MATH 310, SECOND EXAM

(1) (a) Show that \([1 + x^2, x - 1, x^2]\) is a basis for \(P_3\) (the space of polynomials of degree at most two).

A basis is a linearly independent spanning set. Thus we need to check that the set spans and is linearly independent. To see that it spans, we need to check that any polynomial \(a + bx + cx^2\) can be written as a linear combination \(c_1(1 + x^2) + c_2(x - 1) + c_3x^2\). This gives the equation

\[
(c_1 - c_2) + c_2 x + (c_1 + c_3)x^2 = a + bx + cx^2
\]

These are equivalent to the system

\[
\begin{align*}
    c_1 - c_2 &= a \\
    c_2 &= b \\
    c_1 + c_3 &= c
\end{align*}
\]

Solving these equations gives:

\[
\begin{align*}
    c_1 &= a + b \\
    c_2 &= b \\
    c_3 &= c - a - b
\end{align*}
\]

Since there is a solution for every \(a, b,\) and \(c\), the set spans.

To see that the set is linearly independent, we need to see that the only solution to

\[
c_1(1 + x^2) + c_2(x - 1) + c_3x^2 = 0
\]

is the trivial solution, \(c_1 = c_2 = c_3 = 0\). This follows from the formula we found above for \(c_1, c_2,\) and \(c_3\) in terms of \(a, b,\) and \(c\) by setting \(a = b = c = 0\).

(b) Let \(L(P) = P - 2xP' + x^2P''\). Find the matrix representation of \(L\) as a map \(P_3 \to P_3\) in the basis from part (a).

To do this we calculate:

\[
\begin{align*}
    L(1 + x^2) &= 1 + x^2 - 2x(2x) + 2x^2 = 1 - x^2 = 1(1 + x^2) + 0(2x) - 2(x^2) \\
    L(x - 1) &= 1 - x - 2x(-1) + 0 = 1 + x = 2(1 + x^2) + (x - 1) - 2(x^2) \\
    L(x^2) &= x^2 - 2x(2x) + 2x^2 = -x^2 = 0(1 + x^2) + 0(x - 1) - x^2
\end{align*}
\]

So the matrix representing \(L\) in this basis is
(2) Let $A$ be the matrix

$$
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
-2 & -2 & -1
\end{pmatrix}
$$

(a) Find a basis for the row space of $A$.
(b) Find a basis for the column space of $A$.
(c) Find a basis for the null space of $A$.

We first have to put the matrix in row echelon form. This gives:

$$
\begin{pmatrix}
1 & -1 & 2 & 3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(a) The non-zero rows for the row echelon form are a basis for the row space. In this case that gives $(1,-1,2,3), (0,0,1,1),$ and $(0,0,0,1)$. You can use reduced row echelon form instead, which will give a different, but equally valid, answer.
(b) The columns of $A$ which have leading ones in row echelon form are a basis for the column space. Here that is the first, third, and fourth columns : $(1,0,2,3)^T, (2,1,5,0)^T,$ and $(3,1,8,3)^T$.
(c) To get a basis for the null space, we need to solve the equations $Ax = 0$. Since the row echelon form gives an equivalent system, we solve those by back substitution. This means the system:

$$
x - y + 2z + 3w = 0 \\
z + w = 0 \\
w = 0
$$

The solution to this system has one free variable, $y$, and $z = w = 0$, $x = y$. So the solutions are all multiples of the vector $(1,1,0,0)^T$. So this vector is a basis for the null space.

(3) Consider the vector space $C[0,1]$ of continuous functions on the interval with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Let $u = e^x$ and $v = e^{-x}$.

(a) Calculate the lengths of $u$ and $v$ and the angle between them (you may leave your answer in terms of $\cos^{-1}$.)
(b) Find the function in the span of \( u \) and \( v \) which is closest to the function \( x \) (hint: this is a least squares problem).

(a) \[ ||u||^2 = \langle u, u \rangle = \int_0^1 e^{2x} dx = \frac{1}{2}(e^2 - 1) \]

Likewise, \[ ||v|| = \sqrt{\int_0^1 e^{-2x} dx} = \frac{1}{2}(1 - e^{-2}) \].

To find the angle, we use the formula \[ \langle u, v \rangle = \frac{||u|| \cdot ||v|| \cdot \cos \theta}{||u|| \cdot ||v||} \].

The inner product \[ \langle u, v \rangle = \int_1^0 xe^xdx = \frac{1}{1}1. \]

Thus \[ \cos \theta = \frac{e}{e^2 - 1}. \]

So the angle between \( u \) and \( v \) is \( \cos^{-1} \left( \frac{e}{e^2 - 1} \right) \).

(b) There are several ways to solve this problem. The most straightforward is to use the basic fact about closest points: the closest point \( p \) to \( x \) in the span of \( u \) and \( v \) is the point where \( p - x \) is orthogonal to \( u \) and \( v \).

If \( p = c_1 u + c_2 v \) this means we need to solve the equations:

\[ \langle c_1 u + c_2 v - x, u \rangle = 0 \]
\[ \langle c_1 u + c_2 v - x, v \rangle = 0 \]

From the first part, we know \( \langle u, u \rangle = e^2 - 1, \quad \langle v, v \rangle = 1 - e^{-2}, \) and \( \langle u, v \rangle = 1. \) We also need to calculate \( \langle u, x \rangle = \int_0^1 xe^xdx = 1 - 2e^{-1}. \)

Plugging these values in, the system of equations is:

\[ \frac{1}{2}(e^2 - 1)c_1 + c_2 - 1 = 0 \]
\[ c_1 + \frac{1}{2}(1 - e^{-2})c_2 + 1 - 2e^{-1} = 0 \]

This gives \( c_1 = \frac{2(3e^2 - 4e - 1)}{e^2 - 6e + 1} \) and \( c_2 = \frac{2(3e^3 + 2e^2 - e - 2)}{e^2 - 6e + 1} \).

(4) Prove that for every \( m \times n \) matrix \( A \),

\[ \dim(\text{Null}(A)) - \dim(\text{Null}(A^T)) = n - m \]

(Hint: use the rank-nullity theorem).

The rank-nullity theorem says that for any matrix \( M \), \( \dim(\text{null}(M)) + \text{rank}(B) \) is the number of columns of \( B \) (remember that the rank is the dimension of the column space). So, applying this to \( A \), we get

\[ \dim(\text{null}(A)) = n - \text{rank}(A) \]

Applying it again to \( A^T \) gives

\[ \dim(\text{null}(A^T)) = m - \text{rank}(A^T) \]

Since the columns of \( A^T \) are the rows of \( A \), the rank of \( A^T \) is the dimension of the row space of \( A \). One of the basic theorems is that...
row rank equals column rank, so \( \text{rank}(A^T) = \text{rank}(A) \). Using this
and subtracting the two equations above gives

\[
dim(\text{null}(A)) - dim(\text{null}(A^T)) = n - \text{rank}(A) - m + \text{rank}(A^T) = (n - m) - (\text{rank}(A) - \text{rank}(A^T)) = n - m
\]

This proves the given formula.