(1) Prove that if $f$ is continuous but not differentiable at $x = a$ and $g$ is differentiable at $x = a$ then their product $fg$ is differentiable at $x = a$ iff $g(a) = 0$.

(2) Prove directly from the definition that $x^2$ is differentiable everywhere.

(3) Using any of the product, sum, and chain rules (give the precise statements you use) prove that all polynomial functions are differentiable everywhere (hint: induction).

(4) (a) Prove that any twice differentiable function satisfying $f'' + f = 0$ is of the form $A\sin x + B\cos x$ for some constants $A$ and $B$ (alternative hint: consider the quantity $f(x)^2 + f'(x)^2$).

(b) Show that $f(x) = \sin(x + a)$ satisfies this differential equation, and from this prove the formula $\sin(x + a) = \sin x \cos a + \sin a \cos x$.

(5) Prove that any bounded function which is discontinuous at only finitely many points is integrable.

(6) Let $f$ be the function on $[0, 1]$ defined by:

- $f\left(\frac{1}{n}\right) = 1$ for positive integers $n$.
- $f(x) = 0$ for all other $x$.

Where is $f$ continuous? Is $f$ integrable? Prove your answers correct.