \langle 1, 2\rangle$.

   (b) Use the Chain Rule to compute $\dfrac{\partial w}{\partial s}$ when $s = 1$, $t = 2$ if

$$w(x,y) = x + x^2y^3, \;\; x(s,t) = st, \;\; y(s,t) = s^2$$

Solution:

(a) The first partial derivatives of f are:

$$\begin{aligned} f_x &= y^2 + \frac{1}{x} \\ f_y &= 2xy + \frac{1}{y} \end{aligned}$$

The gradient of f at $(1,1)$ is:

$$\overrightarrow{\nabla} f(1,1) = \langle f_x(1,1), f_y(1,1)\rangle = \left\langle 1^2 + \frac{1}{1}, 2(1)(1) + \frac{1}{1} \right\rangle = \langle 2, 3\rangle$$

Next, make $\overrightarrow{\mathbf{v}}$ into a unit vector:

$$\hat{\mathbf{u}} = \frac{\overrightarrow{\mathbf{v}}}{||\overrightarrow{\mathbf{v}}||} = \frac{\langle 1,2\rangle}{\sqrt{5}}$$

The directional derivative is:

$$D_{\mathbf{u}}f = \overrightarrow{\nabla} f \cdot \hat{\mathbf{u}} = \langle 2,3\rangle \cdot \frac{\langle 1,2\rangle}{\sqrt{5}} = \frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}} = \boxed{\frac{8}{\sqrt{5}}}$$

(b) First, we compute the necessary partial derivatives:

$$\frac{\partial w}{\partial x} = 1 + 2xy^3, \;\; \frac{\partial w}{\partial y} = 3x^2y^2, \;\; \frac{\partial x}{\partial s} = t, \;\; \frac{\partial y}{\partial s} = 2s$$

When $s = 1$ and $t = 2$, we have:

$$x(1,2) = (1)(2) = 2, \quad y(1,2) = 1^2 = 1$$

Therefore, when $s = 1$ and $t = 2$ we have:

$$\begin{aligned} \frac{\partial w}{\partial x} &= 1 + 2xy^3 = 1 + 2(2)(1)^3 = 5 \\ \frac{\partial w}{\partial y} &= 3x^2y^2 = 3(2)^2(1)^2 = 12 \\ \frac{\partial x}{\partial s} &= t = 2 \\ \frac{\partial y}{\partial s} &= 2s = 2(1) = 2 \end{aligned}$$

Using the Chain Rule, we have:

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (5)(2) + (12)(2) \\ &= \boxed{34} \end{aligned}$$

2. (20 pts) Consider the function:

$$f(x,y) = x^2 - xy - y^2$$

   (a) Find the equation of the plane tangent to the surface $z = f(x,y)$ at the point $(1,1,-1)$.
   (b) Use the linearization of $f(x,y)$ about $(1,1)$ to estimate $f(1.1, 0.95)$.

   Solution:

   (a) The equation of the tangent plane has the form:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

   where $a = 1$ and $b = 1$. The first derivatives of f are:

$$f_x = 2x - y, \;\; f_y = -x - 2y$$

   Then we have:

$$\begin{aligned} f(1,1) &= -1 \\ f_x(1,1) &= 2(1) - 1 = 1 \\ f_y(1,1) &= -(1) - 2(1) = -3 \end{aligned}$$

   The equation of the tangent plane is:

$$\boxed{z = -1 + 1(x-1) - 3(y-1)}$$

   (b) The linearization is:

$$L(x,y) = -1 + 1(x-1) - 3(y-1)$$

   which is the same answer we obtained in part (a). Plugging in $x = 1.1$ and $y = 0.95$, we get:

$$L(1.1, 0.95) = -1 + 1(1.1-1) - 3(0.95-1) = -1 + 0.1 - 3(-0.05) = \boxed{-0.75}$$

3. (20 pts) Find the critical points of $f(x,y)$ and specify for each whether it corresponds to a local maximum, local minimum or saddle point, given that the partial derivatives of f are:

$$f_x = 2x - 4y, \;\; f_y = -4x + 5y + 3y^2$$

   Solution: To find the critical points, we solve the system:

$$\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= -4x + 5y + 3y^2 = 0 \end{aligned}$$

   Solving the first equation for x, we get $x = 2y$. Plugging this into the second equation, we get:

$$\begin{aligned} -4(2y) + 5y + 3y^2 &= 0 \\ -8y + 5y + 3y^2 &= 0 \\ 3y^2 - 3y &= 0 \\ y^2 - y &= 0 \\ y(y-1) &= 0 \\ y = 0, \;\; y &= 1 \end{aligned}$$

When $y = 0$, we get $x = 0$ from the first equation. When $y = 1$, we get $x = 2$. Therefore, the critical points are:

$$\boxed{(0,0),\ (2,1)}$$

To classify the points, we use the second derivative test. The discriminant $D(x, y)$ is:

$$\begin{aligned}
D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\
D(x,y) &= (2)(5+6y) - (-4)^2 \\
D(x,y) &= 10 + 12y - 16 \\
D(x,y) &= 12y - 6
\end{aligned}$$

Computing $D(x, y)$ at each critical point, we have:

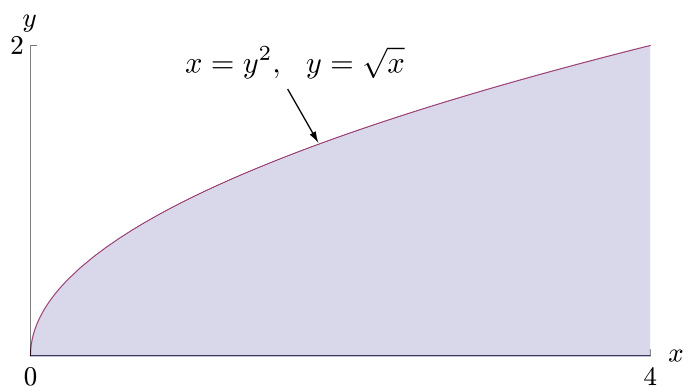
$$\begin{aligned}
&D(0,0) = 12(0) - 6 = -6 < 0 \quad \Rightarrow \quad \text{(0,0) is a saddle point} \\
&D(2,1) = 12(1) - 6 = 6 > 0,\ f_{xx}(2,1) = 2 > 0 \quad \Rightarrow \quad \text{(2,1) corresponds to a local minimum}
\end{aligned}$$

4. (20 pts) For the following integral:

$$\int_0^2 \int_{y^2}^4 \frac{y^3}{x} e^{x^2}\, dx\, dy$$

sketch the region of integration, reverse the order of integration, and evaluate the resulting integral.

The region of integration is shown below.



Reversing the order of integration and evaluating the integral we have:

$$\begin{aligned}
\int_0^2 \int_{y^2}^4 \frac{y^3}{x} e^{x^2}\, dx\, dy &= \int_0^4 \int_0^{\sqrt{x}} \frac{y^3}{x} e^{x^2}\, dy\, dx \\
&= \int_0^4 \left[\frac{1}{4x} y^4 e^{x^2}\right]_0^{\sqrt{x}} dx \\
&= \frac{1}{4}\int_0^4 x e^{x^2}\, dx \\
&= \frac{1}{4}\left[\frac{1}{2}e^{x^2}\right]_0^4 \\
&= \boxed{\frac{1}{8}(e^{16} - 1)}
\end{aligned}$$

5. (20 pts) Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $z = 1$. (Hint: use cylindrical coordinates)

Solution: The intersection of the sphere and the plane is:

$$\begin{aligned} x^2 + y^2 + 1^2 &= 4 \\ x^2 + y^2 &= 3 \end{aligned}$$

Therefore, the projection of the region onto the xy-plane is the disk $x^2 + y^2 \leq 3$. Also, the equation for the sphere in cylindrical coordinates is:

$$r^2 + z^2 = 4 \quad \Rightarrow \quad z = \sqrt{4 - r^2}$$

The volume is then:

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta \\ &= 2\pi \int_0^{\sqrt{3}} r[z]_1^{\sqrt{4-r^2}}\, dr \\ &= 2\pi \int_0^{\sqrt{3}} \left(r\sqrt{4-r^2} - r \right) dr \\ &= 2\pi \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{2}r^2 \right]_0^{\sqrt{3}} \\ &= 2\pi \left[-\frac{1}{3} - \frac{1}{2}(3) + \frac{1}{3}(8) \right] \\ &= \boxed{\frac{5\pi}{3}} \end{aligned}$$