Quasi-actions on trees I, Bounded valence

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Abstract

Given a bounded valence, bushy tree T, we prove that any quasi-action of a group G on T is quasiconjugate to an action of G on another bounded valence, bushy tree T'. This theorem has many applications: quasi-isometric rigidity for fundamental groups of finite, bushy graphs of coarse $\mathrm{PD}(n)$ groups for each fixed n; a generalization to actions on Cantor sets of Sullivan's Theorem about uniformly quasi-conformal actions on the 2-sphere; and a characterization of locally compact topological groups which contain a virtually free group as a cocompact lattice. Finally, we give the first examples of two finitely generated groups which are quasi-isometric and yet which cannot act on the same proper geodesic metric space, properly discontinuously and cocompactly by isometries.

1 Introduction

A *quasi-action* of a group G on a metric space X associates to each $g \in G$ a quasi-isometry $A_g : x \to g \cdot x$ of X, with uniform quasi-isometry constants, so that $A_{\mathrm{Id}} = \mathrm{Id}_X$, and so that the distance between $A_g \circ A_h$ and A_{gh} in the sup norm is uniformly bounded independent of $g, h \in G$.

Quasi-actions arise naturally in geometric group theory: if a metric space X is quasi-isometric to a finitely generated group G with its word metric, then the left action of G on itself can be "quasiconjugated" to give a quasi-action of G on X. Moreover, a quasi-action which arises in this manner is *cobounded* and *proper*; these properties are generalizations of *cocompact* and *properly discontinuous* as applied to isometric actions.

Given a metric space X, a fundamental problem in geometric group theory is to characterize groups quasi-isometric to X, or equivalently, to characterize groups which have a proper, cobounded quasi-action on X. A more general problem is to characterize arbitrary quasi-actions on X up to quasiconjugacy. This problem is completely solved in the prototypical cases $X = \mathbf{H}^2$ or $\mathbf{H}^3$: any quasi-action on $\mathbf{H}^2$ or $\mathbf{H}^3$ is quasiconjugate to an isometric action. When X is an irreducible symmetric space of nonpositive curvature, or an irreducible Euclidean building of rank ≥ 2, then as recounted below similar results hold, sometimes with restriction to cobounded quasi-actions, sometimes with stronger conclusions.

The main result of this paper, Theorem 1, gives a complete solution to the problem in the case when X is a bounded valence tree which is *bushy*, meaning coarsely that the tree is neither a point nor a line. Theorem 1 says that any quasi-action on a bounded valence, bushy tree is quasiconjugate to an isometric action, on a possibly different tree.

We give various applications of this result.

For instance, while the typical way to prove that two groups are quasi-isometric is to produce a proper metric space on which they each have a proper cobounded action, we provide the first examples of two quasi-isometric groups for which there does not exist *any* proper metric space on which they both act, properly and coboundedly; our examples are virtually free groups. We do this by determining which locally compact groups $\mathcal{G}$ can have discrete, cocompact subgroups that are virtually free of finite rank ≥ 2: $\mathcal{G}$ is closely related to the automorphism group of a certain bounded valence, bushy tree T. In [MSW00a] these results are applied to characterize which trees T are the "best" model geometries for virtually free groups; there is a countable infinity of "best" model geometries in an appropriate sense.

Our main application is to quasi-isometric rigidity for homogeneous graphs of groups; these are finite graphs of finitely generated groups in which every edge-to-vertex injection has finite index image. For instance, we prove quasi-isometric rigidity for fundamental groups of finite graphs of virtual $\mathbf{Z}$'s, and by applying previous results we then obtain a complete classification of such groups up to quasi-isometry. More generally, we prove quasi-isometric rigidity for a homogeneous graph of groups Γ whose vertex and edge groups are "coarse" PD(n) groups, as long as the Bass-Serre tree is bushy—any finitely generated group H quasi-isometric to $\pi_1\Gamma$ is the fundamental group of a homogeneous graph of groups Γ' with bushy Bass-Serre tree whose vertex and edge groups are quasi-isometric to those of Γ.

Other applications involve the problem of passing from quasiconformal boundary actions to conformal actions, where in this case the boundary is a Cantor set.

Quasi-actions on $\mathbf{H}^3$ are studied via the theorem that any uniformly quasiconformal action on $S^2 = \partial\mathbf{H}^3$ is quasiconformally conjugate to a conformal action; the countable case of this theorem was proved by Sullivan [Sul81], and the general case by Tukia [Tuk80].[1] Quasi-actions on other hyperbolic symmetric spaces are studied via similar theorems about uniformly quasiconformal boundary actions, as recounted below. Using Paulin's formulation of uniform quasiconformality for the boundary of a Gromov hyperbolic space [Pau96], we prove that when B is the Cantor set, equipped with a quasiconformal structure by identifying B with the boundary of a bounded valence bushy tree, then any uniformly quasiconformal action on B is quasiconformally conjugate to a conformal action in the appropriate sense. Unlike the more analytic proofs for boundaries of rank 1 symmetric spaces, our proofs depend on the low-dimensional topology methods of Theorem 1.

Quasi-actions on $\mathbf{H}^2$ are studied similarly via the induced actions on $S^1 = \partial \mathbf{H}^2$. We are primarily interested in one subcase, a theorem of Hinkkanen [Hin85] which says that any uniformly quasi-symmetric group action on $\mathbf{R} = S^1 - \{\text{point}\}$ is quasisymmetrically conjugate to a similarity action on $\mathbf{R}$; an analogous theorem of Farb and Mosher [FM99] says that any uniform quasisimilarity group action on $\mathbf{R}$ is bilipschitz conjugate to a similarity action. We prove a Cantor set analogue of these results, answering a question posed in [FM99]: any uniform quasisimilarity action on the n-adic rational numbers $\mathbf{Q}_n$ is bilipschitz conjugate to a similarity action on some $\mathbf{Q}_m$, with m possibly different from n.

Theorem 1 has also been applied recently by A. Reiter [Rei00] to solve quasi-isometric rigidity problems for lattices in p-adic Lie groups with rank 1 factors, for instance to show that any finitely generated group quasi-isometric to a product of bounded valence trees acts on a product of bounded valence trees.

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[1] The fact that Sullivan's Theorem implies QI-rigidity of $\mathbf{H}^3$ was pointed out by Gromov to Sullivan in the 1980's [Sul]; see also [CC92].

Contents

2 Statements of results

Theorem 1: Rigidity of quasi-actions on bounded valence, bushy trees[2]

The simplest nonelementary Gromov hyperbolic metric spaces are homogeneous simplicial trees T of constant valence ≥ 3. One novel feature of such geometries is that there is no best geometric model: all trees with constant valence ≥ 3 are quasi-isometric to each other. Indeed, each such tree is quasi-isometric to any tree T satisfying the following properties: T has *bounded valence*, meaning that vertices have uniformly finite valence; and T is *bushy*, meaning that each point of T is a uniformly bounded distance from a vertex having at least 3 unbounded complementary components. In this paper, each tree T is given a geodesic metric in which each edge has length 1; one effect of this is to identify the isometry group $\mathrm{Isom}(T)$ with the automorphism group of T.

Here is our main theorem:

Theorem 1 (Rigidity of quasi-actions on bounded valence, bushy trees). *If $G \times T \to T$ is a quasi-action of a group G on a bounded valence, bushy tree T, then there is a bounded valence, bushy tree T', an isometric action $G \times T' \to T'$, and a quasiconjugacy $f\colon T' \to T$ from the action of G on T' to the quasi-action of G on T.*

Remark Given quasi-actions of G on metric spaces X, Y, a quasiconjugacy is a quasi-isometry $f\colon X \to Y$ which is *coarsely G-equivariant* meaning that $d_Y\big(f(g \cdot x), g \cdot fx\big)$ is uniformly bounded independent of $g \in G$, $x \in X$. Any coarse inverse for f is also coarsely G-equivariant. We remark that properness and coboundedness are each invariant under quasiconjugation.

Theorem 1 complements similar theorems for irreducible symmetric spaces and Euclidean buildings. When $X = \mathbf{H}^n$, $n \geq 4$ [Tuk86], and when $X = \mathbf{CH}^n$, $n \geq 2$ [Cho96], every cobounded quasi-action is quasiconjugate to an action—note that Theorem 1 does not require coboundedness, whereas if $n \geq 4$ then $\mathbf{H}^n$ has a cobounded quasi-action which is not conjugate to any action [Tuk81], [FS87]. When X is a quaternionic hyperbolic space or the Cayley hyperbolic plane [Pan89b], or when X is a nonpositively curved symmetric space or thick Euclidean building, irreducible and of rank ≥ 2

[2] Theorem 1 and several of its applications were first presented in [MSW00c], which also presents results from Part 2 of this paper [MSW00b].

[KL97b], every quasi-action is actually a bounded distance from an action. Theorem 1 complements the building result because bounded valence, bushy trees with cocompact isometry group incorporate thick Euclidean buildings of rank 1. However, the conclusion of Theorem 1 cannot be as strong as the results of [Pan89b] and [KL97b]. A given quasi-action on a bounded valence, bushy tree T may not be quasiconjugate to an action on the same tree T (see Corollary 10); and even if it is, it may not be a bounded distance from an isometric action on T.

The techniques in the proof of Theorem 1 are quite different from the above mentioned results. Starting from the induced action of G on ∂T, first we construct an action on a discrete set, then we attach edges equivariantly to get an action on a locally finite graph quasi-conjugate to the original quasi-action. This graph need not be a tree, however. We next attach 2-cells equivariantly to get an action on a locally finite, simply connected 2-complex quasiconjugate to the original quasi-action. Finally, using Dunwoody's tracks [Dun85], we construct the desired tree action.

Theorem 1 is a very general result, making no assumptions on properness or coboundedness of the quasi-action, and no assumptions whatsoever on the group G. This freedom facilitates numerous applications, particularly for improper quasi-actions.

Application: Quasi-isometric rigidity for graphs of coarse PD(n) groups

From the cobounded, proper case of Theorem 1 it follows that any finitely generated group G quasi-isometric to a free group is the fundamental group of a finite graph of finite groups, and in particular G is virtually free; this result is a well known corollary of work of Stallings [Sta68] and Dunwoody [Dun85]. By dropping properness we obtain a much wider array of quasi-isometric rigidity theorems for certain graphs of groups.

Let Γ be a finite graph of finitely generated groups. There is a vertex group Γ_v for each $v \in \mathrm{Verts}(\Gamma)$; there is an edge group Γ_e for each $e \in \mathrm{Edges}(e)$; and for each end η of an edge e, with η incident to the vertex $v(\eta)$, there is an edge-to-vertex injection $\gamma_\eta \colon \Gamma_e \to \Gamma_{v(\eta)}$. Let $G = \pi_1\Gamma$ be the fundamental group, and let $G \times T \to T$ be the action of G on the Bass-Serre tree T of Γ. See §4 for a brief review of graphs of groups and Bass-Serre trees.

We say that Γ is *geometrically homogeneous* if each edge-to-vertex injection γ_η has finite index image, or equivalently T has bounded valence.

Other equivalent conditions are stated in §4.

Consider for example the class of Poincaré duality n groups or PD(n) groups. If n is fixed then any finite graph of virtual PD(n) groups is geometrically homogeneous, because a subgroup of a PD(n) group K is itself PD(n) if and only it has finite index in K [Bro82]. In particular, if each vertex and edge group of Γ is the fundamental group of a closed, aspherical manifold of constant dimension n then Γ is geometrically homogeneous.

Our main result, Theorem 2, is stated in terms of the (presumably) more general class of "coarse PD(n) groups" defined in §4—such groups respond well to analysis using methods of coarse algebraic topology introduced in [FS96] and further developed in [KK99]. The definition of coarse PD(n) being somewhat technical, for now we state our theorem only for a special case; the general statement is given in Theorem 18.

Theorem 2 (QI-rigidity for graphs of coarse PD(n) groups). *If Γ is a finite graph of groups with bushy Bass-Serre tree, such that each vertex and edge group is virtually a PD(n) group of finite type, and if G is a finitely generated group quasi-isometric to $\pi_1\Gamma$, then G is the fundamental group of a graph of groups with bushy Bass-Serre tree, and with vertex and edge groups quasi-isometric to those of Γ.*

Given a homogeneous graph of groups Γ, the Bass-Serre tree T satisfies a trichotomy: it is either finite, quasi-isometric to a line, or bushy [BK90]. Once Γ has been reduced so as to have no index 1 edge-to-vertex injections, then: T is finite if and only if it is a point, which happens if and only if Γ is a point; and T is quasi-isometric to a line if and only if it is a line, which happens if and only if Γ is a circle with isomorphic edge-to-vertex injections all around or an arc with isomorphic edge-to-vertex injections in the interior and index 2 injections at the endpoints. Thus, in some sense bushiness of the Bass-Serre tree is generic.

Theorem 2 suggests the following problem. Given Γ as in Theorem 2, all edge–to–vertex injections are quasi-isometries. Given $\mathcal{C}$, a quasi-isometry class of coarse PD(n) groups, let $\Gamma\mathcal{C}$ be the class of fundamental groups of finite graphs of groups with vertex and edge groups in $\mathcal{C}$ and with bushy Bass-Serre tree. Theorem 2 says that $\Gamma\mathcal{C}$ is closed up to quasi-isometry.

Problem 3. *Given $\mathcal{C}$, describe the quasi-isometry classes within $\Gamma\mathcal{C}$.*

Here's a rundown of the cases for which the solution to this problem is known to us. Given a metric space X, such as a finitely generated group with the word metric, let $\langle\langle X\rangle\rangle$ denote the class of finitely generated groups quasi-isometric to X.

Coarse PD(0) groups are finite groups, and in this case Theorem 2 reduces to the fact that

$$\langle\langle F_n\rangle\rangle = \Gamma\{\text{finite groups}\} = \{\text{virtual } F_n \text{ groups}, n \geq 2\}$$

where the notation F_n will always mean the free group of rank $n \geq 2$.

Coarse PD(1) groups form a single quasi-isometry class $\mathcal{C} = \langle\langle \mathbf{Z}\rangle\rangle = \{\text{virtual } \mathbf{Z} \text{ groups}\}$. By combining work of Farb and Mosher [FM98], [FM99] with work of Whyte [Why99], these groups are classified as follows:

Theorem 4 (Graphs of coarse PD(1) groups). *If the finitely generated group G is quasi-isometric to a finite graph of virtual $\mathbf{Z}$'s with bushy Bass-Serre tree, then exactly one of the following happens:*

- *There exists a unique power free integer $n \geq 2$ such that G modulo some finite normal subgroup is abstractly commensurable to the solvable Baumslag-Solitar group $\mathrm{BS}(1,n) = \langle a,t \mid tat^{-1} = a^n\rangle$.*
- *G is quasi-isometric to any of the nonsolvable Baumslag-Soliter groups $\mathrm{BS}(m,n) = \langle a,t \mid ta^m t^{-1} = a^n\rangle$ with $2 \leq m < n$.*
- *G is quasi-isometric to any group $F \times \mathbf{Z}$ where F is free of finite rank ≥ 2.*

Proof. By Theorem 2 we have $G = \pi_1\Gamma$ where Γ is a finite graph of virtual $\mathbf{Z}$'s with bushy Bass-Serre tree. If G is amenable then the first alternative holds, by [FM99]. If G is nonamenable then either the second or the third alternative holds, by [Why99]. $\diamondsuit$

For $\mathcal{C} = \langle\langle \mathbf{Z}^n\rangle\rangle$, the amenable groups in $\Gamma\langle\langle \mathbf{Z}^n\rangle\rangle$ form a quasi-isometrically closed subclass which is classified up to quasi-isometry in [FM00b], as follows. By applying Theorem 1 it is shown that each such group is virtually an ascending HNN group of the form $\mathbf{Z}^n *_M$ where $M \in \mathrm{GL}(n,\mathbf{R})$ has integer entries and $|\det(M)| \geq 2$; the classification theorem of [FM00b] says that the absolute Jordan form of M, up to an integer power, is a complete quasi-isometry invariant. For general groups in $\Gamma\langle\langle \mathbf{Z}^n\rangle\rangle$, Whyte reduces the problem to understanding when two subgroups of $\mathrm{GL}_n(\mathbf{R})$ are at finite Hausdorff distance [Why].

For $\mathcal{C} = \langle\langle \mathbf{H}^2\rangle\rangle$, the subclass of $\Gamma\langle\langle \mathbf{H}^2\rangle\rangle$ consisting of word hyperbolic surface-by-free groups is quasi-isometrically rigid and is classified by Farb and Mosher in [FM00a]. The broader classification in $\Gamma\langle\langle \mathbf{H}^2\rangle\rangle$ is open.

If $\mathcal{C}$ is the quasi-isometry class of cocompact lattices in an irreducible, semisimple Lie group L with finite center, then combining Mostow Rigidity for L with quasi-isometric rigidity (see [Far97] for a survey) it follows that for each $G \in \mathcal{C}$ there exists a homomorphism $G \to L$ with finite kernel and discrete, cocompact image, and this homomorphism is unique up to post-composition with an inner automorphism of L. Combining this with Theorem 2 it follows that $\Gamma\mathcal{C}$ is a single quasi-isometry class, represented by the cartesian product of any group in $\mathcal{C}$ with any free group of rank ≥ 2.

Remark In [FM99] it is proved that any finitely generated group G quasi-isometric to $\mathrm{BS}(1,n)$, where $n \geq 2$ is a power free integer, has a finite subgroup F so that G/F is abstractly commensurable to $\mathrm{BS}(1,n)$. Theorem 4 can be applied to give a (mostly) new proof, whose details are found in [FM00b].

Application: Actions on Cantor sets

Quasiconformal actions. The boundary of a δ-hyperbolic metric space X carries a quasiconformal structure and a well behaved notion of uniformly quasiconformal homeomorphisms, which as Paulin showed can be characterized in terms of cross ratios [Pau96]; we review this in Section 5. As such, one ask can for a generalization of the Sullivan–Tukia theorem for $\mathbf{H}^3$: is every uniformly quasiconformal group action on ∂X quasiconformally conjugate to a conformal action?

A bounded valence, bushy tree T has Gromov boundary $B = \partial T$ homeomorphic to a Cantor set, and we answer the above question in the affirmative for B, where "conformal action" is interpreted as the induced action at infinity of an isometric action on some other bounded valence, bushy tree:

Theorem 5 (Quasiconformal actions on Cantor sets). *If the Cantor set B is equipped with a quasiconformal structure by identifying $B = \partial T$ for some bounded valence, bushy tree T, and if $G \times B \to B$ is a uniformly quasiconformal action of a group G on B, then there exists a tree T' and a quasiconformal homeomorphism $\phi\colon B \to \partial T'$ which conjugates the G-action on B to an action on $\partial T'$ which is induced by some isometric action of G on T'.*

In many situations Theorem 5 allows one to make additional conclusions about the algebraic structure of the given group G. Recall that an isometric action of a group G on a δ-hyperbolic metric space X is cocompact if and only if the induced action of G on the space of distinct triples in ∂X is cocompact, and the action of G has bounded orbits in X if and only if the induced action of G on the space of distinct pairs in ∂X has precompact orbits.

Corollary 6. *Under the same hypotheses as Theorem 5, if the action of G on B is cocompact on the triple space of B then G is the fundamental group of a finite graph of groups Γ with finite index edge-to-vertex injections; moreover a subgroup $H < G$ stabilizes some vertex of the Bass-Serre tree of Γ if and only if the action of H on the space of distinct pairs in B has precompact orbits.*

Once the definitions are reviewed, the proofs of Theorem 5 and Corollary 6 are very quick applications of Theorem 1.

Theorem 5 complements similar theorems for the boundaries of all rank 1 symmetric spaces. Any uniformly quasiconformal action on the boundary of $\mathbf{H}^2$ or $\mathbf{H}^3$ is quasiconformally conjugate to a conformal action. Any uniformly quasiconformal action on the boundary of $\mathbf{H}^n$, $n \geq 4$ [Tuk86] or of $\mathbf{CH}^n$ [Cho96], such that the induced action on the triple space of the boundary is cobounded, is quasiconformally conjugate to a conformal action. Any quasiconformal map on the boundary of a quaternionic hyperbolic space or the Cayley hyperbolic plane is conformal [Pan89b].

Uniform quasisimilarity actions on the n-adics. Given $n \geq 2$, let $\mathbf{Q}_n$ be the n-adics, a complete metric space whose points are formal series

$$\xi = \sum_{i=k}^{+\infty} \xi_i \, n^i, \quad \text{where} \quad \xi_i \in \mathbf{Z}/n\mathbf{Z} \quad \text{and} \quad k \in \mathbf{Z}$$

The distance between $\xi, \eta \in \mathbf{Q}_n$ equals n^{-I} where I is the greatest element of $\mathbf{Z} \cup \{+\infty\}$ such that $\xi_i = \eta_i$ for all $i \leq I$. The metric space $\mathbf{Q}_n$ has Hausdorff dimension 1, and it is homeomorphic to a Cantor set minus a point.

Given integers $m, n \geq 2$, Cooper proved that the metric spaces $\mathbf{Q}_m, \mathbf{Q}_n$ are bilipschitz equivalent if and only if there exists integers $k \geq 2$, $i, j \geq 1$ such that $m = k^i$, $n = k^j$ (see Cooper's appendix to [FM98]). Thus, each bilipschitz class of n-adic metric spaces is represented uniquely by some $\mathbf{Q}_m$ where m is not a proper power.

A *similarity* of a metric space X is a bijection $f \colon X \to X$ such that the ratio $\frac{d(f\xi, f\eta)}{d(\xi,\eta)}$ is constant, over all $\xi \neq \eta \in X$. A K-quasisimilarity, $K \geq 1$, is a bijection $f \colon X \to X$ with the property that

$$\frac{d(f\zeta, f\omega)}{d(\zeta,\omega)} \Bigg/ \frac{d(f\xi, f\eta)}{d(\xi,\eta)} \leq K, \quad \forall \zeta \neq \omega, \xi \neq \eta \in X$$

A 1-quasisimilarity is the same thing as a similarity.

In [FM99] it was asked whether any uniform quasisimilarity action on $\mathbf{Q}_n$ is bilipschitz conjugate to a similarity action, as long as n is not a proper power. In retrospect this is not quite the correct question, and in fact there is an easy counterexample: the full similarity group of $\mathbf{Q}_4$ acts as a uniform quasisimilarity group on $\mathbf{Q}_2$, but there is no bilipschitz conjugacy to a similarity action on $\mathbf{Q}_2$ (see the end of §5.2 for details). This can be extended to show that for any m and any $i \neq j$, there is a uniform quasisimilarity action on $\mathbf{Q}_{m^i}$ which is not bilipschitz conjugate to a similarity action on $\mathbf{Q}_{m^j}$. The following theorem resolves the issue in the best possible way, at least for actions satisfying an additional boundedness property:

Theorem 7 (Uniform quasisimilarity actions on n-adic Cantor sets). *Given $n \geq 2$, suppose that $G \times \mathbf{Q}_n \mapsto \mathbf{Q}_n$ is a uniform quasisimilarity action: there exists $K \geq 1$ such that each element of G acts by a K-quasisimilarity. Suppose in addition that the induced action of G on the space of distinct pairs in $\mathbf{Q}_n$ is cocompact. Then there exists $m \geq 2$ and a bilipschitz homeomorphism $\mathbf{Q}_n \mapsto \mathbf{Q}_m$ which conjugates the G action on $\mathbf{Q}_n$ to a similarity action on $\mathbf{Q}_m$.*

This theorem generalizes the similar result of [FM99] for uniform quasisimilarity actions on $\mathbf{R}$, which was in turn an analogue of Hinkkanen's Theorem [Hin85] for uniformly quasisymmetric actions on $\mathbf{R}$. We do not know whether the cocompactness hypothesis is necessary, but it is a useful and commonly occurring boundedness property.

As with Theorem 5, the result of Theorem 7 allows us to make some extra conclusions about the algebraic structure of the given group G, namely an ascending HNN structure whose base group is geometrically constrained:

Corollary 8. *Under the same hypotheses as Theorem 7, there is an ascending HNN decomposition*

$$G = \langle H, t \mid tht^{-1} = \phi(h), \forall h \in H \rangle,$$

where H is a subgroup of G and $t \in G$, such that $\phi\colon H \to H$ is a self-monomorphism with finite index image, and the action of H on $\mathbf{Q}_n$ is uniformly bilipschitz.

Application: Virtually free, cocompact lattices

Given a finitely generated group G, one can ask to describe the *model geometries* for G, the proper metric spaces X on which G acts, properly and coboundedly by isometries. More generally, motivated by $\mathrm{Isom}(X)$, one can ask to describe the locally compact topological groups Γ for which there is a discrete, cocompact, virtually faithful representation $G \to \Gamma$. Then, given a quasi-isometry class $\mathcal{C}$ of finitely generated groups, one can ask:

- Is there a common model geometry X for every group in $\mathcal{C}$?
- Is there a common locally compact group Γ, in which every group of $\mathcal{C}$ has a discrete, cocompact, virtually faithful representation?

For example, the Sullivan-Tukia theorem answers these two questions affirmatively for the quasi-isometry class $\langle\langle \mathbf{H}^n \rangle\rangle$, using the space $X = \mathbf{H}^n$ and the group $\Gamma = \mathrm{Isom}(\mathbf{H}^n)$. Most quasi-isometric rigidity theorems in the literature provide similarly affirmative answers for the quasi-isometry class under consideration, e.g. [BP00], [FM99], [FM00b], [FM00a], [KL97a], [KL97b], [Tab00]; see [Far97] for a survey. On the other hand, it may be true that there are no less than two model geometries for the quasi-isometry class $\langle\langle \mathbf{H}^2 \times \mathbf{R} \rangle\rangle = \langle\langle \widetilde{\mathrm{PSL}_2 \mathbf{R}} \rangle\rangle$ [Rie93].

We show that the above questions have a negative answer for the quasi-isometry class $\langle\langle F_n \rangle\rangle$. Our main tool is the following:

Theorem 9 (Virtually free, cocompact lattices). *Let $\mathcal{G}$ be a locally compact topological group which contains a cocompact lattice in the class $\langle\langle F_n \rangle\rangle$. Then there exists a cocompact action of $\mathcal{G}$ on a bushy tree T of bounded valence, inducing a continuous, proper homomorphism $\mathcal{G} \to \mathrm{Isom}(T)$ with compact kernel and cocompact image.*

Corollary 10. *There exist groups $G, G' \in \langle\langle F_n \rangle\rangle$ such that:*

- *G, G' do not act properly discontinuously and cocompactly by isometries on the same proper geodesic metric space.*
- *G, G' do not have discrete, cocompact, virtually faithful representations into the same locally compact group.*

Theorem 9 reduces the corollary to the statement that there are virtually free groups which cannot act properly and cocompactly on the same tree. It is easy to produce examples of this phenomenon, for example $\mathbf{Z}/p * \mathbf{Z}/p$ and $\mathbf{Z}/q * \mathbf{Z}/q$, for distinct primes $p, q \geq 3$. These are the first examples of quasi-isometric groups which are known not to have a common geometric model.

Theorem 9 complements a recent result of Alex Furman [Fur00] concerning an irreducible lattice G in a semisimple Lie group Γ. Furman's result shows, *except* when $G \in \langle\langle F_n \rangle\rangle$ is noncocompact in $\mathrm{SL}(2,\mathbf{R})$, that any locally compact group $\mathcal{G}$ in which G has a discrete, cocompact, virtually faithful representation is very closely related to the given Lie group Γ.

Remark The techniques of the above results should apply to more general homogeneous graphs of groups. In particular, one ought to be able to determine, using these ideas, which of the Baumslag-Solitar groups $\mathrm{BS}(m,n)$ ([Why99]) are cocompact lattices in the same locally compact group. Also, using the computation of $\mathrm{QI}(\mathrm{BS}(1,n))$ in [FM98], it should be possible to give a conjugacy classification of the maximal uniform cobounded subgroups of $\mathrm{QI}(\mathrm{BS}(1,n))$.

Other applications

Quasi-actions on products of trees. Recently A. Reiter [Rei00] has combined Theorem 1 with results of Kleiner and Leeb on quasi-isometric rigidity for Euclidean buildings [KL97b] to prove:

Theorem 11. *Suppose that G is a finitely generated group quasi-isometric to a product of trees $\Pi_{i=1}^k T_i$, each tree of bounded valence. Then G has a finite index subgroup of index at most $k!$ which is isomorphic to a discrete cocompact subgroup of* $\mathrm{Isom}\left(\Pi_{i=1}^k T_i'\right)$ *where each T_i' is a tree quasi-isometric to the corresponding T_i.* $\diamondsuit$

The finite index arising in this theorem comes from the fact that G is allowed to permute the k factors among themselves. For example, every group quasi-isometric to a product of two bounded valence bushy trees has a subgroup of index ≤ 2 which acts, properly and coboundedly, on a product of two bounded valence, bushy trees. This quasi-isometry class contains all products of two free groups of rank ≥ 2, but it also contains torsion free simple groups [BM97].

Maximally symmetric trees. In light of Theorem 10 showing that there is no single model geometry for an entire quasi-isometry class $\langle\langle F_n \rangle\rangle$, one might still ask for a list of the "best" model geometries for the class. In [MSW00a] we apply Theorem 1 to show that these consist of certain trees which are "maximally symmetric".

Recall that for any metric space X the *quasi-isometry group* $\mathrm{QI}(X)$ is the group of self quasi-isometries of X modulo identification of quasi-isometries which have bounded distance in the sup norm. A subgroup $H < \mathrm{QI}(X)$ is *uniform* if it can be represented by a quasi-action on X. A uniform subgroup

$H < \mathrm{QI}(X)$ is *cobounded* if the induced quasi-action of H on X is cobounded. A bounded valence, bushy tree T is *cocompact* if $\mathrm{Isom}(T)$ acts cocompactly on T; equivalently, the image of the natural homomorphism $\mathrm{Isom}(T) \to \mathrm{QI}(T)$ is cobounded. We say that T is *minimal* if it has no valence 1 vertices; minimality implies that $\mathrm{Isom}(T)$ has no compact normal subgroups, and that $\mathrm{Isom}(T) \to \mathrm{QI}(T)$ is injective, among other nice properties.

Theorem 12 (Characterizing maximally symmetric trees [MSW00a]). *For any bounded valence, bushy, cocompact, minimal tree T, the following are equivalent:*

- $\mathrm{Isom}(T)$ *is a maximal uniform cobounded subgroup of* $\mathrm{QI}(T)$.
- *For any bounded valence, bushy, minimal tree T', any continuous, proper, cocompact embedding* $\mathrm{Isom}(T) \to \mathrm{Isom}(T')$ *is an isomorphism.*
- *For any locally compact group $\mathcal{G}$ without compact normal subgroups, any continuous, proper, cocompact embedding* $\mathrm{Isom}(T) \to \mathcal{G}$ *is an isomorphism.*

Such trees T are called *maximally symmetric*. Theorem 12 says nothing about existence of maximally symmetric trees. In [MSW00a] we also prove:

Theorem 13 (Enumerating maximally symmetric trees). *Fix a bounded valence, bushy tree τ. Every uniform cobounded subgroup of* $\mathrm{QI}(\tau)$ *is contained in a maximal uniform cobounded subgroup. Every maximal uniform cobounded subgroup of* $\mathrm{QI}(\tau)$ *is identified with the isometry group of some maximally symmetric tree T via a quasi-isometry $T \leftrightarrow \tau$, inducing a natural one-to-one correspondence between conjugacy classes of maximal uniform cobounded subgroups of* $\mathrm{QI}(\tau)$ *and isometry classes of reduced maximally symmetric trees T. There is a countable infinity of such isometry classes; and there is a countable infinity of these isometry classes represented by trees T which support a proper, cobounded group action.*

The term "reduced" refers to a simple combinatorial operation that simplies maximally symmetric trees, as explained in [MSW00a].

To summarize, there is a countable infinity of "best" geometries for the quasi-isometry class $\langle\langle F_n \rangle\rangle$, distinct up to isometry. Examples include any homogeneous tree of constant valence ≥ 3, and any bipartite, bihomogeneous tree of valences $p \neq q \geq 3$; these each have proper, cobounded actions, and there is still a countable infinity of other examples.

Theorem 13 should be contrasted with the fact that if X is a rank 1 symmetric space then $\mathrm{QI}(X)$ has a unique maximal uniform cobounded subgroup up to conjugacy, namely $\mathrm{Isom}(X)$; this follows from the fact that every cobounded, uniformly quasiconformal subgroup acting on ∂X is quasiconformally conjugate to a conformal group.

3 Quasi-edges and the proof of Theorem 1

3.1 Preliminaries

Coarse language Let X be a metric space. Given $A \subset X$ and $R \geq 0$, denote $N_R(A) = \{x \in X \mid \exists a \in A \quad \text{such that} \quad d(a,x) \leq R\}$. Given subsets $A, B \subset X$, let $A \overset{[R]}{\subset} B$ denote $A \subset N_R(B)$. Let $A \overset{c}{\subset} B$ denote the existence of $R \geq 0$ such that $A \overset{[R]}{\subset} B$; this is called *coarse containment* of A in B. Let $A \overset{[R]}{=} B$ denote the conjunction of $A \overset{[R]}{\subset} B$ and $B \overset{[R]}{\subset} A$; this is equivalent to the statement $d_{\mathcal{H}}(A,B) \leq R$ where $d_{\mathcal{H}}(\cdot,\cdot)$ denotes Hausdorff distance. Let $A \overset{c}{=} B$ denote the existence of R such that $A \overset{[R]}{=} B$; this is called *coarse equivalence* of A and B.

Given a metric space X and subsets A, B, we say that a subset C is a *coarse intersection* of A and B if we have $N_R(A) \cap N_R(B) \overset{c}{=} C$ for all sufficiently large R. A coarse intersection of A and B may not exist, but if one does exist then it is well-defined up to coarse equivalence.

Given metric spaces X, Y, a map $f\colon X \to Y$ is *coarse Lipschitz* if f stretches distances by at most an affine function: there exist $K \geq 1$, $C \geq 0$ such that

$$d_Y(fx, fy) \leq K d_X(x,y) + C$$

We say that f is a *uniformly proper embedding* if, in addition, f compresses distances by a uniform amount: there exists a proper, increasing function $\rho\colon [0,\infty) \to [0,\infty)$ such that

$$\rho(d_X(x,y)) \leq d_Y(fx, fy)$$

More precisely we say that f is a (K, C, ρ)-uniformly proper embedding. If we can take $\rho(d) = \frac{1}{K}d - C$ then we say that f is a K, C *quasi-isometric embedding.* If furthermore $f(X) \overset{[C]}{=} Y$ then we say that f is a K, C *quasi-isometry between X and Y.* A C'*-coarse inverse* of f is a K, C' quasi-isometry $g\colon Y \to X$ such that $x \overset{[C']}{=} g(f(x))$ and $y \overset{[C']}{=} f(g(y))$, for all $x \in X$,

$y \in Y$. A simple fact says that for all K, C there exists C' such that each K, C quasi-isometry has a C'-coarse inverse.

Let G be a group and X a metric space. A K, C *quasi-action* of G on X is a map $G \times X \to X$, denoted $(g, x) \mapsto A_g(x) = g \cdot x$, so that for each $g \in G$ the map $A_g \colon X \to X$ is a K, C quasi-isometry of X, and for each $x \in X, g, h \in G$ we have

$$g \cdot (h \cdot x) \overset{[C]}{=} (gh) \cdot x$$

In other words, the sup norm distance between $A_g \circ A_h$ and A_{gh} is at most C. A quasi-action is *cobounded* if there exists a constant R such that for each $x \in X$ we have $G \cdot x \overset{[R]}{=} X$. A quasi-action is *proper* if for each R there exists M such that for all $x, y \in X$, the cardinality of the set $\{g \in G \mid (g \cdot N(x, R)) \cap N(y, R) \neq \emptyset\}$ is at most M. Note that if $G \times X \to X$ is an isometric action on a proper metric space, then "cobounded" is equivalent to "cocompact" and "proper" is equivalent to "properly discontinuous".

Given a group G and quasi-actions of G on metric spaces X, Y, a *quasi-conjugacy* is a quasi-isometry $f \colon X \to Y$ such that for some $C \geq 0$ we have $f(g \cdot x) \overset{[C]}{=} g \cdot fx$ for all $g \in G$, $x \in X$. Properness and coboundedness are invariants of quasiconjugacy.

A fundamental principle of geometric group theory says that if G is a finitely generated group equipped with the word metric, and if X is a proper geodesic metric space on which G acts properly discontinuously and cocompactly by isometries, then G is quasi-isometric to X.

A partial converse to this result is the *quasi-action principle* which says that if G is a finitely generated group with the word metric and X is a metric space quasi-isometric to G then there is a cobounded, proper quasi-action of G on X; the constants for this quasi-action depend only on the quasi-isometry constants between G and X.

Ends. Recall the *end compactification* of a locally compact space Hausdorff X. The direct system of compact subsets of X under inclusion has a corresponding inverse system of unbounded complementary components of compact sets, and an *end* is an element of the inverse limit. Letting $\mathrm{Ends}(X)$ be the set of ends, there is a compact Hausdorff topology on $\overline{X} = X \cup \mathrm{Ends}(X)$ in which X forms a dense open set, where for each $e \in \mathrm{Ends}(X)$ there is one basic open neighborhood of e for each compact subset $K \subset X$, consisting of the unbounded component U of $X - K$ corresponding to e together with all ends e' for which U is the corresponding unbounded component of $X - K$.

If $f: X \to Y$ is a quasi-isometry between proper geodesic metric spaces then there is a natural induced homeomorphism $\mathrm{Ends}(X) \to \mathrm{Ends}(Y)$.

If T is a bounded valence, bushy tree then $\mathrm{Ends}(T)$ is a Cantor set. Moreover, there is a natural homeomorphism between $\mathrm{Ends}(T)$ and the Gromov boundary of T.

3.2 Setup

Let G be a finitely generated group quasi-acting on a bounded valence, bushy tree T. To prove Theorem 1 we must construct a quasiconjugacy to an isometric action of G on another tree. When the given quasi-action is cobounded we give the proof in full, leaving for the end a sketch of the noncobounded case; note that all of our applications involve the cobounded case of Theorem 1.

Before continuing, we immediately reduce to the case where every vertex of T has valence ≥ 3. To do this we need only construct a quasi-isometry ϕ from T to a bounded valence tree in which every vertex has valence ≥ 3, for we can then use ϕ to quasiconjugate the given G-quasi-action on T.

Let β be a bushiness constant for T: every vertex of T is within distance β of a vertex with ≥ 3 complementary components. There is a β-bushy subtree $T' \subset T$ containing no valence 1 vertices, such that every vertex of T is within distance β of a vertex of T'. The nearest point projection map $T \to T'$ is a $(1, \beta)$ quasi-isometry. Since each valence 2 vertex of T' is within distance β of a vertex of valence ≥ 3, we may next produce a tree T'' by changing the tree structure on T', removing vertices of valence 2 and conglomerating any path of edges through valence 2 vertices into a single edge. The "identity" map $T' \to T''$ is a β, β quasi-isometry, and T'' is the desired tree in which each vertex has valence ≥ 3. Replacing T by T'', we may henceforth assume every vertex of T has valence ≥ 3.

While our ultimate goal is a quasiconjugacy to an action on a tree, our intermediate goal will be a quasiconjugacy to an action on a certain 2-complex: we construct an isometric action of G on a 2-complex X, and a quasiconjugacy $f: X \to T$, so that X is simply connected and uniformly locally finite. Once this is accomplished we use Dunwoody tracks to construct a quasiconjugacy from the G action on X to a G action on a tree.

The first step in building the 2-comples X is to find the vertex set X^0. We will build the vertex set using the G action on the ends of T (note that even though G only quasi-acts on T it still honestly acts on the ends).

If G actually acts on T then our construction gives for X the complex with 1-skeleton the dual graph of T, with 2-cells attached around the vertices of T so as to make the dual graph simply connected. This picture should make the construction easier to follow.

3.3 Quasi-edges

From the action at infinity we want to get some finite action. Each edge e of T cuts T into two "sides", which are the two components of $T - \text{int}(e)$, each a subtree of T. The end spaces of the two sides of e partition $\text{Ends}(T)$ into an unordered pair of subsets denoted $\mathcal{E}(e) = \{\mathcal{C}_1, \mathcal{C}_2\}$. Each of these $\mathcal{C}_1, \mathcal{C}_2$ is a *clopen* of $\text{Ends}(T)$ which means a subset that is both closed and open.

Generalizing this, we define a *quasi-edge* of T to be a decomposition of $\text{Ends}(T)$ into a disjoint, unordered pair of clopens $\mathcal{E} = \{\mathcal{C}_1, \mathcal{C}_2\}$, and $\mathcal{C}_1, \mathcal{C}_2$ are called the *sides* of the quasiedge $\mathcal{E}$. Although G does not act on the set of edges a priori, clearly G acts on the set $\text{QE}(T)$ of quasi-edges.

We now define the "distortion" of each quasi-edge $\mathcal{E}$ of T. This is a positive integer $R(\mathcal{E})$ which measures how far $\mathcal{E}$ is from being a true edge. In particular, $R(\mathcal{E})$ will equal 1 if and only if $\mathcal{E} = \mathcal{E}(e)$, as described above, for a unique edge e.

Consider a clopen subset $\mathcal{C}$ of $\text{Ends}(T)$. Let $\mathcal{H}(\mathcal{C})$ denote the convex hull of $\mathcal{C}$ in T, the smallest subtree of T whose set of accumulation points in $\text{Ends}(T)$ is $\mathcal{C}$. Equivalently, $\mathcal{H}(\mathcal{C})$ is the union of all bi-infinite geodesics in T whose endpoints lie in $\mathcal{C}$. Under the inclusion $\mathcal{H}(\mathcal{C}) \hookrightarrow T$ we may identify $\text{Ends}(\mathcal{H}(\mathcal{C}))$ with $\mathcal{C} \subset \text{Ends}(T)$.

Consider a quasi-edge $\mathcal{E} = \{\mathcal{C}_1, \mathcal{C}_2\}$ of T. Since each vertex of T has valence at least 3, it follows that $V \subset \mathcal{H}(\mathcal{C}_1) \cup \mathcal{H}(\mathcal{C}_2)$; for if there existed $v \in V - \big(\mathcal{H}(\mathcal{C}_1) \cup \mathcal{H}(\mathcal{C}_2)\big)$ then by convexity at least one of the three or more components of $T - v$ would be disjoint from both $\mathcal{H}(\mathcal{C}_1)$ and $\mathcal{H}(\mathcal{C}_2)$, and all the ends of T reachable by that component would be disjoint from both $\mathcal{C}_1$ and $\mathcal{C}_2$, contradicting that $\mathcal{E} = \{\mathcal{C}_1, \mathcal{C}_2\}$ is a quasi-edge. A similar argument, together with connectivity of T, shows that either $\mathcal{H}(\mathcal{C}_1) \cap \mathcal{H}(\mathcal{C}_2) \neq \emptyset$ or the shortest path in T connecting a point of $\mathcal{H}(\mathcal{C}_1)$ to a point of $\mathcal{H}(\mathcal{C}_2)$ is an edge. It follows that there is at least one edge e of T such that $\partial e \cap \mathcal{H}(\mathcal{C}_i) \neq \emptyset$ for each $i = 1, 2$; moreover, if $\mathcal{H}(\mathcal{C}_1) \cap \mathcal{H}(\mathcal{C}_2) \neq \emptyset$ then there are at least three such edges. Let $\text{Core}(\mathcal{E}) = \text{Core}(\mathcal{C}_1, \mathcal{C}_2)$ denote the union of all such edges e; equivalently,

$$\text{Core}(\mathcal{E}) = N_1(\mathcal{H}(\mathcal{C}_1)) \cap N_1(\mathcal{H}(\mathcal{C}_2))$$

where N_1 denotes the neighborhood of radius 1 in T. Since $N_1(\mathcal{H}(\mathcal{C}_i))$ is convex it follows that $\text{Core}(\mathcal{E})$ is a subtree of T. In fact $\text{Core}(\mathcal{E})$ is a finite subtree: if it were an infinite subtree then it would accumulate on some end of T, producing a point contained in the closures of both $\mathcal{C}_1$ and $\mathcal{C}_2$, contradicting that $\mathcal{E} = \{\mathcal{C}_1, \mathcal{C}_2\}$ is a quasi-edge. We have also seen that $\mathrm{Core}(\mathcal{E})$ contains at least one edge—it is not a single point. The number

$$R(\mathcal{E}) = \mathrm{Diam}(\mathrm{Core}(\mathcal{E}))$$

is therefore a positive integer, called the *quasi-edge distortion* of $\mathcal{E}$. Given a positive integer R, if $R(\mathcal{E}) \le R$ then we say that $\mathcal{E}$ is an *R-quasi-edge*.

It's obvious that if e is an edge, then the associated quasi-edge $\mathcal{E}(e)$ satisfies $R = 1$. Conversely, given a quasi-edge $\mathcal{E} = \{\mathcal{C}_1, \mathcal{C}_2\}$ with $R(\mathcal{E}) = 1$, it follows that $\mathcal{H}(\mathcal{C}_1) \cap \mathcal{H}(\mathcal{C}_2)$ is a single edge e. Moreover the two sides of e must be $\mathcal{H}(\mathcal{C}_1)$ and $\mathcal{H}(\mathcal{C}_2)$, and so $\mathcal{E} = \mathcal{E}(e)$.

As we saw above, a quasi-isometry takes quasi-edges to quasi-edges, and now we investigate how the quasi-edge distortion is affected by a quasi-isometry:

Lemma 14 (Behavior of quasi-edge distortion under a quasi-isometry). *For each K, C there exists a constant A such that if ϕ is a K, C quasi-isometry of $V = \mathrm{Verts}(T)$ and if $\mathcal{E} = \{\mathcal{C}_1, \mathcal{C}_2\}$ is a quasi-edge then*

$$d_{\mathcal{H}}\Big(\phi\big(\mathrm{Core}(\mathcal{C}_1, \mathcal{C}_2) \cap V\big), \mathrm{Core}\big(\phi(\mathcal{C}_1), \phi(\mathcal{C}_2)\big)\Big) \le A$$

It follows that if $\mathcal{E}$ is an R-quasi-edge then $\phi(\mathcal{E})$ is a $KR+C+2A$ quasi-edge.

Proof. There is a constant A' depending only on K, C such that if γ is a bi-infinite geodesic in T with boundary $\partial\gamma \subset \mathrm{Ends}(T)$ then $\phi(\gamma \cap V)$ is within Hausdorff distance A' of the bi-infinite geodesic connecting the two points $\phi(\partial\gamma)$. It follows that

$$d_{\mathcal{H}}\big(\phi(\mathcal{H}(\mathcal{C}_i) \cap V), \mathcal{H}(\phi(\mathcal{C}_i)\big) \le A'$$

and so

$$d_{\mathcal{H}}\big(\phi(N_1(\mathcal{H}(\mathcal{C}_i)) \cap V, N_1(\mathcal{H}(\phi(\mathcal{C}_i)))\big) \le A' + K + C = A$$

from which it follows that

$$d_{\mathcal{H}}\big(\phi(\mathrm{Core}(\mathcal{C}_1, \mathcal{C}_2) \cap V), \mathrm{Core}(\phi(\mathcal{C}_1), \phi(\mathcal{C}_2))\big) \le A$$

$\diamondsuit$

3.4 Construction of the 2-complex X

The 0-skeleton of X. Consider the action of G on $\mathrm{QE}(T)$. The 0-skeleton X^0 consists of the union of G-orbits of 1-quasi-edges of T. By Lemma 14 each element of X^0 is an R-quasi-edge where $R = K + C + 2A$, although perhaps not all R-quasi-edges are in X^0. Clearly G acts on X^0, because X^0 is a union of G-orbits of the action of G on the set of all quasi-edges $\mathrm{QE}(T)$. Define a map $f\colon X^0 \to \mathrm{Verts}(T)$ by taking a quasi-edge $\mathcal{E}$ to any vertex in $\mathrm{Core}(\mathcal{E})$; by Lemma 14 this map is coarsely G-equivariant.

Since each edge $e \subset T$ determines a 1-quasi-edge $\mathcal{E}(e)$, since $\mathcal{E}(e) \in X^0$, and since $e = \mathrm{Core}(\mathcal{E}(e))$, it follows that at least one of the two vertices of e is in $f(X^0)$.

We claim that the cardinality $\left|f^{-1}(v)\right|$ is uniformly bounded independent of v. It suffices to verify that there are boundedly many quasi-edges $\mathcal{E}$ such that $v \in \mathrm{Core}(\mathcal{E})$. To verify this, first note that the number of vertices of T within distance R of v is at most $1 + k(k-1)^{R-1}$, where k is the maximum valence of a vertex of T. It follows that there are boundedly many subtrees of T of diameter $\leq R$ containing v. For each such subtree $\mathcal{C}$, there are boundedly many components of $T - \mathcal{C}$. A quasi-edge with core $\mathcal{C}$ is determined by a partition of the component set of $T - \mathcal{C}$ into two subsets, and there are boundedly many such partitions. This proves the claim.

The 1-skeleton of X. We now extend the G-set X^0 to a 1-dimensional G-complex X^1, by attaching edges to X^0 in two stages: first to make X^1 quasi-isometric to T, and second to extend the G action.

In the first stage, two vertices $v, w \in X^0$ are attached by an edge if $d(f(v), f(w)) = 0$ or 1 in T, or if $d(f(v), f(w)) = 2$ in T and the vertex of T between $f(v)$ and $f(w)$ is not in $f(X^0)$. As noted above, each edge of T contains at least one endpoint in $f(X^0)$. Already at this stage the 1-complex is connected, because for any $v, w \in X^0$, if $f(v) = f(w)$ then there is an edge from v to w, and if $f(v) \neq f(w)$ then the path in T from $f(v)$ to $f(w)$ has at least every other vertex in $\mathrm{image}(f)$. Any further attachment of cells of dimension ≥ 1 will preserve connectedness; we use this fact without comment from now on.

In the second stage, attach additional edges in a G-equivariant manner: given vertices $v, w \in X^0$, if there exists $h \in G$ such that $h \cdot v, h \cdot w$ are attached by a first stage edge, and if v, w are not already attached by a first stage edge, then v, w are to be attached by a second stage edge. This defines a connected 1-complex X^1, and an action of G on X^1.

The map $f\colon X^0 \to T$ is extended over X^1 by taking each edge $e \subset X^1$

to the shortest path in T connecting $f(v)$ to $f(w)$, where $\partial e = \{v, w\}$.

Putting the usual geodesic metric on X^1 where each edge has length 1, we claim the map $f\colon X^1 \to T$ is a G-quasiconjugacy. To prove this, we've already shown that $f \mid X^0$ is coarsely equivariant, and so it suffices to show that $f \mid X^0$ is a quasi-isometry. To see why, note first that each first stage edge of X^1 maps to a path of length ≤ 2 in T, and each second stage edge maps to a path of length $\le 2K + C$, where K, C are quasi-isometry constants for the G-quasi-action on T; this shows that f is coarsely lipschitz, in fact $f(d(v), d(w)) \le (2K + C)d(v, w)$ for $v, w \in \text{Verts}(T)$. To get the other direction, consider $v, v' \in X^0$. If $f(v) = f(v')$ then $d(v, v') \le 1$. If $f(v) \ne f(v')$, let $f(v) = w_0 \to w_1 \to \cdots \to w_k = f(v')$ be the geodesic in T from $f(v)$ to $f(v')$; since at least every other vertex $w_0, \ldots, w_k$ is in $f(X^0)$ it follows there is an edge path in X^1 from v to v', consisting entirely of first stage edges, of length $\le k$. In either case we've shown that $d(v, v') \le d(f(v), f(v')) + 1$.

We note some additional facts about the 1-complex X^1:

(1) For each $w \in \text{Verts}(T)$ the set $f^{-1}(w) \cap X^0$ has uniformly bounded cardinality.

(2) The vertices of X^1 have bounded valence.

(3) For each edge e of T and each $x \in \text{int}(e)$, the set $f^{-1}(x)$ is a finite set of uniformly bounded cardinality and diameter in X^1.

Fact (1) was demonstrated earlier. Facts (2) and (3) both follow from Fact (1), together with the fact that for each edge $v \xrightarrow{e} v'$ in X^1 the distance $d(f(v), f(v'))$ is uniformly bounded, and the fact that an edge in X^1 is determined by its endpoints.

The 2-skeleton of X. We claim that there is a constant B such that attaching 2-cells along all simple loops in X^1 of length $\le B$ results in a simply connected 2-complex. First note that there is a B such that all isometrically embedded loops have length $\le B$, this follows immediately from the fact that X^1 is quasi-isometric to a tree (indeed it holds for any Gromov hyperbolic graph). Any loop is freely homotopic to a concatenation of simple loops, and any simple loop is freely homotopic to a concatenation of isometrically embedded loops. Thus, once 2-cells are attached along all simple loops of length $\le B$, all loops are freely null homotopic, which proves the resulting 2-complex, X, is simply connected.

The action of G on X^1 clearly permutes the set of simple loops of length $\leq B$, and therefore extends to an action of G on X.

It will be convenient, in what follows, to alter the cell-structure on X to obtain a simplicial complex. Since X^1 is already a simplicial complex, this can be done by taking each 2-cell σ, introducing a new vertex in the interior of σ, and connecting this new vertex to each original vertex of σ, thereby cutting σ into b 2-simplices where b is the number of edges of $\partial\sigma$. We put a G-equivariant geodesic metric on X so that each 2-simplex is isometric to an equilateral Euclidean triangle of side length 1.

Now we extend the map f in a G-equivarian manner to obtain a map $f\colon X \to T$. This map is already defined on the 1-skeleton of the original cell-structure on X. For each original 2-cell σ of f, let $v(\sigma)$ be the new vertex in the interior of σ. Map $v(\sigma)$ to any vertex in $f(\partial\sigma)$, and map the new edges to the unique geodesic in T connecting the images of the endpoints.

The inclusion $X^1 \hookrightarrow X$ is a quasi-isometry, and so the map $f\colon X \to T$ is a quasi-isometry. Clearly f quasiconjugates the G action on X to the original quasi-action on T.

3.5 Tracks

In the case where G acts coboundedly on X, the proof of Theorem 1 will be finished quickly once we recall Dunwoody's tracks.

A *track* in a simplicial 2-complex Y is a 1-dimensional complex t embedded in Y such that for each 2-simplex σ of Y, $t \cap \sigma$ is a disjoint union of finitely many arcs, each of which connects points in the interiors of two distinct edges of σ. For each edge e of Y and each $x \in t \cap e$, each 2-simplex σ incident to e therefore contains a component of $t \cap \sigma$ incident to x. A track $t \subset Y$ has a *normal bundle* $p\colon N(t) \to t$, consisting of a regular neighborhood $N(t) \subset Y$ of t and a fiber bundle $p\colon N(t) \to t$ with interval fiber, such that p collapses each fiber to the unique point where that fiber intersects t. If Y is simply connected then the complement of each track in Y has two components; in particular, the track locally separates, i.e. its normal bundle is orientable. A track is *essential* if it separates Y into two unbounded components. Two tracks are *parallel* if they are ambient isotopic, via an isotopy of Y which preserves the skeleta.

In what follows, we shall assume that all tracks are *finite*.

Theorem 15 (Tracks Theorem [Dun85]). *If Y is a locally finite, simply connected, simplicial 2-complex with cobounded isometry group, then there exists a disjoint union of essential tracks $\tau = \coprod_i \tau_i$ in Y which is invariant* *under the action of* $\mathrm{Isom}(Y)$ *such that the closure of each component of* $Y - \tau$ *has at most one end.* $\diamondsuit$

Now we prove Theorem 1 under the additional assumption that the original quasi-action of G on T is cobounded. It follows that the action of G on X is cobounded, and in particular the isometry group of X is cobounded. Now apply Dunwoody's Theorem to X obtaining a disjoint union of tracks $\tau = \coprod_i \tau_i$.

Consider the closure A of a component of $Y - \tau$. By Dunwoody's Theorem A has at most one end. We claim that in fact A is bounded; this follows from the fact that X is quasi-isometric to a tree, by a standard argument which we now recall.

Suppose that A is unbounded. Let $\mathrm{Stab}(A)$ be the subgroup of $\mathrm{Isom}(X)$ that stabilizes A. Since $\mathrm{Isom}(X)$ acts coboundedly on X it follows that $\mathrm{Stab}(A)$ acts coboundedly on A. Choose a sequence of points $x_0, x_1, x_2, \ldots \in A$, all in the same orbit of $\mathrm{Isom}(A)$, such that (x_i) escapes to infinity in X, and so by passing to a subsequence we may assume that (x_i) converges to some end η of X. Since A is connected, its image under the quasi-isometry $X \to T$ contains a ray converging to η, and so we may assume that $x_0, x_1, x_2, \ldots$ lie on a quasigeodesic ray in X converging to η. Choose $g_i \in \mathrm{Stab}(A)$ such that $g_i(x_i) = x_0$. Since X is quasi-isometric to a tree, and since $x_0, x_1, x_2, \ldots$ lie on a quasigeodesic ray converging to η, there exists $R > 0$ such that for all sufficiently large i the compact set $\overline{N}_R(x_i)$ separates x_0 from η. It follows that $C_i = \overline{N}_R(x_i) \cap A$ is a compact subset of A separating x_0 from η. Note that $C_0 = g_i(C_i)$ for all i. Let U_i be the component of $A - C_i$ containing x_0 and let V_i be the component limiting on η, so $U_i \neq V_i$. It follows that $\mathrm{Diam}(U_i) \to \infty$ as $i \to \infty$, and $\mathrm{Diam}(V_i) = \infty$ for all i. Since C_0 has only finitely many complementary components, we may pass to a subsequence so that $g_i(U_i)$ and $g_i(V_i)$ are constant, equal to U, V respectively. But then U, V are distinct components of $A - C_0$, each of unbounded diameter. This shows that A has ≥ 2 ends, a contradiction.

We now construct a map from X to a tree T', equivariant with respect to G, in fact equivariant with respect to the entire isometry group of X. Choose an equivariant system of regular neighborhoods $N_i = N(\tau_i)$, and choose equivariantly a fibration of N_i by tracks parallel to τ_i. The tree T' is the quotient of X obtained by collapsing the closure of each component of $X - \cup N_i$ to a point producing a vertex of T', and collapsing each of the parallel tracks in N_i to a point producing an edge of T'. The quotient map $X \to T'$ is equivariant, and it is a quasi-isometry because the point inverse images are bounded.

This finishes the proof of Theorem 1 in the cobounded case.

3.6 The noncobounded case

Here is a sketch of what needs to be done in the noncobounded case.

Let's recall briefly the proof of the Tracks Theorem 15, which assumes coboundedness of X. First one constructs an essential track which is disjoint from and not parallel to any of its translates; let τ_0 be the resulting equivariant pattern. Continuing by induction, if τ_i has been constructed, and if $X - \tau_i$ has a component A with more than one end, next one augments τ_i to get τ_{i+1} by adding a track, disjoint from τ_i and from all its translates, which separates two ends of A. This inductive process must end, because $X/\operatorname{Isom}(X)$ is a finite complex and so there is an upper bound to the number of nonparallel tracks in the quotient, which thereby gives an upper bound to the number of nonparallel orbits in a pattern of pairwise disjoint tracks in X. This phenomenon is usually referred to as "Kneser finiteness", because of similar finiteness arguments in 3-manifold theory first used by Kneser.

When $\operatorname{Isom}(X)$ does not act coboundedly on X we instead use an argument based on the Hausdorff maximal principle. The inductive process above can be carried out transfinitely: for each ordinal ω one produces an increasing sequence $\tau_0 \subset \tau_1 \subset \tau_2 \subset \cdots \subset \tau_\omega$ of equivariant patterns of pairwise nonparallel, essential tracks. One added ingredient is that this inductive process can be carried out so that there is a uniform upper bound to the diameter of each component of τ_ω, independent of ω. This is true because the 2-complex X is quasi-isometric to a tree, and so each point of X is uniformly close to a uniformly bounded essential track; by closely observing Dunwoody's proofs in [Dun85] one the obtains the desired uniform bound in the inductive step. Another added ingredient is that if $X - \tau_\omega$ has complementary components of arbitrarily large diameter, then the induction can be continued: using again the fact that each point of X is uniformly close to a uniformly bounded essential track, if A is a component of $X - \tau_\omega$ of sufficiently large diameter, one can find an essential track of X contained in A which separates two frontier components of A and which has diameter bounded independent of ω, and then by again closely observing Dunwoody's proofs one can find such a track which is disjoint from all its translates.

By the Hausdorff maximal principle, eventually one obtains τ_ω which cannot be extended. The components of τ_ω and of $X - \tau_\omega$ have uniformly bounded diameter. This is enough to guarantee that the equivariant map to the quotient tree $X \to T'$, defined as before, is a quasi-isometry, completing the proof of Theorem 1.

4 Application: Quasi-isometric rigidity for graphs of coarse $\mathrm{PD}(n)$ groups

4.1 Bass-Serre theory

We review briefly graphs of groups, their Bass-Serre trees, and associated topological spaces [Ser80], [PS79].

A *graph of groups* is a graph or 1-complex Γ, together with the following data: a vertex group Γ_v for each vertex $v \in \operatorname{Verts}\Gamma$; an edge group Γ_e for each $e \in \operatorname{Edges}(\Gamma)$; and for each end η of each edge e, with $v(\eta)$ the vertex incident to η, an injective edge-to-vertex homomorphism $\gamma_\eta \colon \Gamma_e \to \Gamma_{v(\eta)}$. The *fundamental group* $\pi_1\Gamma$ can be defined topologically by first constructing a *graph of spaces* associated to Γ, as follows. For each $v \in \operatorname{Verts}(\Gamma)$ choose a based $K(\Gamma_v, 1)$ space Y_v; for each $e \in \operatorname{Edges}(\Gamma)$ choose a based $K(\Gamma_e, 1)$ space Y_e; and for each end η of each edge e choose a base point preserving map $f_\eta \colon Y_e \to Y_{v(\eta)}$ inducing the homomorphism γ_η. Let Y be the quotient space

$$Y = \left(\coprod_{v \in \operatorname{Verts}(\Gamma)} Y_v \right) \amalg \left(\coprod_{e \in \operatorname{Edges}(\Gamma)} Y_e \times e \right) \Big/ (x, v(\eta)) \sim f_\eta(x)$$

where the indicated gluing is carried out for each edge e, each $x \in Y_e$, and each end η of e. The homotopy type of Y is completely determined independent of the choices of Y_v, Y_e, and f_η, and the fundamental group of Y is defined to be the fundamental group of Γ.

The *Bass-Serre tree* of Γ can also be defined topologically, as follows. Define the *fibers* of Y to be the images under the above quotient of the vertex spaces Y_v and the spaces $Y_e \times x$, $x \in \operatorname{int}(e)$. In the universal cover $\widetilde{Y}$, the connected lifts of the fibers of Y are defined to be the fibers of $\widetilde{Y}$. The quotient space of $\widetilde{Y}$ obtained by collapsing each fiber to a point is a tree T on which $\pi_1\Gamma$ acts, with quotient Γ. The graph of groups structure on Γ can be recovered using the vertex and edge stabilizers of T and the inclusion maps from edge stabilizers to vertex stabilizers.

Let Γ be a graph of groups. The universal cover $\widetilde{Y}$ equipped with its quotient map $\widetilde{Y} \to T$ is an example of a "tree of spaces" for Γ, a concept which we now generalize. A *tree of spaces* for Γ consists of a cell complex X on which $\pi_1\Gamma$ acts properly by cellular automorphisms, together with a $\pi_1\Gamma$-equivariant cellular map $\pi \colon X \to T$, such that the following properties hold:

- For each vertex v of T, the set $X_v = \pi^{-1}(v)$ is a connected subcomplex of X called the *vertex space* of v, and the stabilizer group of v, $\mathrm{Stab}(v) = \{g \in \pi_1\Gamma \mid g \cdot v = v\}$, acts properly on X_v.

- For each edge e of T there is a connected cell complex X_e on which $\mathrm{Stab}(e)$ acts properly, called the *edge space* of e, and there is a $\mathrm{Stab}(e)$-equivariant cellular map $i_e \colon X_e \times e \to X$, such that $\pi \circ i_e$ equals the projection map $X_e \times e \to e$, and such that $i_e \mid X_e \times \mathrm{int}(e)$ is a homeomorphism onto $\pi^{-1}(\mathrm{int}(e))$ taking open cells to open cells.

We will sometimes identify X_e with $\pi^{-1}(\mathrm{mid}(e))$. For each vertex v of each edge e of T, the composition $X_e \approx X_e \times v \hookrightarrow X_e \times e \xrightarrow{i_e} X$ has image contained in X_v and therefore defines a cellular map $\xi_{ev} \colon X_e \to X_v$ called an *edge-to-vertex map*. Regarding $X_e = \pi^{-1}(\mathrm{mid}(e))$, the map ξ_{ev} moves each point of X_e a bounded distance in X to a point of X_v.

If Γ is a finite graph of finitely generated groups, one can construct a tree of spaces $\pi : X \to T$ on which the action of $\pi_1\Gamma$ is cobounded, by choosing the vertex and edge spaces so that the action of the corresponding stabilizer is cobounded, for example by taking Cayley graphs. Then one chooses cellular edge-to-vertex maps $\xi_{ev} \colon X_e \to X_v$, so that $\pi_1\Gamma$ acts equivariantly on this data. Define X by gluing up the edge and vertex spaces, that is:

$$X = \left(\coprod_v X_v \right) \coprod \left(\coprod_e X_e \times e \right) \Big/ \ (x, v) \sim \xi_{ev}(x)$$

The projection map $(\coprod_v X_v) \coprod (\coprod_e X_e \times e) \to T$, which takes X_v to v and $X_e \times e$ to e by projection, agrees with the gluings and therefore defines the map $\pi \colon X \to T$.

4.2 Geometrically homogeneous graphs of groups

A finite graph of finitely generated groups Γ is *geometrically homogeneous* if any of the following equivalent conditions hold:

- T has bounded valence;

- each edge-to-vertex injection γ_η of Γ has finite index image;

- each edge-to-vertex injection γ_η of Γ is a quasi-isometry;

- each edge-to-vertex map ξ_{ev} of X is a quasi-isometry;

- any two edge or vertex spaces in X have finite Hausdorff distance in X.

In the last three statements, we use any finitely generated word metric on the edge and vertex groups, geodesic metrics on the edge and vertex spaces, and a geodesic metric on X, on which the appropriate groups act isometrically. Note that the first two statements are equivalent for any finite graph of groups, regardless of whether the edge and vertex groups are finitely generated. As proved in [BK90], if these properties hold then the Bass-Serre tree T satisfies a trichotomy: either T is bounded; or T is *line-like* meaning that it is quasi-isometric to a line; or T is bushy. In the latter case we will also say that the graph of groups Γ is bushy.

Geometric homogeneity implies that all edge and vertex spaces of X, and all edge and vertex groups of Γ, are in the same quasi-isometry class. The converse does not hold, however: for a counterexample, take a group G having a monomorphism $\phi\colon G \to G$ with infinite index image, for instance a finite rank free group, and consider the HNN amalgamation $G*_\phi$.

4.3 Weak vertex rigidity

Let Γ be a geometrically homogeneous graph of groups with bushy Bass-Serre tree T, and choose a tree of spaces $\pi\colon X \to T$ for Γ.

Let H be a group and let $(h, x) \mapsto h \cdot x$ be a K, C-quasi-action of H on X. We say that the quasi-action satisfies *weak vertex rigidity* if there exists $R \geq 0$ such that for each $h \in H$ and each vertex $v \in \mathrm{Verts}(T)$ there is a vertex $v' \in \mathrm{Verts}(T)$ such that

$$h \cdot X_v \overset{[R]}{=} X_{v'}$$

Choosing one such v' for each v we obtain an induced $(K, C + 2R)$ quasi-isometry $A_h\colon T \to T$.

We claim that $h \mapsto A_h$ is a quasi-action of H on T. To verify this we must estimate the sup distance between $A_{hh'}$ and $A_h \circ A_{h'}$. If we set

$$v' = A_{hh'}(v) \qquad \text{then} \qquad X_{v'} \overset{[R]}{=} hh' \cdot X_v \overset{[C]}{=} h \cdot (h' \cdot X_v)$$

And if we set

$$v_1 = A_{h'}(v), \quad v_2 = A_h(v_1)$$

then

$$h' \cdot X_v \overset{[R]}{=} X_{v_1}$$

$$h \cdot (h' \cdot X_v) \xlongequal{[KR+C]} h \cdot X_{v_1} \overset{[R]}{=} X_{v_2}$$

and so

$$X_{v_2} \xlongequal{[(R)+(KR+C)+(C)+(R)]} X_{v'} S$$

showing that the sup distance between $A_{hh'}$ and $A_h \circ A_{h'}$ is at most $KR + 2C + 2R$.

In other words, any weakly vertex rigid quasi-action of a group H on the tree of spaces X induces a quasi-action of H on the Bass-Serre tree T. In this situation we can apply Theorem 1, obtaining a quasi-conjugacy $f\colon T' \to T$ from an isometric action of H on a bounded valence, bushy tree T' to the induced quasi-action of H on T.

When the original quasi-action of H on X is cobounded and proper, evidently the induced quasi-action of H on T is cobounded, and so the isometric action of H on T' is cobounded. Thus the quotient $\Gamma' = T'/H$ may be regarded as a geometrically homogeneous graph of groups with fundamental group H. In this situation it easily follows that for each vertex v' of T', the stabilizer subgroup $\mathrm{Stab}_H(v')$ is finitely generated, and so we may construct a cobounded tree of spaces $\pi'\colon X' \to T'$ for the graph of groups Γ'.

We now have most of the pieces in place for the following result:

Proposition 16. *Let Γ be a geometrically homogeneous graph of groups, with tree of spaces $X \xrightarrow{\pi} T$. Let H be a finitely generated group and let $(h,x) \mapsto h \cdot x$ be a weakly vertex rigid, proper, cobounded quasi-action of H on X. Let $(h,v) \mapsto h \cdot v$ be an induced quasi-action on T. Then there exists a geometrically homogeneous graph of groups Γ' with fundamental group H and with cobounded tree of spaces $X' \xrightarrow{\pi'} T'$, and there exists a coarsely commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "F"] \arrow[d, "\pi'"'] & X \arrow[r, "\bar F"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"'] \\ T' \arrow[r, "f"] & T \arrow[r, "\bar f"] & T' \end{tikzcd}$$

in which all horizontal arrows are quasiconjugacies, F and $\bar F$ are coarse inverses, and f and $\bar f$ are coarse inverses. As a consequence, all of the following objects are quasi-isometric: the vertex and edge groups of Γ'; the vertex and edge spaces of X'; the vertex and edge spaces of X; the vertex and edge groups of Γ.

Proof. Construct the quasiconjugacy F as follows: choose arbitrarily a point $x' \in X'$ and its image $x = F(x') \in X$. For each point $x'' \in X$ in the H-orbit of x', choose arbitrarily $h \in H$ so that $h \cdot x' = x''$, and define $F(x'') = h \cdot x$. For any other point $x''' \in X$, choose arbitrarily a closest point x'' in the

H-orbit of x', and define $F(x''') = F(x'')$. It is straightforward to check that F is an H-quasiconjugacy, and that the two maps $\pi \circ F, f \circ \pi' \colon X' \to T$ have bounded distance in the sup norm. The coarse inverse $\bar{F}$ is similarly constructed.

To complete the proof it suffices to show that vertex spaces of X and X' are quasi-isometric, and for this it suffices to show that for each vertex $v' \in T'$ we have $F(X'_{v'}) \overset{c}{=} X_{fv'}$. The coarse inclusion $F(X'_{v'}) \overset{c}{\subset} X_{fv'}$ follows from coarse commutativity of the left hand square, for $\pi'(F(X'_{v'}))$ is contained in a uniformly bounded neighborhood of fv', which implies in turn that $F(X'_{v'}) \overset{c}{\subset} X_{fv'}$. Together with coarse commutativity of the right hand square we get $\bar{F}(X_{fv'}) \overset{c}{\subset} X'_{\bar{f}(fv')} \overset{c}{=} X'_{v'}$. Putting these together we see that $F(X'_{v'}) \overset{c}{=} F(\bar{F}(X_{fv'}) \overset{c}{=} X_{fv'}$. $\diamondsuit$

4.4 Coarse Poincaré duality spaces and groups

Proposition 16 has a strong hypothesis, namely weak vertex rigidity. In this section we prove the proposition that any bushy graph of coarse PD(n) groups with fixed n satisfies weak vertex rigidity. Combined with Proposition 16 this will finish the proof of Theorem 2.

In [FM98] it was proved that any bushy graph of $\mathbf{Z}$'s satisfies weak vertex rigidity, using the coarse algebraic topology first developed in [FS96]. In [FM00b] the same argument was generalized to certain bushy graphs of aspherical n-manifold groups for fixed n, with extra hypotheses added to ensure that at no place in the argument did one leave the category of aspherical manifolds, in order that coarse algebraic topology could still apply (these extra hypotheses would be unnecessary if the Borel Conjecture were true). With the theory of coarse Poincaré duality spaces developed by Kapovich and Kleiner [KK99], the same argument can now be further generalized. We begin by recalling some basic concepts from the cohomology of groups [Bro82].

A group G is *of type FP* if $\mathbf{Z}$ admits a finitely generated projective resolution of finite length over $\mathbf{Z}G$; equivalently, there is a finite CW complex which dominates any $K(G,1)$. The *cohomological dimension* $\mathrm{cd}(G)$ of an FP group G is the smallest n for which there exists a finitely generated projective resolution $0 \to P_n \to \cdots \to P_0 \to \mathbf{Z} \to 0$ over $\mathbf{Z}G$.

A group G is *of type FL* if $\mathbf{Z}$ admits a finitely generated free resolution of finite length over $\mathbf{Z}G$. A group G has *finite type* if there is a finite $K(G,1)$ CW complex. The Eilenberg-Ganea Theorem ([Bro82], Theorem 7.1) shows that G has finite type if and only if G is finitely presented and of type FL;

if this is the case, moreover, then there exists a finite $K(G,1)$ of dimension $\max\{3, \mathrm{cd}(G)\}$.

An FP group G is of type $\mathrm{PD}(n)$, $n \geq 1$, if the canonical cohomology $H^i(G; \mathbf{Z}G)$ is $\mathbf{Z}$ when $i = n$ and is trivial otherwise; in this case, $n = \mathrm{cd}(G)$. Mike Davis has examples of $\mathrm{PD}(n)$ groups in every dimension $n \geq 4$ which are not of type FL and hence not of finite type [Dav98].

Next we review the concepts of coarse Poincaré duality spaces from [KK99], extending these concepts from simplicial complexes to CW complexes.

Given a finite dimensional CW complex X, we define what it means for X to have *bounded geometry*. This means that each point of X touches a uniformly bounded number of closed cells, and loosely speaking there is a uniformly finite set of models for the attaching maps of cells. The formal definition uses induction on dimension, as follows. A 1-dimensional CW complex X^1 has bounded geometry if the valence of 0-cells is uniformly bounded; note that for each R there are only finitely many cellular isomorphism classes of connected subcomplexes of X^1 containing $\leq R$ cells. Suppose X^{n+1} is an $n+1$ dimensional CW complex whose n-skeleton X^n has bounded geometry, and assume inductively that for each R there are only finitely many cellular isomorphism classes of connected subcomplexes of X^n containing $\leq R$ cells. Then X^n has bounded geometry if there exists an integer $A > 0$ with the following properties:

- Each point of X^n touches at most A closed cells of dimension $n+1$.
- For each $n+1$ cell e with attaching map $\alpha_e \colon S^n \to X^n$, the set $\mathrm{image}(\alpha_e)$ is a subcomplex of X^n with at most A cells.
- Up to postcomposition by cellular isomorphism, there are at most A different attaching maps $S^n \xrightarrow{\alpha_e} \mathrm{image}(\alpha_e)$, as e varies over all $n+1$ cells.

It follows that for each R there are only finitely many cellular isomorphism classes of connected subcomplexes of X^{n+1} with $\leq R$ cells, completing the induction.

Given two bounded geometry CW complexes X, Y, suppose $f \colon X \to Y$ is a *cellular map*, meaning that $f(X^n) \subset Y^n$ and the image of each cell of X is a subcomplex of Y. We say furthermore that f has bounded geometry if there is a uniformly finite set of models for the maps from cells of X to their images in Y; formalizing this along the above lines is straightforward.

For example, the universal cover $\widetilde{X}$ of a finite CW complex X has bounded geometry. Also, if $f: X \to Y$ is a cellular map between finite CW complexes then any lift $\widetilde{f}: \widetilde{X} \to \widetilde{Y}$ has bounded geometry.

Given a bounded geometry CW-complex X and a subcomplex L, the i-th combinatorial neighborhood $N_i(L)$ is a subcomplex of X defined inductively as follows: $N_0(L) = L$; and $N_{i+1}(L)$ is the union of all closed cells intersecting $N_i(L)$. Intuitively, we imagine that X has a geodesic metric in which each 1-cell is an arc of length 1, each cell has diameter 1, and disjoint cells have distance ≥ 1; with such a geodesic metric, $N_i(L)$ would be the true metric neighborhood about L of radius i, and the inclusion $X^1 \subset X$ would be a quasi-isometry. In fact one can pick a geodesic metric on X having only finitely many isometry types of connected subcomplexes of dimension $\leq R$ for each R, so that the inclusion $X^1 \subset X$ is a quasi-isometry, and so if $B(L, r)$ denotes the metric neighborhood about a subcomplex L then we have $B(L, \frac{i}{K} - C) \subset N_i(L) \subset B(L, Ki + C)$ for constants $K \geq 1, C \geq 0$. We are therefore free to treat the neighborhood $N_i(L)$, for quasi-isometric purposes, as a metric neighborhood.

Suppose X is a bounded geometry CW complex. We say that X is *uniformly contractible* if for each R there exists $R' = R'(R) > R$ such that each subcomplex L with $\mathrm{Diam}(L) \leq R$ is contractible to a point in $N_{R'}(L)$ (here and in the sequel, "distance" measurements in X such as R and R' are implicitly assumed to be non-negative integers).

We say that the CW chain complex $C_*(X)$ is *uniformly acyclic* if for every R there exists $R' = R'(R) > R$ such that for every subcomplex $L \subset X$ with $\mathrm{Diam}(L) \leq R$ the inclusion $L \to N_{R'}(L)$ induces the trivial map on reduced homology.

Let $C^*_c(X)$ be the compactly supported CW cochain complex, let n be the topological dimension of X, and suppose we are given an *n-dimensional augmentation*, which means a surjective homomorphism $C^n_c(X) \xrightarrow{\alpha} \mathbf{Z}$ with the property that the augmented sequence

$$C^0_c(X) \to \cdots \to C^{n-1}_c(X) \to C^n_c(X) \xrightarrow{\alpha} \mathbf{Z}$$

is a cochain complex. We say that the augmented cochain complex $(C^*_c(X), \alpha)$ is *uniformly acyclic* if there exists R_0, and for each $R \geq R_0$ there exists $R' = R'(R) > R$, such that the following hold:

- for all vertices v and all $R \geq R_0$ the image of the restriction map

$$H^*_c(X, \overline{X - N_R(v)}) \to H^*_c(X, \overline{X - N_{R'}(v)})$$

maps isomorphically to $H_c^*(X)$ under the restriction map

$$H_c^*(X, \overline{X - N_{R'}(v)}) \to H_c^*(X)$$

- the induced map $\alpha^*\colon H_c^n(X) \to \mathbf{Z}$ is an isomorphism.
- for all $i \neq n$, $H_c^i(X) \approx 0$.

Following [KK99], a bounded geometry, uniformly acyclic CW complex X is a *coarse* PD(n) *space* if there exist chain maps

$$C_*(X) \xrightarrow{P} C_c^{n-*}(X) \xrightarrow{\overline{P}} C_*(X)$$

such that the maps $\bar{P} \circ P$, $P \circ \bar{P}$ are each chain homotopic to the identity via respective chain homotopies

$$C_*(X) \xrightarrow{\Phi} C_{*+1}(X), \quad C_c^*(X) \xrightarrow{\bar{\Phi}} C_c^{*-1}(X)$$

and there exists a constant $D \geq 0$ such that for all cells σ of X, the supports of each of $P(\sigma)$, $\bar{P}(\sigma)$, $\Phi(\sigma)$, $\bar{\Phi}(\sigma)$ lie in $N_D(\sigma)$.

Lemma 17. *If X is an n-dimensional, bounded geometry CW complex, then X is a coarse* PD(n) *space if and only if there exists an n-dimensional augmentation $\alpha\colon C_c^n(X) \to \mathbf{Z}$ such that $(C_c^*(X), \alpha)$ is uniformly acyclic.*

Proof. The "if" direction comes from [KK99]. For the "only if" direction, take α to be the composition of the duality map $\overline{P}\colon C_*^n(X) \to C_0(X)$ with the ordinary augmentation $C_0(X) \to \mathbf{Z}$. ♢

The main fact we'll need about coarse PD(n) spaces is the coarse separation theorem of [KK99], which says that a uniformly proper embedding of a coarse PD($n-1$) space in a coarse PD(n) space coarsely separates the target into exactly two deep components.

We define a *coarse* PD(n) *group* to be a group G which quasi-acts properly and coboundedly on some uniformly contractible, coarse PD(n) space X of topological dimension n; equivalently, G is quasi-isometric to X. As a shorthand, a uniformly contractible, coarse PD(n) space of topological dimension n is called a *good coarse* PD(n) *space*. Thus, a coarse PD(n) group is one which is quasi-isometric to a good coarse PD(n) space.

For example, if G is a PD(n) group of finite type then G is a coarse PD(n) group. If $n = 2$ this follows from the Eckmann-Müller theorem which implies that G is the fundamental group of a closed, aspherical surface (see [Bro82], page 223). If $n \geq 3$, the Eilenberg-Ganea theorem ([Bro82] Theorem 7.1) says that G has an n-dimensional, finite $K(G,1)$ space K, and the action of G on the universal cover of K demonstrates that G is a coarse $\mathrm{PD}(n)$ group.

The coarse $\mathrm{PD}(n)$ property of a group is by definition a quasi-isometry invariant. In particular it follows that groups which are virtually $\mathrm{PD}(n)$ of finite type are coarse $\mathrm{PD}(n)$.

4.5 Bushy graphs of coarse $\mathrm{PD}(n)$ groups

Here is the result that generalizes Theorem 2:

Theorem 18. *If Γ is a finite graph of coarse $\mathrm{PD}(n)$ groups with bushy Bass-Serre tree, and if G is a finitely generated group quasi-isometric to $\pi_1\Gamma$, then G is the fundamental group of a graph of groups with bushy Bass-Serre tree, and with vertex and edge groups quasi-isometric to those of Γ.*

For expository purposes we will describe the proof in detail under the following:

Standing Assumption: Each vertex and edge group *ACTS*, properly and coboundedly, on a good coarse $\mathrm{PD}(n)$ space.

In other words, while the coarse $\mathrm{PD}(n)$ property for groups only requires the group to quasi-act, our standing assumption replaces this with the stronger requirement that the group acts. Once the proof is complete under this assumption, we sketch the changes needed to cover the general case.

To start with we construct a tree of spaces $\pi\colon X \to T$ by choosing vertex spaces and edge spaces which are good coarse $\mathrm{PD}(n)$ spaces on which the respective vertex and edge groups act properly and coboundedly by cellular automorphisms; also, choose bounded geometry edge-to-vertex maps. This construction is possible, first of all, because of the standing assumption above. Secondly, we need to make use of the fact that in a geometrically homogeneous graph of groups, each edge-to-vertex map is a quasi-isometry; and so in our present situation where the edge and vertex spaces are bounded geometry and uniformly contractible it follows that the edge-to-vertex maps can each be moved a bounded distance in the sup norm to a bounded geometry map. This can be done equivariantly, because of the standing assumption, and by gluing we obtain the required $\pi\colon X \to T$.

Note that X is a bounded geometry, uniformly contractible $n+1$ complex. This follows because the constants for the bounded geometry and uniform contractibility of vertex and edge spaces and the constants for the bounded geometry of edge-to-vertex maps are uniformly bounded (it also follows from contractiblity of X together with the fact that $\pi_1\Gamma$ acts properly and coboundedly on X by cellular automorphisms; but the latter fact will disappear when we drop the standing assumption).

Given a bi-infinite line $L \subset T$, the subcomplex $X_L = \pi^{-1}(L)$ is called the *hyperplane* in X lying over L. Each hyperplane is a bounded geometry, uniformly contractible $n+1$ complex, and is uniformly properly embedded in X. Lemma 20 will show that each X_L is a good coarse $\mathrm{PD}(n+1)$ space.

Proposition 19. *Let P be a good, coarse* $\mathrm{PD}(n+1)$ *space. For every $K \ge 1$, $C \ge 0$, and every proper function $\rho\colon [0,\infty) \to [0,\infty)$ there exists $A \ge 0$ such that if $f\colon P \to X$ is a (K, C, ρ)-uniformly proper embedding, then there is a hyperplane X_L such that $d_{\mathcal{H}}(f(P), X_L) \le A$.*

This is the exact analogue of Theorem 7.3 in [FM00b], and will be proved below.

Proof: Theorem 18 reduces to Proposition 19. The proof follows [FM00b], §7.2, Step 2 very closely; here is a sketch.

Combining with Proposition 16, it remains to show that any quasi-action on X satisfies weak vertex rigidity. For this we need only show the following: if $X' \to T'$ is another tree of spaces associated to a geometrically homogeneous graph of coarse $\mathrm{PD}(n)$ groups, then for each K, C there exists A such that if $f\colon X \to X'$ is a K, C-quasi-isometry and if $v \in T$ is a vertex then there is a vertex $v' \in T'$ such that $d_{\mathcal{H}}(f(X_v), X'_{v'}) \le A$. First note that there exists a proper function $\rho\colon [0,\infty) \to [0,\infty)$, depending only on K, C, such that if X_L is any hyperplane in X then $f \bigm| X_L$ is a ρ-uniformly proper embedding of X_L into X'. Now choose three lines L_1, L_2, L_3 in T whose intersection is v. Applying Proposition 19 there are hyperplanes $X'_{L'_1}, X'_{L'_2}, X'_{L'_3}$ which are uniformly Hausdorff close to $f(X_{L_1}), f(X_{L_2}), f(X_{L_3})$ respectively. The set $f(X_v)$ must therefore be uniformly Hausdorff close to the coarse intersection of the hyperplanes $X'_{L'_1}, X'_{L'_2}, X'_{L'_3}$. The three lines L'_1, L'_2, L'_3 must intersect pairwise in rays, and these three rays have infinite Hausdorff distance in T'; it follows that $L'_1 \cap L'_2 \cap L'_3$ is a vertex v' of T'. This implies that the coarse intersection of $X'_{L'_1}, X'_{L'_2}, X'_{L'_3}$ is $X'_{v'}$, proving that $f(X_v)$ is uniformly Hausdorff close to $X'_{v'}$. $\diamondsuit$

We turn now to the proof of Proposition 19.

The reader who is interested in most real life examples can refer to [FM00b], Theorem 7.3 where the proof is given under the following special circumstances: the vertex and edge groups of Γ are fundamental groups of closed, smooth, aspherical manifolds in a category $\mathcal{C}$ which is closed under finite covers, and which has the property that any homotopy equivalence between manifolds in $\mathcal{C}$ is homotopic to a diffeomorphism. For example: euclidean manifolds; hyperbolic manifolds; irreducible, nonpositively curved symmetric spaces; solvmanifolds; and nilmanifolds.

The proof of Proposition 19 follows the same outline as Theorem 7.3 of [FM00b], but with many changes of details needed to accomodate coarse $\mathrm{PD}(n)$ groups.

Pick a topologically proper embedding of T in an open disc D. For each component U of $D - T$ there is a line $L(U)$ in T such that the projection homeomorphism $L(U) \times 0 \to L(U)$ extends to a homeomorphism of pairs $(L(U) \times [0, \infty), L(U) \times 0) \approx (\mathrm{cl}\, U, L(U))$. Extend the CW structure on T to a CW structure on D, by using the product structure on $L(U) \times [0, \infty) \approx \mathrm{cl}\, U$ for each U, where $L(U)$ has the CW structure it inherits from T and $[0, \infty)$ has the CW structure where each interval $[i, i+1]$ is a 1-cell. Note that with this CW-structure, D is a coarse $\mathrm{PD}(2)$ space.

The tree of spaces $\pi \colon X \to T$ is an example of what might be called a "coarse fibration", and we now extend this to a "coarse fibration" $\pi \colon Y \to D$. The fibers of Y will be good coarse $\mathrm{PD}(n)$ spaces isomorphic to the fibers of X.

For each cell c of D we define a fiber Y_c as follows. If $c \subset T$ then we simply take $Y_c = X_c$. If $c \subset U$ for some component U of $D - T$, it follows that $c = c_1 \times c_2$ for cells $c_1 \subset L(U)$ and $c_2 \subset [0, \infty)$, and we define Y_c to be a disjoint copy of X_{c_1}.

For any two cells $c \supset d$ of D we define an attaching map $\eta_{dc} \colon Y_c \to Y_d$ as follows. If $d, c \subset T$ we simply take $\eta_{dc} = \xi_{dc}$. Otherwise we have $d, c \subset \overline{U}$ for some component U of $D - T$. Let $d = d_1 \times d_2$, $c = c_1 \times c_2$ where d_1, c_1 are cells of $L(U)$ and d_2, c_2 are cells of $[0, \infty)$. If $d_1 = c_1$ then Y_d, Y_c are disjoint copies of Y_{d_1} and we take η_{dc} to be a disjoint copy of the identity map on Y_{d_1}. Otherwise we take η_{dc} to be a disjoint copy of the map $\xi_{d_1 c_1}$.

Note that for cells $c \supset d \supset e$ of D we have $\eta_{ce} = \eta_{de} \circ \eta_{cd}$. We therefore have a well defined quotient space Y as follows:

$$Y = \coprod_c (Y_c \times c) \Big/ (x, p) \sim (\eta_{cd}(x), p)$$

where the pair c, d varies over all cells $c \supset d$ of D, the point x varies over Y_c, and p varies over all points of ∂c that lie on d. The disjoint union of the projection maps $Y_c \times c \to c$ gives a map $\coprod_c (Y_c \times c) \to D$, and the latter is consistent with all gluings, thereby defining the map $\pi \colon Y \to D$. We identify Y_c with $\pi^{-1}(x)$ for a chosen point x in the interior of c, and we call this the *fiber* over c. Once we have a metric in place it is evident that any two fibers in Y have finite Hausdorff distance.

Here's a few facts about Y. First, since the base D and fibers Y_c are bounded geometry cell complexes with uniform bounds, and since the gluing maps η_{cd} have bounded geometry with uniform bounds, it follows that Y has bounded geometry. Furthermore, since the base and fibers are uniformly contractible with uniform bounds, it follows that Y is uniformly contractible. Also, the inclusion map from any fiber Y_c into Y is uniformly proper, the inclusion map $X \hookrightarrow Y$ is uniformly proper, and the inclusion map from any hyperplane X_L into Y is uniformly proper.

Lemma 20. *Y is a good coarse* $\mathrm{PD}(n+2)$ *space. Each hyperplane X_L is a good coarse* $\mathrm{PD}(n+1)$ *space with uniform bounds independent of the line L.*

Accepting this lemma for the moment, we now have:

Proof: Proposition 19 reduces to Lemma 20. The proof follows very closely [FM00b], §7.2, Step 1, which itself follows very closely [FM98], Proposition 4.1, Steps 1 and 2. Here is a sketch.

Applying the coarse Jordan separation theorem of [KK99] together with Lemma 20 it follows that there exists $R > 0$ independent of L such that $Y - N_R(f(P))$ has exactly two components A, A' which are *deep*, meaning that each of A, A' contains points arbitrarily far from $f(P)$.

Since $f(P) \subset X$ and since each component of $Y - X$ is deep, it follows that each component of $Y - X$ is coarsely contained in one of A, A', and each of A, A' coarsely contains at least one component of $Y - X$. We may therefore find two components U, U' of $Y - X$ coarsely contained in A, A' respectively, such that U, U' are adjacent in $Y - X$, meaning that $\overline{U} \cap \overline{U}' = \pi^{-1}(E_0)$ for some edge E_0 of T. Letting E_n denote the neighborhood of radius n about E_0 in T, a simple inductive argument shows that E_n contains an embedded edge path γ_n of length $2n+1$ centered on E_0 such that $\pi^{-1}(\gamma_n)$ is coarsely contained in $f(P)$ with uniform coarse containment constant; if this were not so then as in [FM98] Proposition 4.1, Step 1 one can find a path in $Y - N_R(P)$ connecting A to A', a contradiction. It follows that $f(P)$ coarsely contains some hyperplane X_L. A packing argument as in [FM98] Proposition 4.1, Step 2 shows that X_L is coarsely contained in $f(P)$. $\diamondsuit$

Proof of Lemma 20. The proofs for Y and for X_L are exactly the same. We give the proof for Y, using the E_0 spectral sequence in compactly supported CW cohomology for the coarse fibration $\pi\colon Y \to D$.

Here's some notation. For each cell d of a CW complex Z we let d^* denote the corresponding basis element of the compactly supported CW cochain complex $C_c^*(Z)$. The coboundary operator $\delta\colon C_c^i(D) \to C_c^{i+1}(D)$ can be written as

$$\delta b^* = \sum_c n_{cb} c^*, \quad \text{for } b \text{ an } i\text{-cell of } D$$

where c ranges over cells of dimension $i+1$ and, letting $m_c\colon S^i \to X^i$ be the attaching map for the cell c, the integer n_{cb} is the degree of the map $(S^i, \emptyset) \xrightarrow{m_c} (X^i, X^i - \mathrm{int}(b))$.

Each cell in the CW complex Y has the form $b \times e$ for some cell b of D and some cell e of Y_b, and the corresponding basis element of $C_c^*(Y)$ can therefore be written $b^* \times e^*$. To understand the coboundary operator of Y we look at the E_0 term of the spectral sequence.

The filtration of D by its skeleta $D^0 \subset D^1 \subset D^2 = D$ determines a filtration $Y^0 \subset Y^1 \subset Y^2 = Y$ with $Y^i = \pi^{-1}(D^i)$, which in turn determines an E_0 spectral sequence for $H_c^*(Y)$ as follows. Over the CW complex D we have a coefficient bundle $C_c^*(\mathcal{Y})$: the coefficients over a cell b of D are $C_c^*(Y_b)$; and for cells $b \subset c$ the restriction homomorphism is $\eta_{cb}^{\#}\colon C_c^*(Y_c) \to C_c^*(Y_b)$. We then have

$$E_0^{ij} = C_c^i(D, C_c^j(\mathcal{Y}))$$

The coboundary operator of each coefficient complex $C_c^*(Y_b)$ determines a map

$$\delta\colon E_0^{ij} \to E_0^{i,j+1}$$

given by $\delta(b^* \times e^*) = b^* \times \delta e^*$. The coboundary map of $C_c^*(D)$ together with the restriction maps of the coefficient bundle determine a map

$$\delta\colon E_0^{ij} \to E_0^{i+1,j}$$

given by

$$\delta(b^* \times e^*) = \sum_c n_{cb}\, \eta_{cb}^{\#}(e^*)$$

We have an isomorphism

$$C_c^k(Y) \approx \bigoplus_{i+j=k} C_c^i(D, C_c^j(\mathcal{Y}))$$

and under this isomorphism the cochain map for $C_c^*(Y)$ corresponds to the map

$$d + (-1)^k\delta\colon \bigoplus_{i+j=k} C_c^i(D, C_c^j(\mathcal{Y})) \to \bigoplus_{i+j=k+1} C_c^i(D, C_c^j(\mathcal{Y}))$$

The E_1 term of the spectral sequence is given by

$$E_1^{ij} = C_c^i(D, H_c^j(\mathcal{Y})) = \begin{cases} \mathbf{Z} & \text{if } 0 \leq i \leq 2,\ j = n \\ 0 & \text{otherwise} \end{cases}$$

and the spectral sequence collapses at the E_2 term with

$$E_2^{ij} = \begin{cases} \mathbf{Z} & \text{if } i = 2,\ j = n \\ 0 & \text{otherwise} \end{cases}$$

This shows at least that Y has the correct cohomology for a coarse PD($n+2$) space: $H_c^{n+2}(Y) = \mathbf{Z}$ and $H_c^i(Y) = 0$ for $i \neq n+2$. But we must still verify the uniformity criteria, and for this we look more closely at the representation of $C_c^*(Y)$ using the double complex E_0.

The CW complex Y has topological dimension $n+2$, and so to verify that Y is a good coarse PD($n+2$) space using Lemma 17 we must construct an $n+2$ dimensional augmentation α for $C_c^*(Y)$ and prove that $(C_c^*(Y), \alpha)$ is uniformly acyclic. Since D is a good coarse PD(2) space, there is a 2-dimensional augmentation α_D making $C_c^*(D)$ uniformly acyclic; in fact, we simply define $\alpha_D(\beta) = \sum_b \langle \beta, b\rangle$, summed over 2-cells b of D. For each cell d of D, since Y_d is a good coarse PD(n) space then by Lemma 17 there is an n-dimensional augmentation α_d making $C_c^*(Y_d)$ uniformly acyclic (with constants independent of d). Define $\alpha\colon C_c^{n+2}(Y) \to \mathbf{Z}$ as follows. A basis element of $C_c^{n+2}(Y)$ has the form $b^* \times e^*$, for b a 2-cell of D and e an n-cell of Y_b. We define

$$\alpha(b^* \times e^*) = \alpha_D(b^*)\alpha_b(e^*) = \alpha_b(e^*)$$

Its clear from the spectral sequence that α determines an isomorphism $\alpha^*\colon H_c^{n+2}(Y) \to \mathbf{Z}$. Moreover, each class in $H_c^n(Y_b) \approx \mathbf{Z}$ is represented by a cochain whose support is contained in an arbitrary ball of uniformly bounded radius in Y, and so the same is true for each class in $H_c^{n+2}(Y)$.

It remains to show, given an arbitrary coboundary $\gamma \in C_c^k(Y)$, that $\gamma = \delta\rho$ for some $\rho \in C_c^{k-1}(Y)$ such that $\mathrm{supp}(\rho)$ is contained in a uniformly bounded neighborhood of $\mathrm{supp}(\gamma)$.

Case 1: $k < n+2$. Using the double complex E_0 we may write

$$\gamma = \gamma^{0k} \oplus \gamma^{1,k-1} \oplus \gamma^{2,k-2} \in E_0^{0k} \oplus E_0^{1,k-1} \oplus E_0^{2,k-2}$$

Since γ is a coboundary, there exists $\rho = \rho^{0,k-1} \oplus \rho^{1,k-2} \in E_0^{0,k-1} \oplus E_0^{1,k-2}$ with

$$\begin{aligned} d\rho^{0,k-1} &= \gamma^{0k} \\ (-1)^{k-1}\,\delta\rho^{0,k-1} + d\rho^{1,k-2} &= \gamma^{1,k-1} \\ (-1)^{k-1}\,\delta\rho^{1,k-2} &= \gamma^{2,k-2} \end{aligned}$$

Each $(C_c^*(Y_b), \alpha_b)$ is uniformly acyclic with uniform constants, and so in each stalk of the coefficient bundle $C_c^*(\mathcal{Y})$ we can choose $\rho^{0,k-1}$ to have support contained in a uniformly bounded neighborhood of $\mathrm{supp}(\gamma^{0k})$. Subtracting $(d + (-1)^{k-1}\delta)\rho^{0,k-1}$ from γ it suffices to assume that $\rho^{0,k-1} = 0$ and so we have $d\rho^{1,k-2} = \gamma^{1,k-1}$. Repeating the argument, we can choose $\rho^{1,k-2}$ to have support contained in a uniformly bounded neighborhood of $\mathrm{supp}(\gamma^{1,k-1})$.

Case 2: $k = n+2$. We have $\gamma = \gamma^{2,n}$, and since γ is a coboundary it follows that $\alpha(\gamma) = 0$. We may write $\gamma = \sum_b \gamma_b$ summed over 2-cells b of D where $\gamma_b \in C_*^n(Y_b)$, and so

$$0 = \alpha(\gamma) = \sum_b \alpha_b(\gamma_b)$$

We may reduce to the case that $\mathrm{gcf}\{\alpha_b(\gamma_b)\} = 1$. To see why, for each b choose $\gamma_b' \in C_c^n(Y_b)$ with $\alpha_b(\gamma_b') = 1$ so that $\mathrm{supp}(\gamma_b')$ is in a uniform neighborhood of $\mathrm{supp}(\gamma_b)$. Then replace γ_b with $\alpha_b(\gamma_b) \cdot \gamma_b'$ by subtracting off the coboundary of something in $C_c^{n-1}(Y_b)$ whose support is contained in a uniform neighborhood of $\mathrm{supp}(\gamma_b)$. After this replacement, all coefficients of the cocycle γ are divisible by $\mathrm{gcf}\{\alpha_b(\gamma_b)\}$, and carrying out this division produces a cocycle with integer coefficients, finishing the reduction.

Define $\gamma_D \in C_c^2(D)$ to be the projection of γ to D, namely $\gamma_D = \sum_b \alpha_b(\gamma_b) b^* \in C_c^2(D)$ and so $\alpha_D(\gamma_d) = 0$. Since $(C_c^*(D), \alpha_D)$ is uniformly acyclic we can find $\rho_D \in C_c^1(D)$ with $\delta\rho_D = \gamma_D$ such that $\mathrm{supp}(\rho_D) \subset N_R(\mathrm{supp}(\gamma_D))$ for some uniform R.

We shall now construct, uniformly for each 1-cell $a \in \mathrm{supp}(\rho_D)$, an element $\rho_a \in C_c^n(Y_a)$ such that $\alpha_a(\rho_a) = \langle \rho_D, a \rangle$. To do this, for each inclusion $a \subset b$ of a 1-cell into a 2-cell in D choose a cochain map $\bar\eta_{ab} \colon C_c^*(Y_a) \to C_c^*(Y_b)$ so that $\eta_{ba} \circ \bar\eta_{ab}$ and $\bar\eta_{ab} \circ \eta_{ba}$ are uniformly chain homotopic to the respective identities. For each 1-cell $a \in \mathrm{supp}(\rho_D)$ and 2-cell $b \in \mathrm{supp}(\gamma_D)$, choose an alternating sequence of 2-cells and 1-cells

$$b = b_0, a_0, b_1, a_1, \ldots, b_r, a_r = a$$

so that r is uniformly bounded ($r \leq xR + y$, with x, y depending only on D), and so that $a_i \subset b_i$, $i = 0, \ldots, r$ and $a_{i-1} \subset b_i$, $i = 1, \ldots, r$. Define

$$\gamma_a^b = \eta_{b_r a_r} \circ \bar{\eta}_{a_{r-1} b_r} \circ \cdots \circ \bar{\eta}_{a_0 b_1} \circ \eta_{b_0 a_0}(\gamma_b) \in C_c^n(Y_a)$$

Noting that η_{ba} and $\bar{\eta}_{ab}$ each commute with augmentations, it follows that

$$\alpha_a(\gamma_a^b) = \alpha_b(\gamma_b)$$

and so for each a we have $\mathrm{gcf}_b\{\alpha_a(\gamma_a^b)\} = 1$. This implies that we can find a linear combination in $C_c^n(Y_a)$ of the set of cocycles $\{\gamma_a^b \mid b \subset \mathrm{supp}(\gamma_D)\}$ so that the resulting cocycle maps to 1 under α_a. Multiplying this cocycle by $\langle \rho_D, a \rangle$ we obtain the desired $\rho_a \in C_c^n(Y_a)$. Note that $\mathrm{supp}(a^* \times \rho_a)$ is contained in a uniform neighborhood of $\mathrm{supp}(\gamma)$.

Consider now the $(n+2)$-cocycle $\hat{\gamma} = \gamma - \delta(\sum_a a^* \times \rho_a)$, and note that $\alpha_b(\hat{\gamma}_b) = 0$ for each 2-cell b of D. The cocycle $\hat{\gamma}_b \in C_c^n(Y_b)$ is therefore the coboundary of some $\rho_b \in C_c^{n-1}(Y_b)$ whose support is contained in a uniform neighborhood of $\hat{\gamma}_b$. It follows that

$$\gamma = \delta \left(\sum_{a \subset \mathrm{supp}(\rho_D)} a^* \times \rho_a + \sum_{b \subset \mathrm{supp}(\gamma)} b^* \times \rho_b \right)$$

and the $n+1$-cocycle inside the parentheses has support contained in a uniform neighborhood of $\mathrm{supp}(\gamma)$.

This completes the proof of Lemma 20, Proposition 19, and Theorem 18, under the standing assumption.

Removing the standing assumption For the general proof we choose, for each vertex or edge group, a vertex and edge space on which the corresponding group quasi-acts properly and coboundedly, and which is a good coarse PD(n) space. We can still glue the vertex and edge spaces together to form a tree of spaces $\pi \colon X \to T$, and all the key geometric properties hold: X is bounded geometry and uniformly contractible, and the hyperplanes X_L are still good coarse PD($n+1$) spaces. Moreover, we can extend this to $\pi \colon Y \to D$ as before, where Y is a good coarse PD($n+2$) space.

What we have lost is that the group $\pi_1\Gamma$ no longer acts on X. However, the key point is that $\pi_1\Gamma$ still quasi-acts on X, properly and coboundedly. It follows that $\pi_1\Gamma$, as well as any group quasi-isometric to $\pi_1\Gamma$, is quasi-isometric to X. Moreover, the proof that $\pi \colon X \to T$ satisfies weak vertex rigidity is exactly the same, and so Theorem 18 is proved. $\diamondsuit$

Remark on "coarse fibrations". With the proper notion of a coarse fibration, the proof above ought to generalize to show that any "coarse fibration" over a good coarse PD(m) base space with good coarse PD(n) fiber is a good coarse PD($m+n$) space. This should imply a generalization of a theorem of Bieri [Bie72] and Johnson–Wall [JW72] saying that any extension of a PD(m) group by a PD(n) group is PD($m+n$); the generalized statement would replace "PD(n) group" by "coarse PD(n) group".

5 Application: Actions on Cantor sets

We shall prove Theorem 5 about uniformly quasiconformal actions on Cantor sets, Theorem 7 about uniform quasisimilarity actions on the n-adic rational numbers, and the corollaries to these theorems.

5.1 Uniformly quasiconformal actions

Theorem 5 is an almost immediate corollary of Theorem 1, once we review the results of [Pau96] which prove the equivalence of various notions of quasiconformality for homeomorphisms between boundaries of Gromov hyperbolic spaces, all of which are equivalent to being the extension of a quasi-isometry; see [Pau96] and [Pau95] for the history of these various notions.

First we review a notion of quasiconformality which was developed for rank one symmetric spaces by Pansu [Pan89a] and generalized to word hyperbolic groups by Paulin [Pau96]. Let X, Y be proper, geodesic hyperbolic metric spaces with cobounded isometry groups. Let $\mathcal{H}(\partial X, \partial Y)$ denote the space of homeomorphisms from ∂X to ∂Y with the compact open topology. For each compact set $\mathcal{K} \subset \mathcal{H}(\partial X, \partial Y)$, a homeomorphism $f \colon \partial X \to \partial Y$ is said to be *$\mathcal{K}$-quasiconformal* if for each isometry $\alpha \colon X \to X$ there exists an isometry $\beta \colon Y \to Y$ such that $\partial\beta \circ f \circ \partial\alpha \in \mathcal{K}$.

Next we review a notion of quasiconformality based on cross-ratio. Consider a proper, geodesic, hyperbolic metric space X, and let $\partial^4 X$ denote the space of ordered 4-tuples of pairwise distinct points in ∂X. The *cross-ratio* on ∂X is the function which assigns to each $(a, b, c, d) \in \partial^4 X$ the positive number

$$[a,b,c,d] = \exp\left(\frac{1}{2}\sup_{\substack{x_i\to a,\, y_i\to b\\ z_i\to c, t_i\to d}}\liminf_{i\to\infty}\big(d(x_i,t_i)-d(t_i,z_i)+d(z_i,y_i)-d(y_i,x_i)\big)\right)$$

where the sequences x_i, y_i, z_i, t_i are in X and the convergence to a, b, c, d, respectively, takes place in the Gromov compactification $\overline{X} = X \cup \partial X$. When $X = T$ is a tree, no matter how these sequences are chosen the expression in the parentheses eventually takes on the same constant value, which is described simply as follows. Consider the six geodesic lines in T determined by taking the points a, b, c, d in pairs: $\overline{ab}$, $\overline{ac}$, etc. Then we have

$$|\log[a,b,c,d]| = \max\left\{d(\overline{ab},\overline{cd}), d(\overline{ac},\overline{bd}), d(\overline{ad},\overline{bc})\right\}$$

Given two proper, geodesic, Gromov hyperbolic metric spaces X, Y with cobounded isometry group, and given a proper, increasing function $I\colon [0,\infty)\to[0,\infty)$, a homeomorphism $f\colon \partial X\to\partial Y$ is *I-quasimobius* if for all $(a,b,c,d)\in \partial^4 X$ we have

$$[a,b,c,d] \le I([fa,fb,fc,fd])$$
$$[fa,fb,fc,fd] \le I([a,b,c,d])$$

As noted in [Pau96], it turns out that one can restrict attention to functions I of the form $I(r) = a(\sup\{1,r\})^\kappa$.

Theorem 21 ([Pau96]). *Let X, Y be proper, geodesic, Gromov hyperbolic metric spaces with cobounded isometry group. Given $f\colon \partial X\to\partial Y$, the following are equivalent:*

(1) There exists a compact $\mathcal{K}\subset\mathcal{H}(\partial X,\partial Y)$ such that f is $\mathcal{K}$-quasiconformal.

(2) There exists $K\ge 1, C\ge 0$ such that f is the continuous extension of some K, C quasi-isometry $X\mapsto Y$.

(3) There exists some proper, increasing $I\colon[0,\infty)\to[0,\infty)$ such that f is I-quasimobius.

Moreover, these equivalences are uniform: for example, (1) implies (2) uniformly in the sense that for any compact $\mathcal{K}\subset\mathcal{H}(\partial X,\partial Y)$ there exists $K\ge 1, C\ge 0$ such that if $f\colon\partial X\to\partial Y$ is $\mathcal{K}$-quasiconformal then f is the continuous extension of a K, C quasi-isometry; similarly for the other five implications.

Define a group action $A: G \times \partial X \to \partial X$ to be *uniformly quasiconformal* if the collection of homeomorphisms $A_g: \xi \mapsto A(g, \xi)$ satisfies any of the criteria of Theorem 21 uniformly; for instance, there exists a proper, increasing $I: [0, \infty) \to [0, \infty)$ such that each A_g is I-quasimobius.

Proof of Theorem 5. Let T be a bushy tree of bounded valence, and let $A: G \times B \to B$ be a uniformly quasiconformal action of G on $B = \partial T$. For each $g \in G$ apply Theorem 21 to extend the map $A_g: B \to B$ to a K, C quasi-isometry $\alpha_g: T \to T$, with K, C independent of g. For any $g, g' \in G$, since $\alpha_g \circ \alpha_{g'}$ is a $K^2, KC + C$ quasi-isometry, the following standard lemma shows that $\alpha_g \circ \alpha_{g'}$ is a bounded distance in the sup norm from $\alpha_{gg'}$, proving that $g \mapsto \alpha_g$ is a quasi-action of G on T:

Lemma 22. *For any proper, Gromov hyperbolic metric space X and any $K \geq 1, C \geq 0$ there exists $A \geq 0$ such that if $f, g, \phi: X \to X$ are K, C quasi-isometries of X whose boundary extensions $\partial f, \partial g: \partial X \to \partial X$ are identical, then the sup distance between f and g is at most A.* $\diamondsuit$

Applying Theorem 1 to the quasi-action $g \mapsto A_g$ we finish the proof of Theorem 5. $\diamondsuit$

Proof of Corollary 6. Let $\alpha': G \to \mathrm{Isom}(T')$ be the action produced by Theorem 5, so that B is identified quasiconformally with $\partial T'$. Supposing that the action of G on B is cocompact on the space of distinct triples of B, it follows that the action α' is cobounded, and so the quotient graph of groups $\Gamma = T'/G$ is finite. Since T' has bounded valence, Γ has finite index edge-to-vertex injections. Given a subgroup H of G, suppose the action of H on the space of distinct pairs in B has precompact orbits. This implies that the action of H on T' has bounded orbits, which implies that H fixes some vertex of T'. $\diamondsuit$

Remark In §2.2 of [Pau96] there is another intrinsic notion of a quasiconformal structure on ∂X, based on moduli of annuli, which says roughly that a homeomorphism is quasiconformal if it stretches the moduli of annuli by a uniform amount; the modulus stretching function ϕ is then a measure of uniformity of the homeomorphism. However, this measure of uniformity does not agree uniformly with the ones used in Theorem 21; for example, as noted at the end of [Pau96] §2.2, isometries of X do not even extend with uniform quasiconformality modulus ϕ. Perhaps with some tinkering this notion of quasiconformality could be made to agree uniformly with the others.

Remark It could be interesting to have an intrinsic notion of a conformal structure on a Cantor set B so that a homeomophism $B \xrightarrow{f} \partial T$, where T is a bounded valence, bushy tree, determines a conformal structure on B with the property that conformal homeomorphisms are the same as boundary extensions of isometries of T. Paulin's paper discusses conformal structures on boundaries of Gromov hyperbolic metric spaces, but these do not behave well on Cantor set boundaries or other boundaries which do not have a sufficiently rich set of arcs; for example, using that notion of conformality one can find conformal maps of ∂T whose extension to T can be made isometric outside of a finite subtree of T, but which cannot be made isometric on all of T. Instead, it might be possible to use quasi-edges to come up with a better behaved notion of conformality on B.

5.2 Uniform quasisimilarity actions on n-adic Cantor sets

First we review some basic notions about maps of metric spaces. Recall from the introduction that given a metric space X, a K-quasisimilarity is a bijection $f\colon X \to X$ such that

$$\frac{d(f\zeta, f\omega)}{d(\zeta,\omega)} \Big/ \frac{d(f\xi, f\eta)}{d(\xi,\eta)} \le K, \quad \forall \zeta \ne \omega, \xi \ne \eta \in X$$

A K-bilipschitz homeomorphism is a bijection $f\colon X \to Y$ between metric spaces such that

$$\frac{1}{K} d(\xi,\eta) \le d(f\xi, f\eta) \le K d(\xi,\eta), \quad \xi,\eta \in X$$

If K is unspecified in either of these definitions then f is called simply a *quasisimilarity* or *bilipschitz*, respectively.

Given a compact subinterval $[a,b] \subset (0,\infty)$, a bijection $f\colon X \to X$ is $[a,b]$-*bilipschitz* if

$$\frac{d(f\xi, f\eta)}{d(\xi,\eta)} \in [a,b], \quad \text{for all} \quad \xi \ne \eta \in X$$

We use the notation $r \cdot [a,b]$ for the interval $[ra, rb]$.

As observed in [FM99], each $[a,b]$-bilipschitz homeomorphism of a metric space X is a K-quasisimilarity with $K = \frac{b}{a}$, and each K-quasisimilarity is $[a,b]$-bilipschitz for some interval $[a,b]$ with $\frac{b}{a} \le K^4$.

Next we review results from [FM98] concerning the connection between $\mathbf{Q}_n$ and trees.

Let T_n be the homogeneous directed tree with one edge pointing towards each vertex and n edges pointing away, and put a geodesic metric on T_n where each edge has length 1. In the end space ∂T_n, a Cantor set, there is a unique end denoted $-\infty$ which is the limit point of any ray in T_n obtained by starting at a vertex in T_n and travelling backwards against the direction of edges. There is a *height map* $\mathrm{ht}\colon T_n \to \mathbf{R}$ which takes each directed edge of T_n isometrically to a directed interval $[i, i+1]$ with integer endpoints; the map ht is unique up to postcomposition by translation of $\mathbf{R}$ by an integer amount, and we shall fix once and for all a choice of ht. For each $i \in \mathbf{Z}$ the set $\mathrm{ht}^{-1}(i)$ is called the *level set* of height i in T_n. A *coherent line* in T_n is a continuous section of the height function $T_n \to \mathbf{R}$. There is a one-to-one correspondence between the set of coherent lines and the set $\partial T_n - \{-\infty\}$, where the coherent line ℓ corresponds to the point ξ if $\partial \ell = \{-\infty, \xi\}$; we denote $\ell = \ell_\xi$.

There is a homeomorphism between $\partial T_n - \{-\infty\}$ and $\mathbf{Q}_n$, determined uniquely up to isometry of $\mathbf{Q}_n$ by the property that for all $\xi \neq \eta \in \mathbf{Q}_n$, if the vertex at which ℓ_ξ and ℓ_η diverge from each other has height h then

$$d(\xi, \eta) = n^{-h}$$

We get an explicit picture of T_n as follows.

A *clopen* in a topological space is a subset which is both closed and open. The *separation* of a subset U in a metric space X is defined to be

$$\mathrm{sep}(U) = \inf\{d(x, y) \bigm| x \in U, y \in X - U\}$$

Note that if U is a clopen in $\mathbf{Q}_n$ then $\mathrm{sep}(U) > 0$.

A *clone* in $\mathbf{Q}_n$ is a certain kind of clopen, defined as follows. Given an integer h, define an equivalence relation on $\mathbf{Q}_n$ where $\xi, \eta \in \mathbf{Q}_n$ are equivalent if $\xi_i = \eta_i$ for all $i \le h$; the equivalence classes are called *clones of height* h, and each of them is a clopen in $\mathbf{Q}_n$. Thus, there is one clone of height h for every sequence $\omega = (\omega_i)_{i=-\infty}^{h}$ in $\mathbf{Z}/n$ satisfying the property that ω_i is eventually zero as $i \to -\infty$; the integer h, which specifies that the domain of the sequence ω is the interval $(-\infty, h]$, is also called the height of the sequence ω.

There is a one-to-one, height preserving correspondence between vertices of T_n and clones of $\mathbf{Q}_n$, where the vertex v corresponds to the clone U_ω if and only if the positive endpoints of the coherent lines passing through v are exactly the points of the clone U_ω; we write $v = v_\omega$ and $\omega = \omega_v$ to emphasize this correspondence. Note also that the structure of T_n as an oriented tree corresponds to the inclusion structure among clones: there is a directed edge $v_\omega \to v_{\omega'}$ if and only if $U_{\omega'} \subset U_\omega$ and no other clone is nested strictly between U_ω and U_ω; moreover this occurs if and only if ω has some height h, ω' has height $h+1$, and $\omega_i = \omega'_i$ for all $i \le h$.

Now we describe certain isometries and quasi-isometries of T_n and their effect on $\mathbf{Q}_n$.

A *height translation* of T_n is an isometry $f\colon T_n \to T_n$ with the property that $h_0 = \mathrm{ht}(f(v)) - \mathrm{ht}(v)$ is constant for $v \in T_n$; the constant h_0 is called the *height translation length* of f, and f is called more specifically an h_0-*height translation.* Height translations form a subgroup of the full isometry group of T_n, in fact they are exactly the orientation preserving isometries of T_n. Height translations of T_n are related to similarities of $\mathbf{Q}_n$ as follows:

Proposition 23. *Continuous extension defines an isomorphism between the height translation group of T_n and the similarity group of $\mathbf{Q}_n$. The extension to $\mathbf{Q}_n$ of an h_0-height translation of T_n is an n^{-h_0} similarity of $\mathbf{Q}_n$. In particular, the height preserving isometry group of T_n corresponds to the isometry group of $\mathbf{Q}_n$.* $\diamondsuit$

We also need a quasification of Proposition 23. A quasi-isometry $f\colon T_n \to T_n$ is called a *coarse height translation* if there exists $A \ge 0$ such that $\mathrm{ht}(f(v)) - \mathrm{ht}(v)$ varies over some subinterval $[m, m+A] \subset \mathbf{R}$ of length $\le A$ as v varies over T_n. Any value $h_0 = \mathrm{ht}(f(v)) - \mathrm{ht}(v) \in [m, m+A]$ is called a *coarse height translation length* of f. We shall incorporate the coarseness constant A and the height translation length h_0 into the terminology by referring to f as an *A-coarse h_0-height translation.*

The quasification of Proposition 23 says roughly speaking that continuous extension defines a "uniform" isomorphism between the coarse height translation group of T_n and the quasisimilarity group of $\mathbf{Q}_n$:

Proposition 24. *For each $K' \ge 1$ there exist constants $K \ge 1, C \ge 0, A \ge 0$, $R \ge 1$ with the following properties:*

- *Each K'-quasisimilarity $F\colon \mathbf{Q}_n \to \mathbf{Q}_n$ extends to a (K, C) quasi-isometric A-coarse height translation $f\colon T_n \to T_n$;*
- *If f is a coarse h_0-height translation then F is $n^{-h_0} \cdot [R^{-1}, R]$-bilipschitz.*

Conversely, for each $K \ge 1, C \ge 0, A \ge 0$ there exists $K' \ge 1$ such that the continuous extension of each (K, C)-quasi-isometric A-coarse height translation of T_n is a K'-quasisimilarity of $\mathbf{Q}_n$. $\diamondsuit$

Proof of Theorem 7. Fix a uniform quasisimilarity action of a group G on $\mathbf{Q}_n$, $n \geq 2$, and suppose that the induced action of G on the space of distinct doubles of $\mathbf{Q}_n$ is cocompact.

We must produce a bilipschitz homeomorphism $\mathbf{Q}_n \to \mathbf{Q}_p$ which conjugates the G action on $\mathbf{Q}_n$ to a similarity action on $\mathbf{Q}_p$.

Step 1: A topological conjugacy. Using Proposition 24 together with Lemma 22, the uniform quasi-similarity action of G on $\mathbf{Q}_n$ extends to a quasi-action of G on T_n by uniformly coarse height translations. Moreover, coboundedness of the G-action on the double space of $\mathbf{Q}_n$ translates to coboundedness of the G-quasi-action on T_n.

Apply Theorem 1 to get a new tree T', a cobounded isometric action of G on T', and a quasiconjugacy $f : T' \to T_n$ with continuous extension $F : \partial T' \to \partial T_n$. Since the quasi-action of G on T_n fixes the end $-\infty$, the action of G on T' fixes the end $F^{-1}(-\infty)$ which we also denote as $-\infty$. Orient T' away from $-\infty$, and so the G-action respects this orientation. The downward valence at each vertex of T' with respect to this orientation equals 1.

Now we make some simplifications to T'.

First, we may assume:

(*) T' has no valence 1 vertices.

For if there is a valence 1 vertex, incident to an edge e, we may equivariantly collapse all edges in the orbit $G \cdot e$ producing a new tree T'' on which G acts, and a G-equivariant collapsing map $T' \mapsto T''$. The collapsing map $T' \mapsto T''$ is a quasi-isometry because each component of the forest $\cup G \cdot e$ is a graph of uniformly finite size: in fact, each component is a star graph with k valence 1 vertices for some constant k and with one vertex of valence k. Replacing T' with T'' reduces the number of vertex orbits, of which there are finitely many. Continuing inductively, eventually (*) is established.

At this stage, the quotient graph T'/G is a connected directed graph, with inward valence 1 at each vertex, and outward valence at least 1. It follows that T'/G is a circle with one or more vertices, all of whose directed edges agree with some global orientation on the circle.

For the second simplification, we may assume:

- T' has exactly one orbit of vertices and one orbit of edges, or in other words the directed graph T'/G is a circle with one vertex and one edge.

If this is not so, consider any edge e of T', pointing toward a vertex $\partial_+ e$, and pointing away from a vertex $\partial_- e$ in a different orbit than $\partial_+ e$. Since $\partial_+ e \neq \partial_- e$, and since no other edge in the orbit $G \cdot e$ points toward $\partial_+ e$, it follows that the components of $\cup G \cdot e$ are graphs of uniformly finite size. Collapsing all edges in the orbit $G \cdot e$ produces a tree T'' on which G acts and a G-equivariant, quasi-isometric collapse $T' \to T''$. Replacing T' by T'' and continuing inductively, eventually (**) is established.

With (*) and (**) established, T' is an oriented tree on which G acts, transitively on vertices and on edges, with downward valence 1 and constant upward valence p for some integer $p \geq 2$. In other words, we may identify $T' = T_p$ as oriented trees. Extension of the G-action on T_p gives a similarity action on $\mathbf{Q}_p$. Since the quasiconjugacy $f \colon T_p \to T_n$ takes the $-\infty \in \partial T_p$ to $-\infty \in \partial T_n$, it follows that f induces a topological (indeed quasiconformal) conjugacy $F \colon \mathbf{Q}_p \to \mathbf{Q}_n$ between G-actions.

Step 2: F is bilipschitz We will reduce this statement to Claim 25 which describes the effect of F on clones of $\mathbf{Q}_p$; in step 3 we will prove Claim 25.

Let $\mathcal{U}_h$ be the set of height h clones in $\mathbf{Q}_p$, and let $F(\mathcal{U}_h) = \{F(U) \bigm| U \in \mathcal{U}_h\}$ be the set of image clopens in $\mathbf{Q}_n$.

Claim 25. *There exists $\lambda > 1$ and an interval $[a,b] \subset (0,\infty)$ such that for all $h \in \mathbf{Z}$ and all $U \in \mathcal{U}_h$,*

$$\mathrm{Diam}(F(U)),\ \mathrm{sep}(F(U)) \in \lambda^{-h} \cdot [a,b]$$

Using this claim we show that F is bilipschitz.

First we evaluate λ. Since each element of $\mathcal{U}_h$ subdivides into p disjoint elements of $\mathcal{U}_{h+1}$, it follows that each element $F(U) \in F(\mathcal{U}_h)$ is partitioned into p disjoint elements of $F(\mathcal{U}_{h+1})$; moreover, by Claim 25 the diameters of these partition elements shrink by a factor of λ relative to the diameter of $F(U)$, up to a bounded multiplicative error. A simple calculation shows that the Hausdorff dimension of $\mathbf{Q}_n$ equals $\log_\lambda p$. But we know that the Hausdorff dimension of $\mathbf{Q}_n$ equals 1, proving that

$$\lambda = p.$$

Consider now two points $x, y \in \mathbf{Q}_p$, and let $U \in \mathcal{U}_h$ be the smallest clone of $\mathbf{Q}_p$ containing x, y, so

$$d_{\mathbf{Q}_p}(x,y) = p^{-h}$$

We also have:

$$d_{\mathbf{Q}_n}(Fx, Fy) \leq \mathrm{Diam}(F(U)) \leq b\lambda^{-h} = bp^{-h}$$

And, since x, y are contained in *distinct* level $h+1$ clones U_x, U_y we have

$$d_{\mathbf{Q}_n}(Fx, Fy) \geq \operatorname{sep}(F(U_x)) \geq a\lambda^{-h} = ap^{-h}$$

proving that

$$ad_{\mathbf{Q}_p}(x, y) \leq d_{\mathbf{Q}_n}(Fx, Fy) \leq bd_{\mathbf{Q}_p}(x, y)$$

and so F is bilipschitz.

Step 3: Proof of Claim 25 The similarity action of G on $\mathbf{Q}_p$ induces a stretch homomorphism $s\colon G \to (0, \infty)$ defined by the property that g is an $s(g)$-similarity for each $g \in G$. Let $G_0 = \operatorname{Ker}(s)$, and so G_0 is the subgroup of G acting isometrically on $\mathbf{Q}_p$. The quotient group $G/G_0 \approx \operatorname{image}(s)$ is infinite cyclic. Choose an infinite cyclic subgroup $Z = \langle z \rangle$ of G such that the projection from Z to G/G_0 is an isomorphism. To prove the claim we shall show:

(1) The action of G_0 on $\mathbf{Q}_n$ is uniformly bilipschitz: there exists $R \geq 1$ such that each $g \in G_0$ is a R-bilipschitz homeomorphism of $\mathbf{Q}_n$.

(2) There exists $\lambda > 1$ and an interval $[a_0, b_0] \in (0, \infty)$ such that for each integer k, the action of z^k on $\mathbf{Q}_n$ is a $\lambda^{-k} \cdot [a_0, b_0]$-bilipschitz homeomorphism.

To see why this proves the claim, note first that G acts transitively on vertices of T_p and so G_0 acts transitively on each level of T_p, in particular on level zero. This shows that G_0 acts transitively on the clones $\mathcal{U}_0$ of $\mathbf{Q}_p$. Since $F\colon \mathbf{Q}_p \to \mathbf{Q}_n$ is G_0-equivariant it follows that G_0 acts transitively on the clopens $F(\mathcal{U}_0)$ of $\mathbf{Q}_n$. Together with (1) above it follows that the diameters and separations of the clopens in $F(\mathcal{U}_0)$ are all bounded multiples of the diameter and separation of a single clopen in $F(\mathcal{U}_0)$, all lying in some fixed interval $[R^{-1}, R]$. Next, note that the action of z^k on vertices of T_p induces a bijection from level 0 vertices to level k vertices; it follows that the action of z^k on $\mathbf{Q}_p$ induces a bijection from $\mathcal{U}_0$ to $\mathcal{U}_k$, and so the action of z^k on $\mathbf{Q}_n$ induces a bijection from $F(\mathcal{U}_0)$ to $F(\mathcal{U}_k)$. Using (2) above it follows that the diameters and separations of all the clopens in $F(\mathcal{U}_k)$ lie in the interval $\lambda^{-k}[a_0 R^{-1}, b_0 R]$. This establishes Claim 25.

The proofs of (1) and (2) rely on some general results about infinite cyclic uniform quasisimilarity actions on metric spaces. First we have the following result from [FM99], which occurs in Proposition 3.3, Step 4:

Lemma 26. *If $f_n \colon X \to X$, $n \in \mathbf{Z}$, is a uniform quasisimilarity action of the infinite cyclic group $\mathbf{Z}$ on a metric space X, then there is a unique number $\lambda \in (0, \infty)$ with the following property: for each $n \in \mathbf{Z}$ the map f_n is $\lambda^n \cdot [K^{-1}, K]$-bilipschitz, where K depends only on the quasisimilarity constant of the action (f_n).* $\diamondsuit$

The map $n \mapsto \lambda^n$ is called the *stretch homomorphism* of the infinite cyclic uniform quasisimilarity action (f_n). Note that (f_n) is uniformly bilipschitz if and only if the stretch homomorphism is trivial.

Next we have a generalization of [FM99], Proposition 3.3, giving a topological characterization of when an infinite cyclic uniform quasisimilarity action is uniformly bilipschitz. Given a topological space X and an infinite cyclic action $f_n \colon X \to X$, $n \in \mathbf{Z}$, we say that the action is *locally homothetic at the point* $x \in X$ if $f_n(x) = x$ for all n, and in either the positive or negative direction points near x converge to x. That is: for some $\epsilon \in \{-1, +1\}$, and for some neighborhood U of x, we have

$$f_n(y) \to x \quad \text{as} \quad n \to \epsilon \cdot \infty, \quad \text{for all} \quad y \in U$$

If the neighborhood U can be taken to be all of X then we say that f is *globally homothetic.*

Lemma 27. *Let $f_n \colon X \to X$, $n \in \mathbf{Z}$ be a uniform quasisimilarity action of the infinite cyclic group $\mathbf{Z}$ on a complete metric space X. Then exactly one of the following happens:*

- *The action f_n is uniformly bilipschitz, with bilipschitz constant depending only on the quasisimilarity constant.*
- *There exists a point x at which the action is locally homothetic. When this occurs, the local homothety point x is unique, and in fact f is globally homothetic.*

Proof. If there exists a point of local homothety x for the action f_n then, for points y near x, the ratios

$$\frac{d(f_n(x), f_n(y))}{d(x,y)}, n \in \mathbf{Z}$$

converge to 0 for $n \to -\infty$ or $+\infty$, and so the ratios

$$\frac{d(f_{-n}(x), f_{-n}(y))}{d(x,y)}$$

are not bounded, proving that f_n is not uniformly bilipschitz.

Conversely, suppose f_n is not uniformly bilipschitz, with stretch homomorphism $n \mapsto \lambda^n$. Choose an interval $[a,b] \subset (0,\infty)$ so that f_n is $\lambda^n \cdot [a,b]$-bilipschitz for all n. Since $\lambda \neq 1$ it follows that for sufficiently large n we have $1 \notin \lambda^n \cdot [a,b]$ and so there is at most one fixed point. The Contraction Mapping Theorem shows the existence of a unique fixed point x, and it is evident that f_n is locally homothetic at x, indeed f_n is globally homothetic. $\diamondsuit$

Now we prove (1) and (2) above.

To prove (1), note that the action of G_0 on $\mathbf{Q}_p$ is uniformly bilipschitz, indeed it is isometric. It follows that no cyclic subgroup of G_0 has a local homothety point in $\mathbf{Q}_p$. The actions of G_0 on $\mathbf{Q}_p$ and on $\mathbf{Q}_n$ are topologically conjugate, and so no cyclic subgroup of G_0 has a local homothety point in $\mathbf{Q}_n$. From Lemma 27 it follows that each cyclic subgroup of G_0 is uniformly bilipschitz, with bilipschitz constant independent of the cyclic subgroup since the quasisimilarity constant is independent. In other words, the action of G_0 on $\mathbf{Q}_n$ is uniformly bilipschitz.

To prove (2), let $k \mapsto \lambda^k$ be the stretch homomorphism of the action of $Z = \langle z \rangle$ on $\mathbf{Q}_n$. Since z *does* have a local homothety point in $\mathbf{Q}_p$ it also has one in $\mathbf{Q}_n$, and so $\lambda \neq 1$. By replacing z with z^{-1} if necessary we obtain $\lambda > 1$.

This completes the proof of Claim 25 and of Theorem 7. The proof of Corollary 8 is immediate. $\diamondsuit$

As remarked in the introduction, the motivation for Theorem 7 comes from the end of [FM99] where it is asked whether a uniform quasisimilarity action on $\mathbf{Q}_n$ is always conjugate to a similarity action, at least when n is not a proper power. But this is false. Consider for example $\mathrm{Sim}(\mathbf{Q}_4)$, the full similarity group of $\mathbf{Q}_4$. Conjugating by a bilipschitz homeomorphism $\mathbf{Q}_4 \to \mathbf{Q}_2$ we obtain a faithful, uniform quasisimilarity action of $\mathrm{Sim}(\mathbf{Q}_4)$ on $\mathbf{Q}_2$. However, this action is not bilipschitz conjugate to a uniform similarity action of $\mathrm{Sim}(\mathbf{Q}_4)$ on $\mathbf{Q}_2$. To see why, $\mathrm{Sim}(\mathbf{Q}_4)$ contains a copy of the symmetric group on 4 symbols, a finite group of order $4! = 24$. Every similarity action of a finite group on $\mathbf{Q}_2$ is an isometric action. But every finite subgroup of the isometry group of $\mathbf{Q}_2$ is a 2 group, because it acts by direction preserving isometries on the tree T_2 fixing some vertex.

6 Application: Virtually free, cocompact lattices

In this section we prove Theorem 9 which shows that a locally compact group containing a cocompact lattice which is free of finite rank is closely related to the automorphism group of a bounded valence, bushy tree. Corollary 10 which gives simple examples of virtually free groups that cannot be cocompact lattices in the same locally compact group.

We want to understand when two virtually free groups can both act properly discontinuously and cocompactly by isometries on the same proper metric space X. Slightly more generally, we ask whether they are both cocompact lattices in the same locally compact group.

Lemma 28. *Let $\mathcal{G}$ be a locally compact topological group, Γ a finitely generated, cocompact lattice in $\mathcal{G}$, and equip Γ with the word metric. Then $\mathcal{G}$ quasi-acts coboundedly on Γ.*

Proof. We prove this by constructing a metric on $\mathcal{G}$ which is left invariant and so that the inclusion map $\Gamma \hookrightarrow \mathcal{G}$ is a quasi-isometry, with respect to the word metric d_A for a fixed finite generating set A of Γ. We may then quasiconjugate the left action of $\mathcal{G}$ on itself to a quasi-action on Γ using the inclusion map $\Gamma \to \mathcal{G}$. The metric we construct on $\mathcal{G}$ does not induce the given topology on $\mathcal{G}$, but as we are only concerned with the large scale geometry of $\mathcal{G}$ this is not important.

Let K be a symmetric compact set in $\mathcal{G}$ containing the identity and containing the generating set A for Γ, so that $\Gamma K = \mathcal{G}$. It follows that K is a generating set for $\mathcal{G}$. Let d_K be the left invariant word metric on $\mathcal{G}$ defined in the usual manner using the generating set K: $d_K(g, g')$ is the minimal $n \geq 0$ for which there exists a K-chain of length n, meaning a sequence $g = g_0, g_1, \ldots, g_n = g'$ with $g_{i+1} \in g_i K$ for all i. By the choice of K, every element of $\mathcal{G}$ is within d_K distance 1 of an element of Γ. Thus we need only show that on Γ the word metric d_A and the restriction of d_K are quasi-isometric.

Since K contains the generating set A for Γ, any path in the Cayley graph with respect to A is a K-chain, and so d_K is bounded above by the word metric d_A.

Given any K-chain $\{g_i\}$ from γ_1 to γ_2, define a sequence $\{\gamma_i\}$ in Γ by choosing, for each i, an element of $g_i K \cap \Gamma$. The intersection is non-empty by the definition of K. Now $\gamma_{i+1} \in g_{i+1} K \subset g_i K^2 \subset \gamma_i K^3$, so this sequence in Γ is a path in the Cayley graph with respect to the generating set $K^3 \cap \Gamma$, which is finite as K^3 is compact. Thus d_K is bounded below by the word metric for $K^3 \cap \Gamma$.

We have shown d_K is pinched between two finitely generated word metrics on Γ, which proves that it is in the same quasi-isometry class as the word metrics. $\diamondsuit$

Proof of Theorem 9. Given a locally compact group $\mathcal{G}$ which has a free group Γ as a cocompact lattice, we need to construct a tree T with a $\mathcal{G}$ action, so that the induced map $\phi\colon \mathcal{G} \to \mathrm{Isom}(T)$ is continuous, closed, has compact kernel, and has cocompact image.

Lemma 28 gives a cobounded quasi-action of $\mathcal{G}$ on the Cayley graph of Γ. Moreover, the proof shows that if we fix a compactly generated word metric on $\mathcal{G}$ then the left action of $\mathcal{G}$ on itself is quasiconjugate to resulting quasi-action of $\mathcal{G}$ on the Cayley graph of Γ. Theorem 1 produces the tree T with a $\mathcal{G}$ action. This $\mathcal{G}$ action is quasi-conjugate to the quasi-action of $\mathcal{G}$ on the Cayley graph of Γ, and so the left action of $\mathcal{G}$ on itself is quasiconjugate to the action of $\mathcal{G}$ on T. It remains to show using this quasi-conjugacy that the map $\phi\colon \mathcal{G} \to \mathrm{Isom}(T)$ has all the desired properties. For example, the left action of $\mathcal{G}$ on itself is cobounded, implying that the $\mathcal{G}$-action on T is cobounded, and so the image of $\phi\colon \mathcal{G} \to \mathrm{Isom}(T)$ is cocompact.

Lemma 29. *Given a bounded valence, bushy tree T, a sequence (f_i) converges in* $\mathrm{Isom}(T)$ *if and only if (f_i) satisfies the following property:*

Coarse convergence *There is some D so that for any v there is n so that $\{f_i(v) \mid i \geq n\}$ has diameter at most D.*

Proof. "Only if" is obvious.

Using the local finiteness of T, pass to a subsequence of the f_i which converges to some f. Further, if, for some v, there are infinitely many f_i for which $f_i(v) \neq f(v)$, then we can find another convergent subsequence with a limit, f', different from f. Clearly one has that $d(f, f') \leq D$. As isometries of T are unique in their bounded distance classes, this is a contradiction proving the lemma. $\diamondsuit$

Continuing the proof of Theorem 9, consider a sequence of elements $\{g_i\}$ in $\mathcal{G}$. Given two quasiconjugate quasi-actions of $\mathcal{G}$, the sequence g_i satisfies the coarse convergence condition of Lemma 29 for one of the quasi-actions if and only if it satisfies the condition for the other quasi-action. In particular we can apply this to the left action of $\mathcal{G}$ on itself and to the quasi-conjugate action of $\mathcal{G}$ on T. If g_i converges in $\mathcal{G}$ then it satisfies coarse convergence with respect to the left action, and so $\phi(g_i)$ satisfies coarse convergence in

T; applying Lemma 29 it follows that g_i converges in $\mathrm{Isom}(T)$. This proves that ϕ is continuous.

Also, ϕ is proper, for take a compact subset $C \subset \mathrm{Isom}(T)$ and a sequence g_i in $\phi^{-1}(C)$. Passing to a subsequence, $\phi(g_i)$ converges in C, and so by Lemma 29, $\phi(g_i)$ satisfies coarse convergence in T. This implies that g_i satisfies coarse convergence in $\mathcal{G}$. Passing to a subsequence it follows that g_i converges in $\mathcal{G}$; let g be the limit. Continuity implies that $\phi(g) = \lim \phi(g_i)$ and so $\phi(g) \in C$ and $g \in \phi^{-1}(C)$.

The kernel of ϕ, by definition, consists of those elements which act trivially on T. This means that there is some R so that, in the quasi-action on Γ, every element of the kernel moves no point more than R. This means, as in the proof of Lemma 28, that there is a compact set K in $\mathcal{G}$ so that for any g in the kernel of ϕ, and any $g' \in \mathcal{G}$, $gg' \in g'K$. In particular, $g \in K$. As the kernel of ϕ is closed and contained in a compact set, it is compact.

$\diamondsuit$

Proof of Corollary 10. If two virtually free groups G and G' are both cocompact lattices in the same locally compact group, Theorem 9 produces a tree T on which they both act properly discontinuously and cocompactly.

Consider a group $G = \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/p\mathbb{Z}$ for a prime p. Let G act properly and cocompactly on a bounded valence, bushy tree T, with all vertices of valence at least three. We claim T must be the Bass-Serre tree arising from the given splitting.

Let X be the quotient graph of groups. We must have X a tree, as G has no surjections to $\mathbf{Z}$. Consider an extreme vertex x of X, and let e be the incident edge. As there are no valence one vertices in T, the edge group of e must be a proper subgroup of the vertex group of x. Since the action on T is proper, all of the groups are finite. Every finite subgroup of G is contained in, and hence equal to, a conjugate of one of the free factors. Conversely, every such conjugate must fix a vertex of T. Thus X has exactly two extreme points, corresponding to the free factors of G, with the incident edges, and all other vertices of X, having trivial stabilizers. A finite tree with two extreme points is a subdivided interval, and as T has no valence two vertices, X is a single edge, as claimed.

The groups $\mathbf{Z}/p\mathbf{Z} * \mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/q\mathbf{Z} * \mathbf{Z}/q\mathbf{Z}$ for distinct primes p and q, do not have isomorphic Bass-Serre trees, so the claim above and Theorem 9 prove that they are not both cocompact lattices in any locally compact group. $\diamondsuit$

References

[Bie72] R. Bieri, *Gruppen mit Poincaré-dualität*, Comment. Math. Helv. **47** (1972), 373–396.

[BK90] H. Bass and R. Kulkarni, *Uniform tree lattices*, Jour. AMS **3** (1990), no. 4, 843–902.

[BM97] M. Burger and S. Mozes, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 7, 747–752.

[BP00] M. Bourdon and H. Pajot, *Rigidity of quasi-isometries for some hyperbolic buildings*, preprint, February 2000.

[Bro82] K. Brown, *Cohomology of groups*, Graduate Texts in Math., vol. 87, Springer, 1982.

[CC92] J. Cannon and D. Cooper, *A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three*, Trans. AMS **330** (1992), 419–431.

[Cho96] R. Chow, *Groups quasi-isometric to complex hyperbolic space*, Trans. AMS **348** (1996), no. 5, 1757–1769.

[Dav98] M. W. Davis, *The cohomology of a Coxeter group with group ring coefficients*, Duke math. J. **91** (1998), 297–314.

[Dun85] M. J. Dunwoody, *The accessibility of finitely presented groups*, Invent. Math. **81** (1985), 449–457.

[Far97] B. Farb, *The quasi-isometry classification of lattices in semisimple Lie groups*, Math. Res. Letters **4** (1997), no. 5, 705–717.

[FM98] B. Farb and L. Mosher, *A rigidity theorem for the solvable Baumslag-Solitar groups*, Invent. Math. **131** (1998), no. 2, 419–451, Appendix by D. Cooper.

[FM99] B. Farb and L. Mosher, *Quasi-isometric rigidity for the solvable Baumslag-Solitar groups, II*, Invent. Math. **137** (1999), no. 3, 613–649.

[FM00a] B. Farb and L. Mosher, *The geometry of surface-by-free groups*, preprint, 2000.

[FM00b] B. Farb and L. Mosher, *On the asymptotic geometry of abelian-by-cyclic groups*, Acta Math. **184** (2000), no. 2, 145–202.

[FS87] M. Freedman and R. Skora, *Strange actions of groups on spheres*, J. Diff. Geom. **25** (1987), 75–98.

[FS96] B. Farb and R. Schwartz, *The large-scale geometry of Hilbert modular groups*, J. Diff. Geom. **44** (1996), no. 3, 435–478.

[Fur00] A. Furman, *Mostow-Margulis rigidity with locally compact targets*, Geom. Funct. Anal. (2000), to appear.

[Hin85] A. Hinkkanen, *Uniformly quasisymmetric groups*, Proc. London Math. Soc. **51** (1985), 318–338.

[JW72] F. E. A. Johnson and C. T. C. Wall, *On groups satisfying Poincaré duality*, Ann. of Math. **96** (1972), 592–598.

[KK99] M. Kapovich and B. Kleiner, *Coarse Alexander duality and duality groups*, preprint, 1999.

[KL97a] M. Kapovich and B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math. **128** (1997), no. 2, 393–416.

[KL97b] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and euclidean buildings*, IHES Publ. Math. **86** (1997), 115–197.

[MSW00a] L. Mosher, M. Sageev, and K. Whyte, *Maximally symmetric trees*, in preparation, 2000.

[MSW00b] L. Mosher, M. Sageev, and K. Whyte, *Quasi-actions on trees II: Bass-Serre trees*, in preparation, 2000.

[MSW00c] L. Mosher, M. Sageev, and K. Whyte, *Quasi-actions on trees, research announcement*, preprint, 2000.

[Pan89a] P. Pansu, *Dimension conforme et sphère à l'infini des variété à courbure négative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), no. 2, 177–212.

[Pan89b] P. Pansu, *Metriques de Carnot-Caratheodory et quasiisometries des espaces symmetriques de rang un*, Ann. of Math. **129** (1989), 1–60.

[Pau95] F. Paulin, *De la géométrie et la dynamique des groupes discrets*, Mémoire d'Habilitation à Diriger les Recherches, ENS Lyon, Juin 1995.

[Pau96] F. Paulin, *Un groupe hyperbolique est déterminé par son bord*, J. London Math. Soc. (2) **54** (1996), no. 1, 50–74.

[PS79] C. T. C. Wall P. Scott, *Topological methods in group theory*, Homological group theory, Proceedings of Durham symposium, Sept. 1977, London Math. Soc. Lecture Notes, vol. 36, 1979, pp. 137–203.

[Rei00] A. Reiter, *Groups quasi-acting on products of trees*, preprint, 2000.

[Rie93] E. Rieffel, *Groups coarse quasi-isometric to* $\mathbf{H}^2 \times \mathbf{R}$, Ph.D. thesis, UCLA, 1993.

[Ser80] J. P. Serre, *Trees*, Springer, New York, 1980.

[Sta68] J. Stallings, *On torsion free groups with infinitely many ends*, Ann. of Math. **88** (1968), 312–334.

[Sul] D. Sullivan, private correspondence.

[Sul81] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann surfaces and related topics, Proceedings of the 1978 Stony Brook Conference, Ann. Math. Studies, vol. 97, Princeton University Press, 1981, pp. 465–496.

[Tab00] J. Taback, *Quasi-isometric rigidity for* $\mathrm{PSL}(2, \mathbf{Z}[1/p])$, Duke Math. J. **101** (2000), no. 2, 335–357.

[Tuk80] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 73–78.

[Tuk81] P. Tukia, *A quasiconformal group not isomorphic to a Möbius group*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 149–160.

[Tuk86] P. Tukia, *On quasi-conformal groups*, J. D'Anal. Math. **46** (1986), 318–346.

[Why] K. Whyte, *Geometries which fiber over trees*, in preparation.

[Why99] K. Whyte, *The large scale geometry of the higher Baumslag-Solitar groups*, preprint, 1999.

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